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APPROXIMATE MODEL MATCHING WITH RELAXED CONSTRAINTS ON THE REFERENCE MODEL[1]

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Abstract

It is suggested that if the overall transfer function, $O(s)$, of a certain control system is allowed to match *approximately* a reference model, $H(s)$, that is $O(j\omega) \approx H(j\omega)$, then several constraints on the structure of $H(s)$ can be relaxed. The main ideas are illustrated by means of examples.

Keywords: model matching, model approximation, padé techniques, control systems design.

1. Introduction and Statement of the Problem

Consider the control system of *Fig. 1.* The typical *model matching problem* can be stated as follows: Given $G(s)$, obtain $C(s)$ such that the *overall transfer function* (OTF), $O(s)$, matches a given reference model, $H(s)$, that is,

$$O(s) \equiv H(s). \tag{1}$$

The matching in *Eq.* (1) can be accomplished using a number of techniques such as the linear algebraic method (CHEN, 1987), pole placement approaches, etc. For the sake of the discussion which follows, it suffices to remark that *Eq.* (1) represents some kind of *exact* model matching which aims to achieve $O(j\omega) \approx H(j\omega)$.

Algebraic model matching is not a new idea. As a matter of fact, due to its appealing interpretation and ease of implementation it has been present since early works in the field (SHIPLEY, 1963) and currently continues to draw some attention (CHEN, 1987; CHEN and SEO, 1990; CHEN and SEO, 1991).

CHEN (1987) has shown that because of physical constraints $H(s)$ cannot be arbitrarily chosen. Indeed, the choice of an appropriate reference

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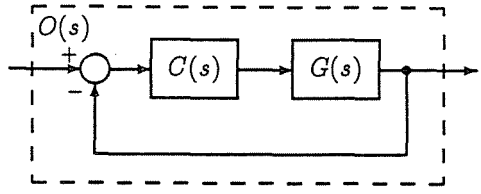

Fig. 1. Closed-loop overall transfer function

model is not a simple task and it is often critical (CHEN, 1987; CHEN and SEO, 1990; CHEN and SEO, 1991; HOSTETTER and SANTINA, 1988).

Consider the following transfer functions

$$G(s) = \frac{g_0 + g_1 s + \ldots + g_q s^q}{h_0 + h_1 s + \ldots + h_n s^n}, \tag{2}$$

$$C(s) = \frac{a_0 + a_1 s + \ldots + a_p s^p}{b_0 + b_1 s + \ldots + b_m s^m}. \tag{3}$$

The closed-loop OTF considered in this paper is (see *Fig. 1*)

$$O(s) = \frac{N_o(s)}{D_o(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}, \tag{4}$$

where

$$N_o(s) = x_o + x_s + \ldots + x_{q+p} s^{q+p}, \tag{5}$$

$$D_o(s) = \alpha_o + \alpha_1 s + \cdots + \alpha_{n+m} s^{n+m}. \tag{6}$$

It has been shown that an adequate OTF should be implementable, in other words $O(s)$ should satisfy (CHEN, 1987)

i) $D_o(s)$ is Hurwitz
ii) $(n+m) - (p+q) \geq n - q$
iii) All the *right half-plane zeros* (RHPZ) of $G(s)$ are retained in $N_o(s)$.

It should be noted that condition (ii) reduces to $m - p \geq 0$ which is the condition for the controller being implementable (proper). From the definition of the OTF, see *Eq.* (4), it is clear that condition (iii) will be satisfied as long as the poles of $C(s)$ do not cancel the RHPZ of $G(s)$. Therefore if $C(s)$ is known, stable and implementable, conditions (i)–(iii) are likely to be met.

However, in the model matching problem, $C(s)$ is to be determined and therefore is not known beforehand. The desired $C(s)$ should be such

that *Eq.* (1) holds and consequently it becomes clear that if $O(s)$ is to match $H(s)$ and be implementable, then $H(s)$ should also be implementable. In other words, if

$$H(s) = \frac{N_h(s)}{D_h(s)}, \tag{7}$$

then

i') $D_h(s)$ should be Hurwitz
ii') $\deg\{D_h(s)\} - \deg\{N_h(s)\} \geq n - q$
iii') All the RHPZ of $G(s)$ should be retained in $N_h(s)$

where $\deg\{\cdot\}$ is the degree of the respective polynomial. Once a reference model is chosen to satisfy (i')–(iii'), some algorithm may be used to carry out the matching in *Eq.* (1) and thus yield $C(s)$. One way of matching $O(s)$ to $H(s)$ is to choose $C(s)$ with $p = m = n - 1$ and to take

$$N_h(s) = \eta_0 + \eta_1 s + \ldots + \eta_{q+n-1} s^{q+n-1}, \tag{8}$$

$$D_h(s) = \delta_o + \delta_1 s + \ldots + \delta_{2n-1} s^{2n-1}. \tag{9}$$

In this case, $C(s)$ has $2n$ unknowns which can be uniquely determined using the following constraints

$$\alpha_j = \delta_j, \qquad j = 0, 1, \ldots, 2n-1. \tag{10}$$

It is clear that this is a pole placement procedure since the poles of the OTF will match those of the reference model. However, the numerator of $H(s)$ is not taken into account in the design and consequently there is no control on the zeros of $O(s)$. Thus this matching is of the form

$$\text{den}\{O(s)\} \equiv \text{den}\{H(s)\}, \tag{11}$$

where $\text{den}\{\cdot\}$ is the denominator of the respective transfer function.

Concerning the above procedure two things can be noted, namely i) $2n - 1$ poles must be chosen for the OTF, or in other words $H(s)$ must be of order $2n-1$, and ii) if $p \leq m < n-1$, arbitrary pole assignment cannot be achieved. Hence, the exact matching in *Eq.* (11) imposes constraints on the structure of $H(s)$ and of $C(s)$.

Clearly, if the plant model, $G(s)$ is of high order then $H(s)$ should also be a high order transfer function. As stated before, the choice of $H(s)$ is seldom trivial and being able to select simple reference models is thus highly desirable.

To this end, LEPSCHY and VIARO (1985) have proposed a procedure in which $H(s)$ and $C(s)$ can be of low order (typically of second-order).

To accomplish the design they suggest using a reduced-order plant model, $G_r(s)$, in the matching

$$O_r(s) \equiv H(s), \tag{12}$$

where $O_r(s)$ is the OTF obtained using $G_r(s)$ instead of $G(s)$. In that paper, the matching was achieved by choosing particular structures for $C(s), G_r(s)$ and $H(s)$. Such a method enables the use of simple reference models at the expense of having to reduce the plant order. This has the inconvenience of carrying over the inaccuracies associated with the approximation $G_r(s) \approx G(s)$ to the design step (ANDERSON and MOORE, 1989).

It is now clear that, for the given control configuration, the ability of matching the OTF to the reference model is limited by the structures of the transfer functions $C(s), G(s)$ and $H(s)$. The objective of this paper is to show that if the matching is allowed to be *approximate*, that is,

$$O(j\omega) \approx H(j\omega), \tag{13}$$

as opposed to the *exact* matchings in *Eqs.* (1), (11) and (12), then some constraints on the structures of $C(s)$ and $H(s)$ can be relaxed. This is motivated by the convenience of obtaining simple (low-order) systems and by the difficulty, in some cases, of choosing appropriate high-order implementable reference models.

It is noted that in general much can be gained if certain constraints in a typical design problem are conveniently relaxed. In fact, *approximation* methods have recently been used successfully in other design problems (CHU, 1993).

In this paper, the matchings performed in *Eqs.* (1), (11) and (12) are considered *exact* in the sense that the transfer functions involved are required to be of the same order (see footnote 1)[2]. It will become clear that in *Eq.* (13) no such a restriction exists. Particularly, it will be illustrated by way of examples that the OTF of the designed system matches *approximately* the reference model (that is *Eq.* (13) is satisfied in some way) and that constraints on $H(s)$ and $C(s)$ can be relaxed in the sense that

a) $H(s)$ need not be implementable in the sense of (i')–(iii')
b) If $G(s)$ has a pure delay, there is no need to include it in $H(s)$, and
c) Given $G(s)$ and an implementable reference model, systems of different orders may be obtained.

This paper is therefore concerned with the *approximate model matching problem* (AMMP) and although this has clear implications for design

[2]In this paper *exact* will indicate that the transfer functions involved are of the same order. Although this does not necessarily imply an exact matching of the dynamics it may enable, for instance, *exact* pole matching.

(AGUIRRE, 1992c), control system design is quite complex a subject, and is concerned with many other important aspects such as disturbance rejection (HANG, 1989; CHEN and SEO, 1990) and the properties of different control configurations (CHEN, 1987; CHEN and SEO, 1991). These aspects are not covered in the present paper.

2. Background and Algorithms

In this section some background is provided and the algorithm used to perform the approximate matching of *Eq.* (13) is presented. Most of the discussion in the preceding section is also valid for open-loop model matching thus an algorithm for such a case is also included.

2.1 Two-Point Padé Approximation

The transfer function

$$H(s) = \frac{d_0 + d_1 s + \ldots + d_r s^r}{e_0 + e_1 s + \ldots + e_r s^r} \tag{14}$$

can be expanded into the following infinite series

$$H(s) = c_0 + c_1 s + c_2 s^2 + \ldots , \tag{15}$$

$$H(s) = m_0 + m_1 s^{-1} + m_2 s^{-2} + \ldots . \tag{16}$$

The respective coefficients can be computed using the following recursive equations

$$\begin{cases} c_0 = \dfrac{d_0}{e_0}, & k = 0, \\ c_k = \dfrac{d_k - \sum\limits_{j=1}^{k} e_j c_{k-j}}{e_0}, & k > 0, \end{cases} \tag{17}$$

$$\begin{cases} m_0 = \dfrac{d_r}{e_r}, & k = 0, \\ m_k = \dfrac{d_{r-k} - \sum\limits_{j=1}^{k} e_{r-j} m_{k-j}}{e_r}, & k > 0. \end{cases} \tag{18}$$

The coefficients $\{c_i\}_{i=0}^{\infty}$ are known as Padé coefficients and are proportional to the time moments which characterize the slow dynamics of a system

(ALEXANDRO, 1984). The coefficients $\{m_i\}_{i=0}^{\infty}$ are the Markov parameters and characterize $H(s)$ at high frequencies.

Models which approximate (match) the original system at low and high frequencies can be obtained by matching a finite set of Padé coefficients and Markov parameters (BULTHEEL and VAN BAREL, 1986). Therefore two models which have a common (finite) set of Padé coefficients and Markov parameters are likely to share similar dynamic properties at both low and high frequencies. This property has motivated hundreds of methods for model order reduction and in what follows will be explored to solve the AMMP.

2.2 Approximate Model Matching Algorithms

The following algorithms are such that the OTF, $O(s)$, matches a certain number of Padé coefficients and Markov parameters of a reference model $H(s)$. In this way the frequency responses of such transfer functions are approximated. In other words, the algorithms perform the matching indicated in *Eq.* (13) by means of matching the Taylor series of the respective models around two different frequencies. Moreover, the matching is said to be approximate because only a finite number of terms in the Taylor series are matched.

Algorithm 1 (Closed-loop model matching)

Consider $G(s)$ and $C(s)$ as in *Fig. 1.* Assuming $G(s)$ is known, the transfer function $C(s)$ which will make the OTF, $O(s)$, match the specified set of Padé coefficients and Markov parameters $\{c_0, c_1, \ldots, c_{P-1}, m_\nu, m_{\nu+1}, \ldots, m_{\nu+M-1}\}$, where $P + M = p + m + 1$, is uniquely given by the solution of the following set of linear equations (AGUIRRE, 1991)

$$\begin{cases} \alpha_0 c_0 = x_0 \,, \\ \alpha_0 c_k = x_k - \sum_{j=1}^{k} \alpha_j c_{k-j} \,, & k = 1, 2, \ldots, P-1 \,, \\ 1 = \alpha_{n+m} \,, \\ m_k = x_{n+m-k} - \sum_{j=1}^{k} \alpha_{n+m-j} m_{k-j} \,, & k = \nu, \nu+1, \ldots, \nu+M-1 \,, \end{cases} \tag{19}$$

where c_i is the ith Padé coefficient, m_i is the ith Markov parameter, m_ν is the first non-zero Markov parameter, and

$$\alpha_i = x_i + y_i \,, \qquad i = 0, 1, \ldots, n+m \,, \tag{20}$$

$$x_i = \sum_{j=0}^{i} a_j g_{i-j}\,, \qquad i = 0, 1, \ldots, q+p\,, \tag{21}$$

$$y_i = \sum_{j=0}^{i} b_j h_{i-j}\,, \qquad i = 0, 1, \ldots, n+m\,. \tag{22}$$

Remark 2.1

If $h_0 = 0$ and $c_0 = 1$ then the first equation in (19) should not be used and an extra constraint can be obtained by letting $k = 1, 2, \ldots, P$ in the second equation of (19) or $k = \nu, \nu+1, \ldots, \nu+M$ in the fourth equation of (19).

Consider the block diagram shown in *Fig. 2.* The design problem in the open-loop case can then be stated as follows: Given $G(s)$, a transfer function $C(s)$ is sought such that the open-loop transfer function, $C(j\omega)G(j\omega)$, approximates an open-loop reference model $H(j\omega)$.

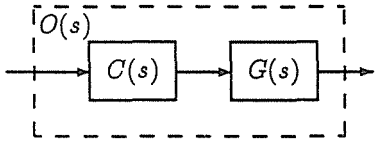

Fig. 2. Open-loop overall transfer function

Algorithm 2 (Open-loop model matching)

Assuming $G(s)$ is known, the transfer function $C(s)$ which will make the OTF, $O(s) = C(s)G(s)$, match the specified set of Padé coefficients and Markov parameters $\{c_0, c_1, \ldots, c_{P-1}, m_\nu, m_{\nu+1}, \ldots, m_{\nu+M-1}\}$, where $P + M = p + m + 1$, is uniquely given by the solution of the following set of linear equations (AGUIRRE, 1992a)

$$\begin{cases} y_0 c_0 = x_0 C, \\ y_0 c_k = x_k - \sum_{j=1}^{k} y_j c_{k-j}\,, & k = 1, 2, \ldots, P-1\,, \\ 1 = y_{n+m}\,, \\ m_k = x_{n+m-k} - \sum_{j=1}^{k} y_{n+m-j} m_{k-j}\,, & k = \nu, \nu+1, \ldots, \nu+M-1\,, \end{cases} \tag{23}$$

where c_i, m_i, x_i and y_i are defined as for algorithm 1.

In order to use such algorithms in model matching problems, the following steps may be taken

Step 1 Choose the structure of $C(s)$, that is choose p and m

Step 2 Choose a reference model which satisfies the control specifications

Step 2o *(For open-loop matching only).* Obtain the auxiliary transfer functions $\overline{H}(s)$ and $\overline{G}(s)$ by removing the poles that $H(s)$ and $G(s)$ might have at the origin

Step 3 Using *Eqs.* (17) and (18) compute P Padé coefficients and $M+\nu$ Markov parameters of the reference model ($\overline{H}(s)$ in the case of open-loop matching)

Step 4 Using *Eq.* (19), or respectively *Eq.* (23) for open-loop matching, obtain the coefficients of $C(s)$ (or $\overline{C}(s)$ for open-loop cases)

Step 4o *(For open-loop matching only).* The desired transfer function $C(s)$ is obtained by adding to $\overline{C}(s)$, obtained in step 4, poles at the origin in order to satisfy

$$\text{Type of } C(s) + \text{Type of } G(s) = \text{Type of } H(s) \tag{24}$$

Step 5 If the approximation $O(j\omega) \approx H(j\omega)$ is not satisfactory three design parameters may be changed, namely, i) p and m, and/or ii) $H(s)$, and/or iii) P and M (AGUIRRE, 1992b).

The type zero transfer functions $\overline{H}(s)$ and $\overline{G}(s)$ are needed because the Padé coefficients in *Eq.* (17) are not defined for a type I ($I > 0$) system.

Remark 2.2

In the algorithms above the transfer function $C(s)$ is calculated such that the OTF, $O(s)$, matches a certain set of Padé and Markov parameters. In an AMMP such parameters are calculated from the reference model, $H(s)$. The choice of $H(s)$ is another important aspect of the design with which algorithms 1 and 2 are not concerned.

Remark 2.3

If $\overline{C}(j\omega)\overline{G}(j\omega)$ is a good approximation[3] to $\overline{H}(j\omega)$ then $C(j\omega)G(j\omega)$ will also be a good approximation to $H(j\omega)$.

It is clear that several possible combinations of P and M satisfy $P + M = p + m + 1$. If $M > 0$, however, the further restriction holds

$$(n - q) + (m - p) \leq \nu . \tag{25}$$

There seems to be no way of choosing *a priori* the best combination of P and M. In practice, a few different values are tested and the combination which yields the best (according to some cost function) OTF is selected. This is feasible given the simplicity of the algorithms.

3. Comments on the Solution of the AMMP

The restriction of *Eq.* (25) is equivalent to restriction (ii') for the case where $m = p$ (it is noted that ν is the pole-zero excess of the reference model).

The price paid for being able to relax the constraints on the structure of $C(s)$ and $H(s)$ is that the matching of Padé coefficients and Markov parameters does not necessarily guarantee a good approximation between the final system and the reference model, especially over the mid-frequency range. Indeed, the main cause of poor approximations is the choice of reference models which are too demanding or unnecessarily complex. If sensible choices of the reference model are made then accurate approximations can be achieved and, in this respect, well known 'rules of thumb' might prove helpful (MIDDLETON, 1991; CHEN, 1993).

If the approximation is unacceptable over the range of mid-frequencies a couple of things can be tried, namely i) constraints for the retention of the dominant poles of the reference model can be included in the algorithms presented in § 2, and ii) constraints for the matching of the frequency responses at specific frequencies can also be used. These simple solutions will not be dealt with in this paper because of space limitations. However, related examples in the context of model simplification can be found elsewhere (TAO et al., 1993).

Clearly, the solution of the AMMP includes a matrix inversion. The conditioning of the related matrix depends basically on two factors, namely i) the convergence of the series (15) – (16) and ii) the number of unknown

[3]By 'good approximation' it is meant that the OTF matches (in the frequency domain) or follows (in the time domain) the reference model within acceptable limits which are peculiar to each application.

coefficients of $C(s)$. LUCAS and MUNRO (1990) have argued that the convergence of the aforementioned infinite series depend on the pole distribution of the model. Because $H(s)$ can be chosen rather freely, in general there will be no need to include fast poles in $H(s)$ just to attain a suitable order. Therefore the convergence of the sequences of Padé coefficients and Markov parameters will not hamper numerical conditioning. On the other hand, algorithms 1 and 2 enable the design of typically low order systems. Therefore the matrices to be inverted will also be of low dimension (typically less than three) and consequently such matrices will be well conditioned in most cases.

Although the reference models need not be implementable in the sense of (i') – (iii'), the OTF for stable closed-loop systems will *always* be implementable in the sense of (i) – (iii). Thus the results by CHEN (1987, 1993) are still valid for the final OTF but can be relaxed in choosing the structure of both the reference model and the controller. This will be illustrated in the examples.

As mentioned in § 2.1, the matching of Padé coefficients increases the approximation accuracy at low frequencies whereas matching Markov parameters tends to improve the approximation at high frequencies. Consequently the choice of P and M influences the overall accuracy of the approximation $O(j\omega) \approx H(j\omega)$. It should be noted, however, that the control requirements should be taken into account in selecting the reference model, $H(s)$, and not in choosing P and M (AGUIRRE, 1992c). These should be selected in order to attain a satisfactory approximation.

Finally, it is noted that there is a fundamental difference between the matchings represented in *Eq.* (12) and that in *Eq.* (13) (which is actually accomplished by algorithms 1 and 2). In the former, an order reduction procedure is performed in order to yield a plant model of a certain order which will enable *exact* matching. On the other hand, no 'order adjustment' is required by the aforementioned algorithms which, in turn, perform the matching *approximately*. In other words, in the former approach a simplified plant model is used whereas in the latter the entire model is taken into account in the design.

4. Numerical Examples

In this section examples are provided to illustrate how some constraints on the structure of $H(s)$ and $C(s)$ can be relaxed in the sense of (a) – (c), see introduction. Examples 1 and 2 illustrate item (a), example 3 item (b) and example 4 refers to item (c).

Example 1 (non-implementable $H(s)$ - closed-loop)

Consider the well-known four-disk system with transfer function (ENNS, 1984)

$$G_1(s)=\frac{0.01(0.64s^5+0.235s^4+7.13s^3+100.02s^2+10.45s+99.55)}{s^2(s^6+0.161s^5+6.004s^4+0.5822s^3+9.9835s^2+0.4073s+3.982)}. \tag{26}$$

For plants which have a pole at the origin, algorithm 1 should be altered according to Remark 2.1. In cases where the plant has N_i integrators ($N_i \geq 2$), an auxiliary plant model is used which has only one integrator and $N_i - 1$ poles at $-\epsilon$, where $0 < \epsilon \ll 1$. In this example, the design will be carried out using the auxiliary transfer function

$$\overline{G}_{1c}(s) = \frac{sG_1(s)}{(s+0.001)}. \tag{27}$$

Choosing $p = 1, m = 1, P = 3, M = 0$ and the reference model

$$H_{1c}(s) = \frac{0.01}{s^2 + 0.14s + 0.01} \tag{28}$$

gives

$$C_{1c}(s) = \frac{3.9986 \times 10^{-2}(s + 1 \times 10^{-3})}{s + 1.3995 \times 10^{-1}}. \tag{29}$$

The first five Padé coefficients of $H_{1c}(s)$ and of the OTF are respectively {1, −14, 96, 56, −1.0384e4} and {1, −14, 96, 55.997, −2.1822e5}. It is noted that although $P = 3$, four Padé coefficients were matched as a consequence of remark 2.1. *Fig. 3* shows the step responses of $H_{1c}(s)$ and of the closed-loop system with $C_{1c}(s)$. It is clear that $H_{1c}(s)$ does not satisfy (ii') – (iii') and, therefore, is non-implementable in the sense of (i') – (iii').

Example 2 (non-implementable $H(s)$ - open-loop)

In this example the four-disk system transfer function is considered anew. The open-loop frequency response specification is as shown in *Fig. 4* (ANDERSON and MOORE, 1989; LIU and ANDERSON, 1986). A transfer function which satisfies such constraints is

$$H_{1o}(s) = \frac{\frac{1}{40}(s+0.07)^2}{s^2(s+1)(s+0.15)^2}. \tag{30}$$

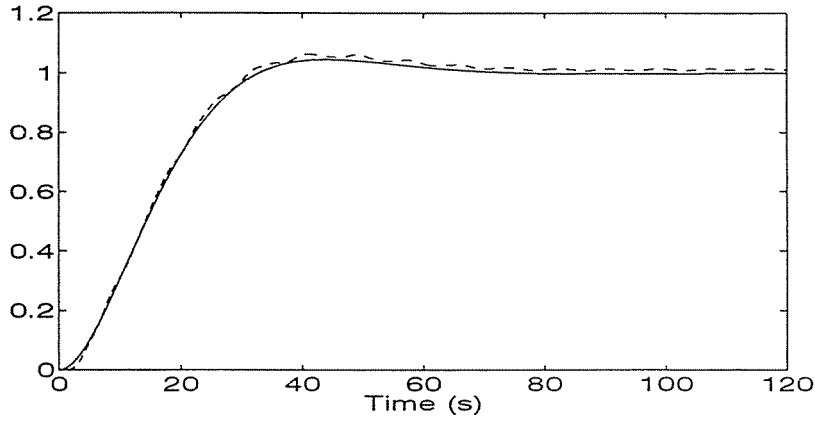

Fig. 3. Step responses of — $H_{1c}(s)$ and of - - closed-loop with $C_{1c}(s)$

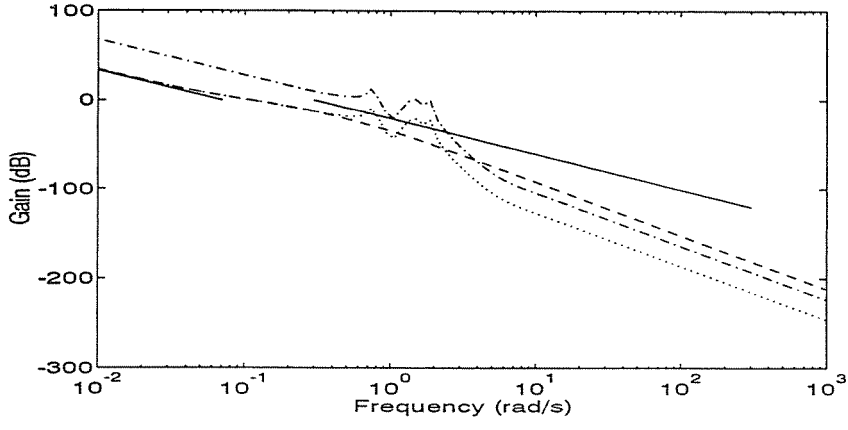

Fig. 4. Design constraints and gain plots of - - $H_{1o}(j\omega)\ldots C_{1o}(j\omega)G_1(j\omega)$ and - · - $G_1(j\omega)$

Taking $P = 5, M = 0, p = m = 2$ and using $\overline{G}_{1o}(s) = s^2 G_{1o}(s)$ and $\overline{H}_{1o}(s) = s^2 H_{1o}(s)$, algorithm 2 yields

$$\overline{C}_{1o}(s) = \frac{0.0749s^2 + 0.0113s + 0.0004}{s^2 + 0.2469s + 0.0191}. \tag{31}$$

From *Eq.* (24) it is clear that $C(s)$ must be type zero and therefore $C(s) = \overline{C}_{1o}(s)$ is the desired transfer function. *Fig. 4* shows the plots of $|G_1(j\omega)|$, $|H_{1o}(j\omega)|$ and $|C_{1o}(j\omega)G_1(j\omega)|$.

As can be seen, $H_{1o}(s)$ is non-implementable in the sense of (i') – (iii') as it fails to satisfy (iii') because $G_1(s)$ has a pair of zeros at $2.2616 \pm j5.1916$ which are not in $H_{1o}(s)$.

Example 3 (Plant with a pure delay)

Industrial processes are commonly approximated by the following transfer function (HANG, ÅSTRÖM and HO, 1991)

$$G_2(s) = \frac{e^{-\theta s}}{(s+1)(\tau s+1)}. \tag{32}$$

In this example $\theta = 0.5s$ and $\tau = 1.5$ were used. In order to write $G_2(s)$ in the form of *Eq.* (2), the following nth order approximation for the pure delay is used (GLADER et al., 1991)

$$e^{-\theta s} \approx R_{nn}(s) = \frac{Q_{nn}(-\tau s)}{Q_{nn}(\tau s)}, \tag{33}$$

where

$$Q_{nn}(\tau s) = \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!}(\tau s)^{n-j}. \tag{34}$$

The chosen closed-loop reference model, $H_2(s)$, was a second order transfer function with damping ratio $\zeta = 0.85$ and natural frequency $\omega_n = 1.0$ rad/s. Taking $p = 2, m = 2, P = 4$ and $M = 1$ yields

$$C_2(s) = \frac{4s^2 + 1.9277s + 0.1489}{2.6667s^2 + 0.25312s}. \tag{35}$$

The step responses of $G_2(s)$, $H_2(s)$ and of $O_2(s) = C_2(s)G_2(s)/[1 + C_2(s)G_2(s)]$ are plotted in *Fig. 5*. Although the step response of $O_2(s)$ does not match the reference model response accurately, it is remarkable that such a reference model, without the plant delay, could have been used to yield a stable OTF.

Example 4 (Systems of diverse orders)

In this example implementable reference models are employed in the design of systems of various orders to illustrate that there are no severe constraints on the order of $C(s)$.

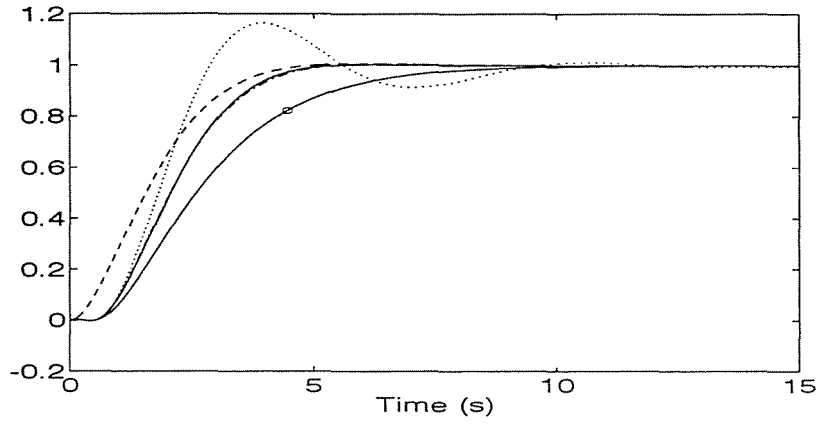

Fig. 5. Step responses of -o- $G_2(s)$, - - reference model without delay, that is $H_2(s)$, ... closed-loop with $C_2(s)$, that is $O_2(s)$, -.-. reference model with delay, that is $H_{2_{delay}}(s)$, — closed-loop with $C_{2_{delay}}(s)$

Consider the transfer function of the structure block of a flexible missile (CHEN et al., 1980)

$$G_3(s) = \\ = \frac{-(s+53)(s-53)(s^2-152.2s+14500)(s^2+153.8s+14500)}{(s^2+s+605)(s^2+45.5s+2660)(s^2+2.51s+3900)(s^2+3.99s+22980)}. \tag{36}$$

In this example the numerator of $G_3(s)$ was multiplied by a constant such that $G_3(0) = 1$. Using the same reference model

$$H_3(s) = \frac{-5.6213 \times 10^{-4}(s-53)(s^2-152.2s+14500)}{(s+3)^3(s+4)^2}, \tag{37}$$

then following transfer functions were obtained

$$C_{31}(s) = \frac{2.3018 \times 10^{-6}s + 0.65387}{s} \tag{38}$$

for $p = m = 1, P = 2$ and $M = 1$,

$$C_{32}(s) = \frac{2.3018 \times 10^{-6}s^2 - 3.8929 \times 10^{-4}s + 1.0987}{s^2 + 1.6803s} \tag{39}$$

for $p = m = 2, P = 3$ and $M = 2$ and

$$C_{33}(s) = \frac{2.3018 \times 10^{-6}s^3 - 3.8169 \times 10^{-4}s^2 - 1.0653s + 7.2572}{s^3 + 4.9797s^2 + 11.099s} \tag{40}$$

for $p = m = 3, P = 5$ and $M = 2$. It is interesting to note that the same reference model was used in the design of systems of different orders. This would have not been possible in *exact* model matching problems. *Fig. 6* shows the step responses of $G_3(s)$, $H_3(s)$ and of the closed-loops using $C_{31}(s)$, $C_{32}(s)$ and $C_{33}(s)$.

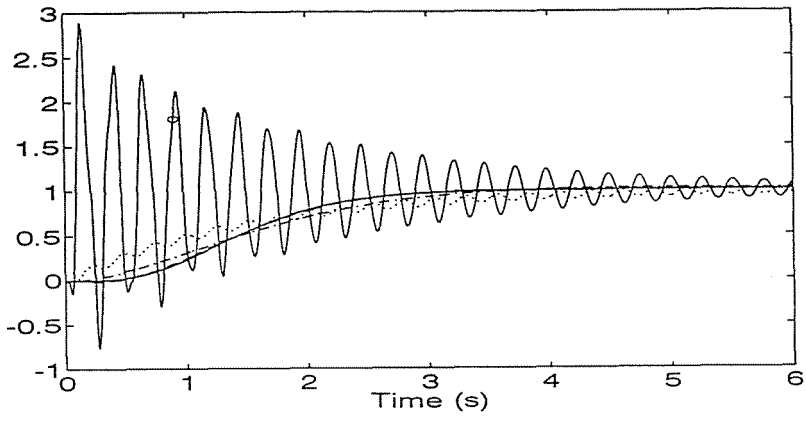

Fig. 6. Step responses of -o- $G_3(s)$, — $H_3(s)$, and closed-loops with ... $C_{31}(s)$, -.-. $C_{32}(s)$ and - - $C_{33}(s)$

5. Final Remarks and Conclusions

It has been shown that performing approximate model matching has some advantages. In particular, some constraints on the structures of the designed transfer function and reference model can in a sense be, relaxed. In this paper, the terms *approximate* and *exact* matching were employed to indicate the matching of transfer functions with distinct and similar structures, respectively.

One of the advantages of algorithms 1 and 2 is that the transfer functions being matched (that is $O(s)$ and $H(s)$) need not be of the same order. For instance, in example 1, $O(s)$ was of the form 6/9 (that is $O(s)$ had six zeros and 9 poles) and $H(s)$ was of the form 0/2. Consequently, the choice of an appropriate reference model is greatly simplified. For instance, in examples 1 and 3 the reference models were chosen according to the well known form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{41}$$

Example 3 illustrated that the reference model need not include the pure delay present in the plant. However, including such a delay will usually be beneficial. For example, including $e^{-\theta s}$ (approximated by *Eqs.* (33) and (34) with $n = 2$) in the reference model of example 3 to yield $H_{2_{delay}}(s) \approx R_{22}(s)H_2(s)$, the following transfer function was found

$$C_{2_{delay}}(s) = \frac{4s^2 + 5.876s + 2.2185}{2.6667s^2 + 4.8807s} \tag{42}$$

for the same values of p, m, P and M as before. The step responses of $H_{2_{delay}}(s)$ and of the corresponding OTF are also shown in *Fig. 5.* Thus in some instances it might be useful to choose reference models according to well-established rules (MIDDLETON, 1991; CHEN, 1993).

The main aim of this paper has been to show that relaxing some constraints in typical model matching problems has two main advantages. Firstly, the reference model can be chosen more freely. This is welcome since the choice of the reference model is often critical. Secondly, there are no severe constraints on the structure of the final system. Thus simple transfer functions are often the solution of the matching problem. This is also highly desirable since the implementation of low order systems is cheaper, easier, more accurate and less time-consuming.

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