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WHISTLER MODE PROPAGATION: SOLUTION IN HOMOGENEOUS AND WEAKLY INHOMOGENEOUS, LOSSY PLASMAS

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Abstract

While investigating the whistler mode propagation the modelling of loss in the media and its description as exactly as possible turned out to be absolutely necessary. Starting directly from Maxwell's Equations, as it is to be seen in the paper, using homogeneous and inhomogneneous lossy, superimposed plasma model, the exact space-time dependence of the propagating whistler mode is to be determined, which gives the possibility for more detailed description and comparison of calculated and measured fine structural characteristics of the whistlers. The method applies the multidimensional Laplace-transform and the Method of Inhomogeneous Basic Modes.

Introduction

As it is known, in the magnetosphere, – caused by a lightning discharge in the troposphere – the electromagnetic signal propagating in the ELF-VLF frequency band is called as whistler. In this paper our goal is determining the space-time dependence of this signal as exactly as possible.

The previous papers (FERENCZ, 1994a,b, FERENCZ et al., 1994, BOGNÁR et al., 1994) presented the exact space-time dependence of the solution in homogeneous or inhomogeneous lossless anisotropic plasma, respectively, obtained from Maxwell's Equations when the electromagnetic waves were supposed to be non-monochromatic. Now, our goal is to find the solution containing even the effect of loss. In the model plane waves propagating longitudinally along the superimposed field and (electron) plasma of one component will be considered.

In the lossy model the same plasma and wave pattern as above will be supposed again, where the loss originates from the dissipated energy caused by collisions between electrons. It will be characterized by the ν_c collision frequency. The superimposed magnetic field is supposed to be $\overline{B}_F = B_{F_0} \cdot \overline{e}_x$, directed along the $+x$ axis. The electron density of the plasma is N, and the electron mass and charge m and q, respectively. The average electron velocity excited by the electromagnetic field be $\overline{v}$.

In the present model ν_c, N and $\overline{B}_{F_0}$ are constant in the homogeneous case, while $\nu_c(x)$, $N(x)$, and $\overline{B}_{F_0}(x)$ depend on space in the inhomogeneous case. Investigation of more sophisticated ν_c-functions may be the goal of a further improved model. The change of the direction of $\overline{B}_{F_0}$ will be neglected as a first approximation in the inhomogeneous case.

1. Homogeneous Solution

First of all the homogeneous case is considered and the starting equations are the Maxwell's Equations:

$$\begin{aligned} \overline{\nabla} \times \overline{H}(\overline{r},t) &= \overline{J}(\overline{r},t) + \varepsilon_0 \frac{\partial \overline{E}(\overline{r},t)}{\partial t} , \\ \overline{\nabla} \times \overline{E}(\overline{r},t) &= -\mu_0 \frac{\partial \overline{H}(\overline{r},t)}{\partial t} , \\ \overline{\nabla}\, \overline{H}(\overline{r},t) &= 0 , \\ \overline{\nabla}\, \overline{E}(\overline{r},t) &= \frac{\rho}{\varepsilon_0} . \end{aligned} \tag{1.1}$$

Besides that the further equations must hold describing the interaction between the signal and the medium:

$$\begin{aligned} m\frac{\partial \overline{v}}{\partial t} + \overline{F} &= q(\overline{E} + \overline{v} \times \overline{B}) , \\ \overline{F} &= m\nu_c\overline{v} , \\ \overline{J} &= qN\overline{v} , \\ \overline{\nabla J} + \frac{\partial \rho}{\partial t} &= 0 . \end{aligned} \tag{1.2}$$

The effect of loss is contained in the additive term, $\overline{F}$. Introducing the commonly used gyrofrequency (ω_b) and plasma frequency (ω_p):

$$\omega_b = \frac{q}{m} B_{F_0} \quad \text{and} \quad \omega_p^2 = \frac{q^2 N}{\varepsilon_0 m} . \tag{1.3}$$

The frequency band of the signal searched for is $0 < \omega < \omega_b$, because the propagation will be in whistler mode.

As usual, one may start from the following:

$$m\frac{\partial \overline{v}}{\partial t} + m\nu_c\overline{v} = q(\overline{E} + \overline{v} \times \overline{B}) . \tag{1.4}$$

Rearranging this equation, the velocity vector $\overline{v}$ can be found, and for the x, y and z components an equation-system will be obtained, which is reducible to common differential equations (KAMKE, 1965), all of the same form. The solution of these exists in closed form (KAMKE, 1956).

Knowing the velocity components one may write the components of the current density, using the

$$\overline{J} = q \cdot N \cdot \overline{v} , \tag{1.5}$$

equation. Substituting *Eq.* (1.5) into Maxwell's Equations (1.1), it may be seen that if the appearance of changes describable solely by distributions can be excluded (İDEMEN, 1973) – the divergence equations will be fulfilled automatically if the rotational equations are satisfied (FERENCZ, 1978a).

As for plane-wave solutions are supposed now, as it was mentioned earlier,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x} \equiv 0 , \tag{1.6}$$

therefore the E_x and H_x components of the electromagnetic field are zero.

The equation system takes now the following form in time domain:

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial x^2} =& \frac{1}{c^2} \Bigg\{ \omega_p^2 \int_0^1 \frac{\partial E_y}{\partial \tau} e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau + \\
&+ \nu_c \omega_p^2 \int_0^1 E_y(\tau) e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau + \\
&+ \omega_b \omega_p^2 \int_0^1 E_z(\tau) e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau + \frac{\partial^2 E_y}{\partial t^2} \Bigg\} , \\
\frac{\partial^2 E_z}{\partial x^2} =& \frac{1}{c^2} \Bigg\{ \omega_p^2 \int_0^1 \frac{\partial E_z}{\partial \tau} e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau + \\
&+ \nu_c \omega_p^2 \int_0^1 E_z(\tau) e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau - \\
&- \omega_b \omega_p^2 \int_0^1 E_y(\tau) e^{-\nu_c(t-\tau)} \cos \omega_b(t-\tau) d\tau + \frac{\partial^2 E_z}{\partial t^2} \Bigg\} .
\end{aligned} \tag{1.7}$$

It was taken into consideration that $\nu_c \ll \omega_b$, i.e. the approximation can be taken for a real model in the magnetosphere.

2. Solution of the Equations

The Laplace-transform may be applied for the E_y and E_z field components according to space and time (FERENCZ, 1994a) even in lossy model.

By the aid of the relations

$$s \longleftrightarrow t \quad \text{and} \quad p \longleftrightarrow x \,, \tag{2.1}$$

the new transform-variables s and p are introduced, and the necessary initial conditions are as follows:

$$\begin{aligned} E_i(x, t=0) &= e_{i_{t0}}(x) \longleftrightarrow e_{i_{t0}}(p) \,, \\ E_i(x=0, t) &= e_{i_{x0}}(t) \longleftrightarrow e_{i_{x0}}(s) \,, \\ \left.\frac{\partial E_i(x,t)}{\partial x}\right|_{x=0} &= e'_{i_{x0}}(t) \longleftrightarrow e'_{i_{x0}}(s) \,, \\ \left.\frac{\partial E_i(x,t)}{\partial t}\right|_{t=0} &= e'_{i_{t0}}(x) \longleftrightarrow e'_{i_{t0}}(p) \,, \\ i &= y, z \,. \end{aligned} \tag{2.2}$$

As for describing the coupling of the two field components into the magnetospheric model two initial conditions will be sufficient, and the exciting signal is a lightning impulse, (i.e. the $t = 0$ and $x = 0$ initial values exist), further it can be supposed that the excitation is plane-polarized, at the plasma boundary so with the suitable choice of the $t = +0$ moment and the y and z coordinate axes the following can be obtained:

$$\begin{aligned} &e_{yt0}(x) = e_{yx0}(t) = e'_{yt0}(x) = e'_{yx0}(t) = 0 \,, \\ &e_{zt0}(x) = e'_{zt0}(x) = 0 \end{aligned} \tag{2.3}$$

and

$$e_{zx0}(s) = A(s) \,, \quad e'_{zx0}(s) = B(s) \,. \tag{2.4}$$

Taking all these into consideration, the field components may be expressed from the transformed equations:

$$\begin{aligned} u &= s + \nu_c \,, \\ E_y(p,s) &= c^2 \frac{\omega_b \omega_p^2 u (u^2+\omega_b^2)[pA(s)+B(s)]}{[c^2p^2(u^2+\omega_b^2) - \omega_p^2 u^2 - s^2(u^2+\omega_b^2)]^2 + [u\omega_b\omega_p^2]^2} \,, \\ E_z(p,s) &= E_y(p,s) \frac{c^2p^2(u^2+\omega_b^2) - \omega_p^2 u^2 - s^2(u^2+\omega_b^2)}{u\omega_p^2\omega_b} \,. \end{aligned} \tag{2.5}$$

Separating the originally space- and time-dependent lossy terms one arrives at the field components which have*formally* approximately identical form with the lossless case (FERENCZ, 1994a) :

$$E_y(p,s)=c^2\frac{a_0(s)a_\infty(s)a_2(s)}{b_3(s)}\cdot\frac{p+\frac{a_1(s)}{a_2(s)}}{p^4-\frac{b_2(s)}{b_3(s)}p^2+\frac{b_1(s)}{b_3(s)}}\ ,$$

$$E_z(p,s)=c^2\frac{a_\infty(s)a_4(s)a_2(s)}{b_3(s)}\cdot\frac{\left(p+\frac{a_1(s)}{a_2(s)}\right)\left(p^2-\frac{a_3(s)}{a_4(s)}\right)}{p^4-\frac{b_2(s)}{b_3(s)}p^2+\frac{b_1(s)}{b_3(s)}}\ , \tag{2.6}$$

where

$$\begin{aligned}
a_0(s)&=\omega_b\omega_p^2u\ , & a_2(s)&=A(s)\ ,\\
a_\infty(s)&=(u^2+\omega_b^2)\ , & a_3(s)&=\omega_p^2u^2+s^2(u^2+\omega_b^2)\ ,\\
a_1(s)&=B(s)\ , & a_4(s)&=(u^2+\omega_b^2)c^2\ ,\\
b_1(s)&=(u\omega_b\omega_p^2)^2+\omega_p^4u^4+s^4(u^2+\omega_b^2)^2+\\
&\quad+2\omega_p^2u^2s^2(u^2+\omega_b^2)\ ,\\
b_2(s)&=2c^2(u^2+\omega_b^2)[\omega_p^2u^2+s^2(u^2+\omega_b^2)]\ ,\\
b_3(s)&=c^4(u^2+\omega_b^2)^2\ .
\end{aligned} \tag{2.7}$$

However, we can find several essential differences between the lossy and lossless cases. It can be seen that, s and $(s+\nu_c)$ appear together, in the polynomials above, so if the Laplace-transform were applied mechanically to the lossless case, which contains attenuation – i.e. loss – (which would mean simply an $s\rightarrow(s+\nu_c)$ substitution) – that would mean a completely wrong result.

Executing the inversion of the Laplace-transform according to p as usual, the poles are:

$$p_{1,2}=$$

$$=\pm\frac{1}{c}j\sqrt{\frac{\omega^2(\omega_p^2+\omega_b^2-\omega^2)+\omega\omega_b\omega_p^2}{\omega_b^2-\omega^2}}\cdot\sqrt{\frac{1-\frac{\nu_c^2(\omega_b^2-\omega^2)+j\nu_c[2\omega(\omega_p^2-\omega^2)+\omega_p^2\omega_b]}{\omega_p^2\omega_b\omega+\omega^2(\omega_p^2+\omega_b^2-\omega^2)}}{1+\frac{\nu_c^2+2j\omega\nu_c}{\omega_b^2-\omega^2}}}\ ,$$

$$p_{3,4}=$$

$$=\mp\frac{1}{c}\sqrt{\frac{(\omega\omega_b\omega_p^2+\omega^4)-\omega^2(\omega_p^2+\omega_b^2)}{\omega_b^2-\omega^2}}\cdot\sqrt{\frac{1+\frac{\nu_c^2(\omega_b^2-\omega^2)+j\nu_c[2\omega(\omega_p^2-\omega^2)-\omega_p^2\omega_b]}{(\omega\omega_b\omega_p^2+\omega^4)-\omega^2(\omega_p^2+\omega_b^2)}}{1+\frac{\nu_c^2+2j\omega\nu_c}{\omega_b^2-\omega^2}}} \tag{2.8}$$

where the $s \to j\omega$ substitution leads us into the ω-domain, and for evaluating the effect of the loss, the originally lossless factors $(k_1(\omega), \alpha_1(\omega))$ are separated from the ones, generated by the loss.

All of these can be analysed in more details:

$$p_{12} = \mp j k_1(\omega)\eta_1(\omega) \ ,$$

where

$$\eta_1(\omega) = \sqrt[4]{\frac{1+\nu_c^2 M_1(\omega)+\nu_c^4 M_2(\omega)}{1+\nu_c^2 N_1(\omega)+\nu_c^4 N_2(\omega)}}e^{j\frac{1}{2}[-\mathrm{arctg}\ \nu_c\Phi_1-\mathrm{arctg}\ \nu_c\Phi_2]} \ ,$$

$$M_1(\omega) = \\ = \frac{[2\omega(\omega_p^2-\omega^2)+\omega_p^2\omega_b]^2-2(\omega_p^2-\omega^2)[\omega^2(\omega_p^2+\omega_b^2-\omega^2)+\omega_p^2\omega_b\omega]}{[\omega^2(\omega_p^2+\omega_b^2-\omega^2)+\omega_p^2\omega_b\omega]^2} \ ,$$

$$M_2(\omega) = \frac{(\omega_p^2-\omega^2)^2}{[\omega^2(\omega_p^2+\omega_b^2-\omega^2)+\omega_p^2\omega_b\omega]^2} \ ,$$

$$N_1(\omega) = \frac{2(\omega_b^2+\omega^2)}{(\omega_b^2-\omega^2)^2} \ ,$$

$$N_2(\omega) = \frac{1}{(\omega_b^2-\omega^2)^2} \ ,$$

$$\Phi_1(\omega) = \frac{2\omega(\omega_p^2-\omega^2)+\omega_b\omega_p^2}{\omega^2(\omega_p^2+\omega_b^2-\omega^2)+\omega_p^2\omega_b\omega-\nu_c^2(\omega_p^2-\omega^2)} \ ,$$

$$\Phi_2(\omega) = \frac{2\omega}{(\omega_b^2-\omega^2)+\nu_c^2} \tag{2.9}$$

and

$$p_{3,4} = \mp\alpha_3(\omega)\lambda_3(\omega) \ ,$$

where

$$\lambda_3(\omega) = \sqrt[4]{\frac{1+\nu_c^2 M_1'(\omega)+\nu_c^4 M_2'(\omega)}{1+\nu_c^2 N_1'(\omega)+\nu_c^4 N_2'(\omega)}}\cdot e^{j\frac{1}{2}[\mathrm{arctg}\ \nu_c\Phi_1'-\mathrm{arctg}\ \nu_c\Phi_2']} \ ,$$

$$M_1'(\omega) = \\ = \frac{2(\omega_p^2-\omega^2)[\omega\omega_b\omega_p^2+\omega^4-\omega^2(\omega_p^2+\omega_b^2)]+[2\omega(\omega_p^2-\omega^2)-\omega_p^2\omega_b]^2}{[\omega\omega_b\omega_p^2+\omega^4-\omega^2(\omega_p^2+\omega_b^2)]^2} \ ,$$

$$M_2'(\omega) = \frac{(\omega_p^2-\omega^2)^2}{[\omega\omega_b\omega_p^2+\omega^4-\omega^2(\omega_p^2+\omega_b^2)]^2} \ ,$$

$$N_1'(\omega) = \frac{2(\omega_b^2 + \omega^2)}{(\omega_b^2 - \omega^2)^2} ,$$

$$N_2'(\omega) = \frac{1}{(\omega_b^2 - \omega^2)^2} ,$$

$$\Phi_1'(\omega) = \frac{2\omega(\omega_p^2 - \omega^2) - \omega_p^2\omega_b}{(\omega\omega_b\omega_p^2 + \omega^4) - \omega^2(\omega_p^2 + \omega_b^2) + \nu_c^2(\omega_p^2 - \omega^2)} ,$$

$$\Phi_2'(\omega) = \frac{2\omega}{(\omega_b^2 - \omega^2) + \nu_c^2} . \tag{2.10}$$

For low loss approximation:

$$p_{1,2} = \mp jk_1(\omega)\eta_1(\omega) ,$$
$$\eta_1(\omega) = \sqrt{1 + \frac{\nu_c^2}{2}[M_1(\omega) - N_1(\omega)]} \cdot \left\{1 - j\frac{1}{2}\nu_c[\Phi_1(\omega) + \Phi_2(\omega)]\right\} \tag{2.11}$$

and

$$p_{3,4} = \mp j\alpha_3(\omega)\lambda_3(\omega) ,$$
$$\lambda_3(\omega) = \sqrt{1 + \frac{\nu_c^2}{2}[M_1'(\omega) - N_1'(\omega)]} \cdot \left\{1 + j\frac{1}{2}\nu_c[\Phi_1'(\omega) - \Phi_2'(\omega)]\right\} . \tag{2.12}$$

The *complex* form of the loss can be well seen in each term. The influence of this is a more realistic result for the lossy case – as it could be expected. The poles in the lossless case – i.e. the propagating factors and the attenuation factors are:

$$p_1 = -jk_1 , \quad p_2 = jk_1 , \quad p_3 = -\alpha_3 , \quad p_4 = \alpha_3 , \tag{2.13}$$

so we can see one forward propagating and one reflected but not attenuated mode, and one in the positive and one in the negative direction, which is not propagating, but solely attenuated mode; but here

$$p_1 = -jk_1' - \alpha_1', \quad p_2 = jk_1' + \alpha_1', \quad p_3 = -\alpha_3' - jk_3', \quad p_4 = \alpha_3' + jk_3' , \tag{2.14}$$

where all the poles are *complex*, that is, – in different extent – they will propagate and decrease simultaneously. (When interpreting these poles it must be taken into account that the frequency domain is $0 < \omega < \omega_b$.)

The magnetosphere is supposed to be infinite in this model, the edge of which is at the boundary between the troposphere and magnetosphere. The multi-hop whistlers will not be dealt with in this paper.

After all the space dependent spectra can be written as follows:

$$E_y(x,\omega)=c^2\frac{a_0(\omega)a_\infty(\omega)a_2(\omega)}{2b_3(\omega)}\cdot\frac{1}{k_1^2(\omega)\eta_1^2(\omega)+\alpha_3^2(\omega)\lambda_3^2(\omega)}\cdot$$
$$\cdot\left\{\frac{\left[\frac{a_1(\omega)}{a_2(\omega)}-jk_1(\omega)\eta_1(\omega)\right]}{jk_1(\omega)\eta_1(\omega)}\cdot e^{-jk_1(\omega)\eta_1(\omega)x}-\right.$$
$$-\frac{\left[\frac{a_1(\omega)}{a_2(\omega)}+jk_1(\omega)\eta_1(\omega)\right]}{jk_1(\omega)\eta_1(\omega)}\cdot e^{+jk_1(\omega)\eta_1(\omega)x}-$$
$$-\frac{\left[\frac{a_1(\omega)}{a_2(\omega)}-\alpha_3(\omega)\lambda_3(\omega)\right]}{\alpha_3(\omega)\lambda_3(\omega)}\cdot e^{-\alpha_3(\omega)\lambda_3(\omega)x}+$$
$$\left.+\frac{\left[\frac{a_1(\omega)}{a_2(\omega)}-\alpha_3(\omega)\lambda_3(\omega)\right]}{\alpha_3(\omega)\lambda_3(\omega)}\cdot e^{\alpha_3(\omega)\lambda_3(\omega)x}\right\}, \tag{2.15.a}$$

$$E_z(x,\omega)=c^2\frac{a_\infty(\omega)a_4(\omega)a_2(\omega)}{2b_3(\omega)}\cdot\frac{1}{k_1^2(\omega)\eta_1^2(\omega)+\alpha_3^2(\omega)\lambda_3^2(\omega)}\cdot$$
$$\cdot\left\{\frac{\left[k_1(\omega)\eta_1(\omega)+j\frac{a_1(\omega)}{a_2(\omega)}\right]\left[k_1^2(\omega)\eta_1^2(\omega)+\frac{a_3(\omega)}{a_4(\omega)}\right]}{k_1(\omega)\eta_1(\omega)}\cdot e^{-jk_1(\omega)\eta_1(\omega)x}+\right.$$
$$\left.+\frac{\left[\alpha_3(\omega)\lambda_3(\omega)-\frac{a_1(\omega)}{a_2(\omega)}\right]\left[\alpha_3^2(\omega)\lambda_3^2(\omega)-\frac{a_2(\omega)}{a_4(\omega)}\right]}{\alpha_3(\omega)\lambda_3(\omega)}\cdot e^{-\alpha_3(\omega)\lambda_3(\omega)x}\right\}. \tag{2.15.b}$$

It follows elementarily, that:

$$E_{yw}(x,\omega)=-jE_{zw}(x,\omega)\ , \tag{2.16}$$

that is, the signal is rotating right-handed, as it could be expected. Knowing the field components E_y and E_z one may obtain H_y and H_z unambiguously from Maxwell's Equations:

$$H_y(p,s)=\frac{p}{\mu_0 s}E_z(p,s)\ ,$$
$$H_z(p,s)=-\frac{p}{\mu_0 s}E_y(p,s)\ . \tag{2.17}$$

So the space-time dependence of the electrical and magnetic field components in the magnetosphere is as follows:

$$E_i(x,t) = \frac{1}{2\pi} \int\limits_{\omega_{\min}}^{\omega_{\max}} E_{i0}(x,\omega) \cdot e^{j[\omega t - \mathcal{K}_1(\omega)x]} d\omega \ ,$$

$$H_i(x,t) = \frac{1}{2\pi} \int\limits_{\omega_{\min}}^{\omega_{\max}} H_{i0}(x,\omega) \cdot e^{j[\omega t - \mathcal{K}_1(\omega)x]} d\omega \ , \tag{2.18}$$

$$0 < \omega_{\min} \quad \text{and} \quad \omega_{\max} < \omega_b \ , \quad \mathcal{K}_1(\omega) = k_1(\omega)\eta_1(\omega) \ .$$

It is important to notice that the initial conditions remain in the solution in $A(\omega)$ and $B(\omega)$, and by the aid of this fact the coupling can be described in a simple way between the exciting signal in the troposphere and the propagating whistler mode in the magnetosphere.

3. The Signal Propagating in the Troposphere and the Coupling

In the lossless case a method was presented (FERENCZ, 1994a) by the aid of which, and using the Method of the Inhomogeneous Basic Modes (FERENCZ, 1978a,b) one may obtain a simple form for the propagating signal in the troposphere. Now, – as the tropospheric model has not changed – this solution is identical with the previous one (FERENCZ, 1994a), i.e. it is:

$$E_{1z}(x,t) = -\frac{Z_0}{2} \int\limits_0^{x_0} J_0\left(\xi, t - \frac{x-\xi}{c}\right) d\xi \ ,$$

$$H_{1y}(x,t) = \frac{1}{2} \int\limits_0^{x_0} J_0\left(\xi, t - \frac{x-\xi}{c}\right) d\xi \ , \tag{3.1}$$

where

$$\overline{J}_1(x,t) = J_0(x,t)\overline{e}_z \ , \tag{3.2}$$

is supposed for the exciting current density, Z_0 is the characteristic impedance.

Taking into consideration the complete coupling, the exact space-time dependence of the propagating whistler mode in *homogeneous* and *lossy* plasma

$$E_{2zw}(x,t) = -\frac{Z_0}{4\pi} \int\limits_{\omega_{\min}}^{\omega_{\max}} \frac{k_0(\omega)}{k_0(\omega)+\mathcal{K}_1(\omega)} I_{x_0}(\omega) \cdot e^{j[\omega t - \mathcal{K}_1(\omega)(x-x_0)]} d\omega , \quad (3.3)$$

where

$$I_{x_0}(\omega) = \int\limits_{-\infty}^{\infty} \left[\int\limits_{0}^{x_0} J_0\left(\xi, t - \frac{x_0-\xi}{c}\right) d\xi \right] \cdot e^{-j\omega t} d\omega \ , \quad (3.4)$$

$$k_0 = \frac{\omega}{c} \ ,$$

from which all the further field components can be determined. The commonly known whistler spectrum pattern can be obtained from this space-time function at a given space by FFT or matched filtering.

4. Solution in Weakly Inhomogeneous Plasma

Now the magnetosphere is taken into consideration as a tempered, cold, anisotropic, lossy plasma, which is, however, weakly inhomogeneous. The inhomogeneity is supposed to be parallel with the signal propagation ($+x$ direction). Further let the whistler mode we are looking for be a plane wave propagating longitudinally along x axis defined by the $\overline{e}_x$ unit vector. The inhomogeneity of the plasma will be taken into consideration by the space dependence of the functions $N(x)$, $B_{F_0}(x)$ and $\nu_c(x)$, that is, the time invariance of the medium holds in the following, too. The equations describing the interaction between the signal and the medium in the weakly inhomogeneous plasma are as follows:

$$\begin{aligned} m\frac{\partial \overline{v}}{\partial t} + \overline{F} &= q(\overline{E} + \overline{v} \times \overline{B}_F) \ , \\ \overline{F} &= q\nu_c(x)\overline{v} \ , \\ \overline{J} &= qN(x)\overline{v} \ , \\ \overline{\nabla J} + \frac{\partial \rho}{\partial t} &= 0 \ . \end{aligned} \quad (4.1)$$

Further

$$\omega_b(x) = \frac{q}{m} B_{F_0}(x) \quad \text{and} \quad \omega_p^2(x) = \frac{q^2 N(x)}{\varepsilon_0 m} \ . \quad (4.2)$$

If we rewrite these equations according to the method applied in the homogeneous model (1.7), the time-depending equations to be solved:

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\Bigg\{\omega_p^2(x)\int_0^t \frac{\partial E_y}{\partial \tau} e^{-\nu_c(x)(t-\tau)} \cos\omega_b(x)\cdot(t-\tau)d\tau +$$
$$\nu_c(x)\omega_p^2(x)\int_0^t E_y(\tau)e^{-\nu_c(x)(t-\tau)}\cos\omega_b(x)\cdot(t-\tau)d\tau +$$
$$\omega_b(x)\omega_p^2(x)\int_0^t E_z(\tau)e^{-\nu_c(x)(t-\tau)}\cos\omega_b(x)\cdot(t-\tau)d\tau + \frac{\partial^2 E_y}{\partial t^2}\Bigg\} ,$$
$$\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\Bigg\{\omega_p^2(x)\int_0^t \frac{\partial E_z}{\partial \tau} e^{-\nu_c(x)(t-\tau)} \cos\omega_b(x)\cdot(t-\tau)d\tau +$$
$$\nu_c(x)\omega_p^2(x)\int_0^t E_z(\tau)e^{-\nu_c(x)(t-\tau)}\cos\omega_b(x)\cdot(t-\tau)d\tau -$$
$$\omega_b(x)\omega_p^2(x)\int_0^t E_y(\tau)e^{-\nu_c(x)(t-\tau)}\cos\omega_b(x)\cdot(t-\tau)d\tau + \frac{\partial^2 E_z}{\partial t^2}\Bigg\} . \tag{4.3}$$

However, the inhomogeneity changed the structure of the equations in way that they are not Laplace-transformable according to place and it is not expedient either to do that. Let the equations be transformed according to time, so with the

$$t \longleftrightarrow s \tag{4.4}$$

substitution one obtains that:

$$c^2\frac{\partial^2 E_y}{\partial x^2} = \left\{\omega_p^2(x)\frac{[s+\nu_c(x)]^2}{[s+\nu_c(x)]^2+\omega_b^2(x)} + s^2\right\}E_y(x,s) +$$
$$\omega_p^2(x)\omega_b(x)\frac{[s+\nu_c(x)]}{[s+\nu_c(x)]^2+\omega_b^2(x)}E_z(x,s) ,$$
$$c^2\frac{\partial^2 E_z}{\partial x^2} = \left\{\omega_p^2(x)\frac{[s+\nu_c(x)]^2}{[s+\nu_c(x)]^2+\omega_b^2(x)} + s^2\right\}E_z(x,s) -$$
$$\omega_p^2(x)\omega_b(x)\frac{[s+\nu_c(x)]}{[s+\nu_c(x)]^2+\omega_b^2(x)}E_y(x,s) . \tag{4.5}$$

As for quasi-longitudinal case is considered, the W.K.B. solution will be used to solve for the field components (FERENCZ, 1977), keeping in mind that now the signal is non monochromatic. Let the generalized propagation vector be introduced, that is, the homogeneous solution will be extended to the inhomogeneous case. So the poles obtained earlier will hold further, e.g. the poles belonging to the whistler mode propagating along the $+x$ direction are:

$$p_i(x,s) = \pm\frac{1}{c}\sqrt{\frac{\omega_p^2(x)[s+\nu_c(x)]^2 + s^2[(s+\nu_c(x))^2+\omega_b^2(x)] \pm j\omega_p^2(x)\omega_b(x)[s+\nu_c(x)]}{[s+\nu_c(x)^2]+\omega_b^2(x)}}$$

$$i = 1,2,3,4 \; , \tag{4.6}$$

which can be rewritten as:

$$p_1(\omega) = -jk_1(\omega)\eta_1(\omega) = -j\mathcal{K}_1(\omega) \; , \tag{4.7}$$

and for the fields:

$$\begin{aligned} E_{zw}(x,\omega) &= \frac{1}{4\mathcal{K}_1(\omega)}[a_2(\omega)\mathcal{K}_1(\omega)+ja_1(\omega)]e^{-j\int_{x_0}^{x}\mathcal{K}_1(\xi,\omega)d\xi} = E_{z_0}^{+}e^{-j\int_{x_0}^{x}\mathcal{K}_1(\xi,\omega)d\xi} \\ H_{yw}(x,\omega) &= -\frac{\mathcal{K}_1(\omega)}{\mu_0\omega}E_{z_0}^{+}e^{-j\int_{x_0}^{x}\mathcal{K}_1(\xi,\omega)d\xi} = \\ &= -\frac{E_{z_0}^{+}}{Z_0}\sqrt{\frac{\omega^2(\omega_p^2(x)+\omega_b^2(x)-\omega^2)+\omega\omega_b(x)\omega_p^2(x)}{\omega^2(\omega_b^2(x)-\omega^2)}}\eta_1(x,\omega)e^{-j\int_{x_0}^{x}\mathcal{K}_1 d\xi} \; . \end{aligned} \tag{4.8}$$

It is important to note that in our case the spectral Poynting vector, the $S(x,\omega)$ must be constant, as a consequence of the W.K.B. condition, which follows from the non-monochromacy of the signal:

$$H_{x,w}(x,\omega) = -\frac{E_{z_0}^{+}}{Z_0}n_1(x)\cdot e^{j\int_{x_0}^{x}\mathcal{K}_1(\xi,\omega)d\xi} \; , \tag{4.9}$$

where the characteristic impedance is as usual:

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi[\Omega] \; . \tag{4.10}$$

For the Poynting vector has the form:

$$S \sim (E_{zw})(H_{yw})^{*} = (E_{z_0}^{+})^2\frac{n_1(x)}{Z_0} = \text{const} \; = (A_0^{+})^2 \; , \tag{4.11}$$

where the space-dependent refraction coefficient is:

$$n_1(x) = \sqrt{\frac{\omega^2(\omega_p^2(x) + \omega_b^2(x) - \omega^2) + \omega\omega_b(x)\omega_p^2(x)}{\omega^2(\omega_b^2(x) - \omega^2)}} \cdot \eta_1(x,\omega) \ . \qquad (4.12)$$

Using these results the exact space-time dependence of the field components is:

$$E_{zw}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_0^+(\omega) \sqrt[4]{\frac{\omega^2(\omega_b^2(x) - \omega^2)}{\omega^2(\omega_p^2(x) + \omega_b^2(x) - \omega^2) + \omega\omega_b(x)\omega_p^2(x)}} \cdot \sqrt{\frac{1}{\eta_1(x)}} e^{j\left[\omega t - \int_{x_0}^{x} \mathcal{K}_1(\xi,\omega)d\xi\right]} d\omega \ ,$$

$$H_{yw}(x,t) = -\frac{1}{2\pi Z_0} \int_{-\infty}^{\infty} A_0^+(\omega) \sqrt[4]{\frac{\omega^2(\omega_b^2(x) - \omega^2)}{\omega^2(\omega_p^2(x) + \omega_b^2(x) - \omega^2) + \omega\omega_b(x)\omega_p^2(x)}} \cdot \sqrt{\eta_1(x)} e^{j\left[\omega t - \int_{x_0}^{x} \mathcal{K}_1(\xi,\omega)d\xi\right]} d\omega \ . \qquad (4.13)$$

As for the earlier solution for the troposphere holds even now, and the coupling equations may be written and solved in the usual manner, for the total solution, i.e. for the whistler mode propagating in lossy and weakly inhomogeneous plasma one obtains:

$$E_{2zw}(x,t) = -\frac{Z_0}{4\pi} \int_{\omega_{min}}^{\omega_{max}} I_{x_0}(\omega) \sqrt{\frac{\mathcal{K}_1(x,\omega)}{\mathcal{K}_1(x_0,\omega)}} \frac{\mathcal{K}_1(x_0,\omega)}{k_0(\omega) + \mathcal{K}_1(x_0,\omega)} e^{j\left[\omega t - \int_{x_0}^{x} \mathcal{K}_1(\xi,\omega)d\xi\right]} d\omega$$

$$(4.14)$$

$$H_{2yw}(x,t) = \frac{1}{4\pi} \int_{\omega_{min}}^{\omega_{max}} I_{x_0}(\omega) \sqrt{\frac{\mathcal{K}_1(x,\omega)}{\mathcal{K}_1(x_0,\omega)}} \frac{\mathcal{K}_1(x_0,\omega)}{k_0(\omega) + \mathcal{K}_1(x_0,\omega)} e^{j\left[\omega t - \int_{x_0}^{x} \mathcal{K}_1(\xi,\omega)d\xi\right]} d\omega \ ,$$

where $k_0 = \frac{\omega}{c}$ and $\mathcal{K}_1(x,\omega)$ is identical with the homogeneous solution.

5. Results

The (3.3), (3.4), (4.14) functions obtained with analysing methods published in (BOGNÁR et al.,1994, FERENCZ et al., 1994) may be evaluated by numerical processing. These are the FFT procedure and the matched filtering (HAMAR et al., 1982, 1992).

In the calculations different exciting spectra were used.

The results of calculations executed with the values of x = 30.000 km, ω_b = 55 krad/s, ω_p = 0.734 Mrad/s using the formulas obtained for the lossy model, are shown in *Figs. 1–7*. *Fig. 1* presents the amplitude-time functions at a given location with double-impulse excitation for lossless and low-lossy models (the distance of the two impulses is 1.7 ms, the length of the impulse is 0.15 ms, the distance of the magnetic field line from the centre of the Earth, using earth radius units, in the plane of the equator: L=4.654, the electron density at an altitude of 1000 km is N=2550 cm^{-3}, $\nu_c = 0$ and 0.1). It is important to notice that the lossy signal is damped but its character is unchanged, compared to the non-damped case.

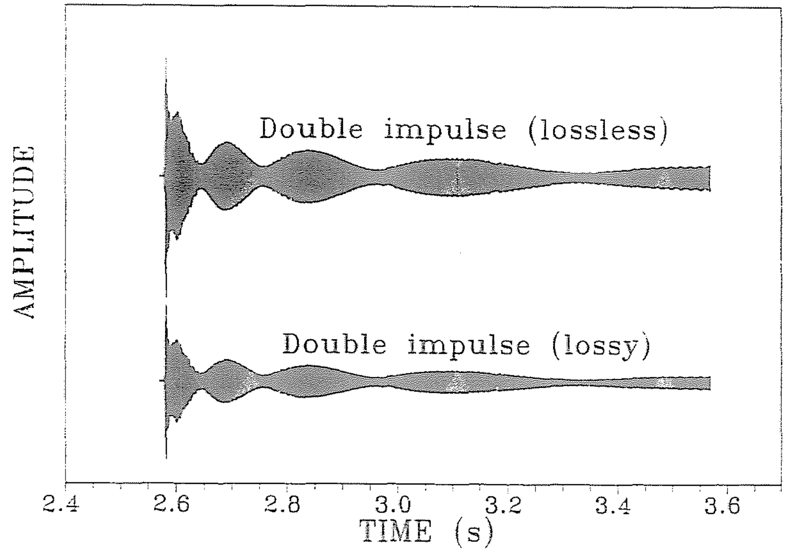

Fig. 1. Time-dependence of the electric field intensity of whistler at a given place in lossless and lossy cases

Fig. 2 shows the first section of the above, calculated whistlers. It can be seen that the position of the null phases does not change when the loss is low, while by increasing losses it will change as a consequence of the formulas.

Fig. 3 shows the characteristic shape of time-dependence of a calculated whistler (L=4.654, N=2550 cm^{-3}, $\nu_c = 0$ and 0.1). As for the fine structure, the characters of the lossy and lossless solutions are similar.

Fig. 4 shows the FFT analysis of one of the calculated whistlers shown in *Fig. 1*.

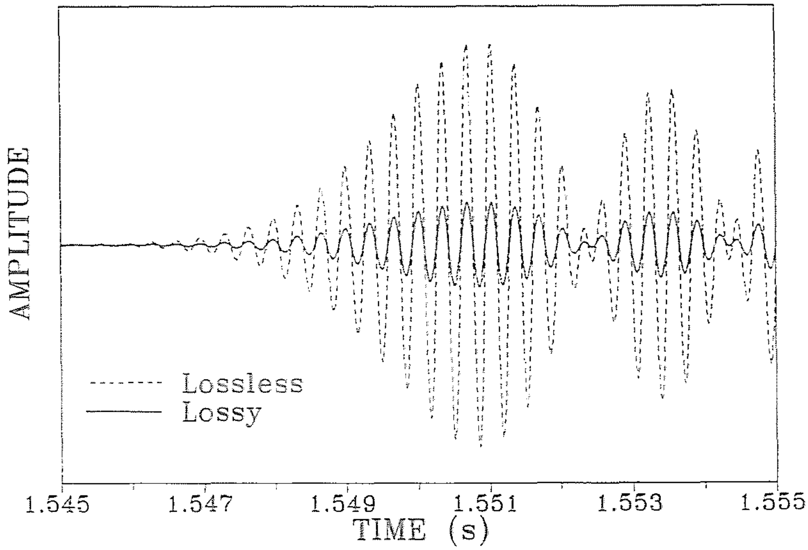

Fig. 2. Detailed time-dependence of the leading edge of calculated whistlers

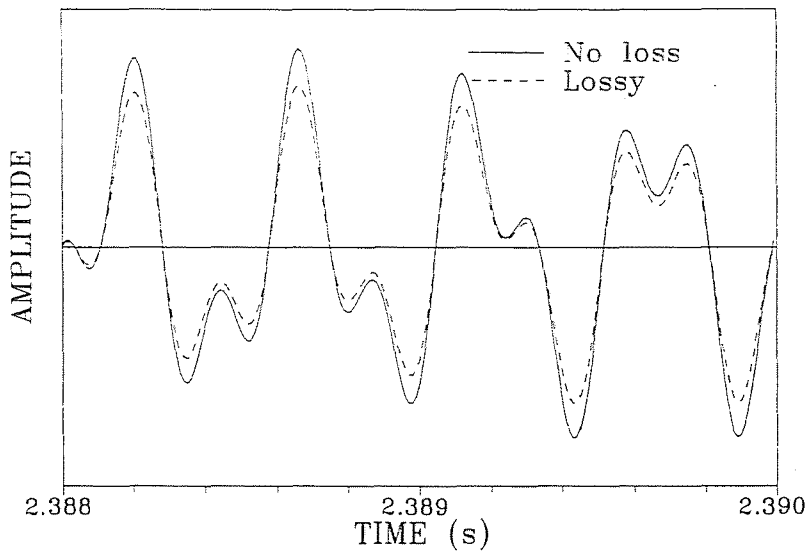

Fig. 3. Detailed time-dependence of whistlers

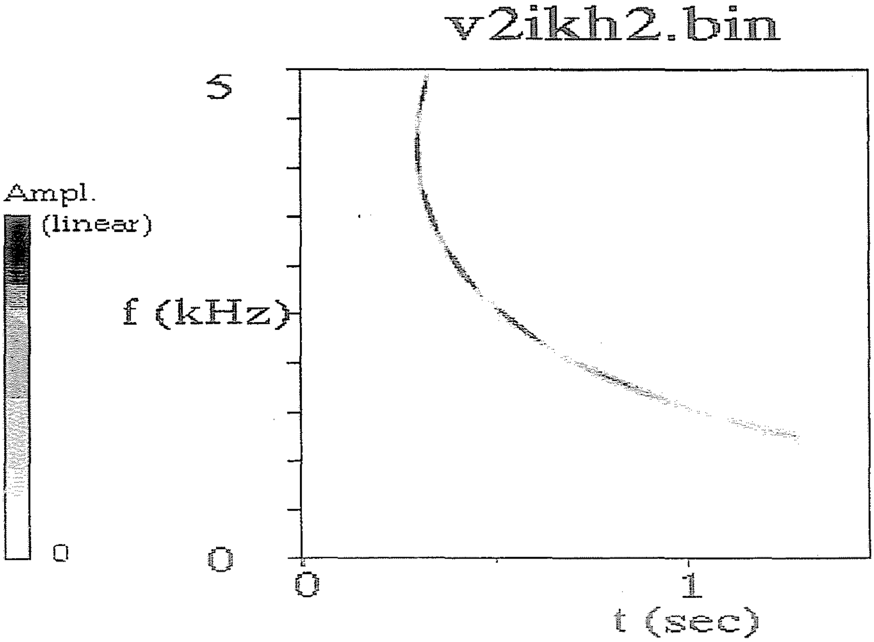

Fig. 4. Typical frequency-time function (FFT pattern) of whistlers

Fig. 5 shows the FFT pattern of a measured whistler (NO. 84010. ns., 21. July 1984 17.14.UT Halley, Antarctica; Courtesy of Dr. A. J. Smith, British Antarctic Survey, Cambridge, U.K.). It can be seen that the calculated result correlates significantly to the measured spectra.

Fig. 5 shows the same whistler after matched filtering. It is obvious that the former analysing processes can be used even for lossy formulas as well as earlier.

In *Fig. 6* a comparison is presented between impulse responses belonging to Dirac delta excitation for lossy and lossless cases (the amplitude is normalized to 2500 Hz).

Fig. 7 shows the amplitude density function of one measured whistler (NO.89221 AA, 9. August 1989, 16.17.UT Halley, Antarctica, Courtesy of Dr. A. J. Smith, British Antarctic Survey, Cambridge, U. K.) and two calculated whistlers (lossless and lossy), for $\nu_c = 0.2$. It is important that the results of the lossy case describe the amplitude function much more accurately when compared with measured values, than the lossless case.

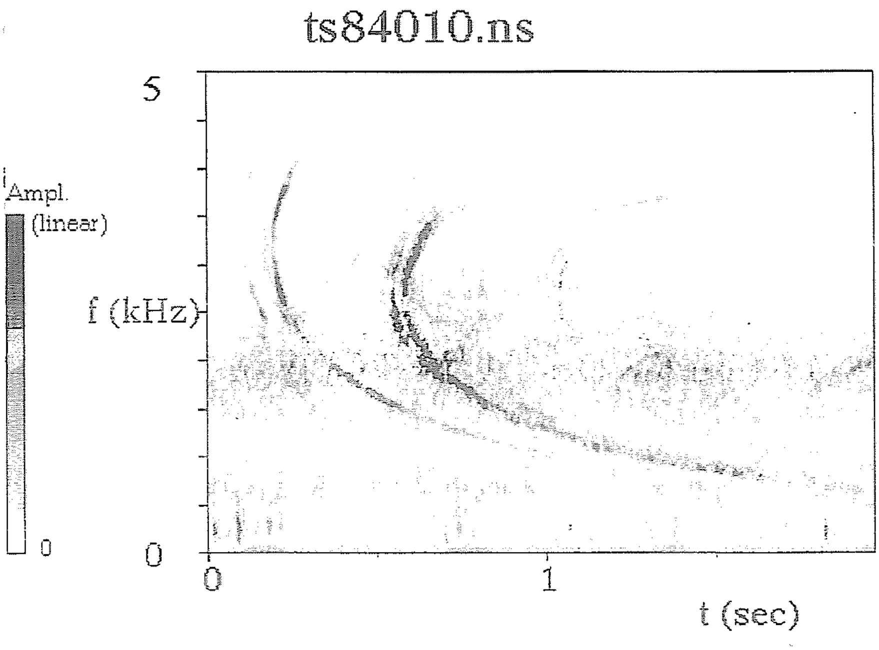

Fig. 5. FFT analysis of a measured whistler

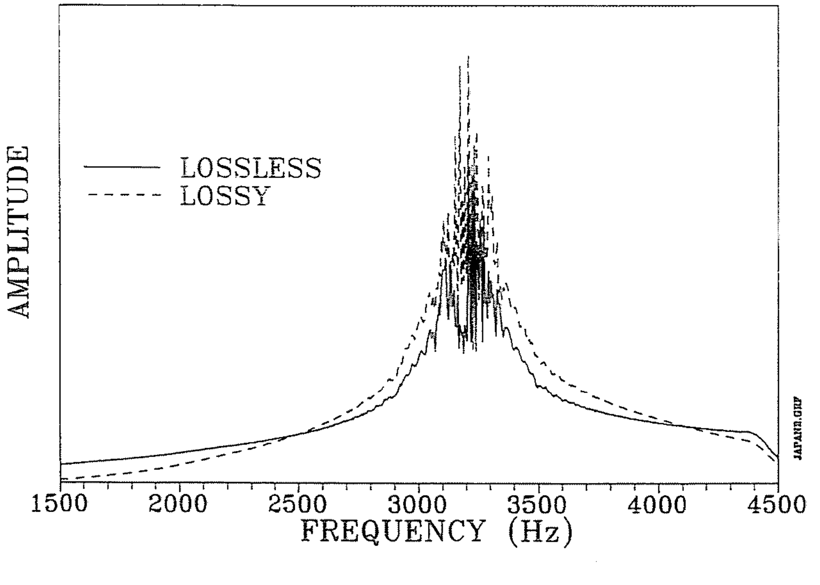

Fig. 6. Amplitude spectra as function of frequency

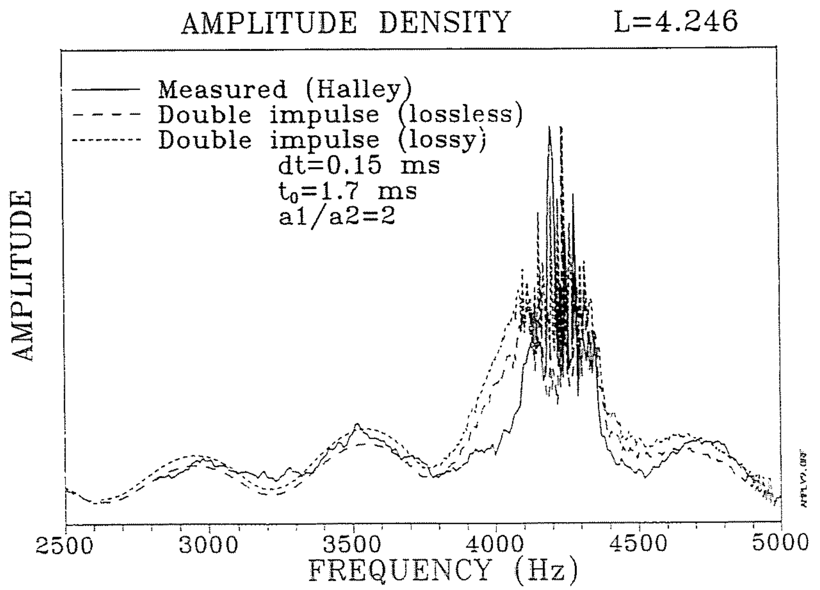

Fig. 7. Comparison of calculated and measured results

6. Conclusions

It can be concluded that whistlers may be successfully modelled in a lossy plasma, assuming a non-monochromatic solution, as presented in this paper.

The used method can be generalized for other cases and other frequency bands when non-monochromatic solution is needed.

Taking into consideration the effect of losses the possibility arises to describe the fine structure of whistlers, more accurately and to compare the results of the model and of real whistlers.

The method can be generalized for a magnetic induction vector which is not perpendicular to the boundary of media, as for quasi-longitudinal and general propagation, for boundary surfaces which are not planes, and for a multicomponent plasma model, which takes into consideration the dispersion due to protons, too (case of proton whistlers).

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