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SPECTRAL CORRELATION MEASUREMENT

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Abstract

A concise description of the correlation theory for cyclostationary random signals is given. It is based on a time-frequency cyclic correlation function and on a bifrequent spectral correlation function (SCF). The relations to conventional stationary correlation, Wigner distribution and spectrogram analysis are emphasized. The cyclic transfer properties of linear time-invariant and linear periodically time-variant systems are outlined. Simple examples give a feeling for performance and applications of spectral correlation measurement. The basic schemes of SCF estimation are mentioned briefly.

Keywords: spectral correlation, cyclostationary signals, Wigner distribution.

Introduction

Signal analysis is performed for various applications in order to detect relevant features that characterize an underlying physical or technical process. Above all spectral analysis is used. Algorithms and strategies of their application as well as the interpretation of the results strongly depend on the type of the signal involved. In the case of stationary random signals, for instance, the usual approach to power spectral density (PSD) estimation consists of averaging short time spectral estimates that are determined from consecutive sections of the recorded signal. But if the statistical parameters of the analyzed random process vary with time (this means if some sort of nonstationarity is present), this procedure causes the time dependence to be averaged out. Therefore, only the stationary part is measured. In many cases, however, it is just the time varying feature that carries the interesting information.

There are different time-frequency analysis procedures that are usually applied to detect the time varying spectral content of a signal. The most common representations are the short-time spectrogram and the Wigner distribution (HLAWATSCH and BOUDREAUX-BARTELS, 1992). These

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methods work well with deterministic nonstationary signals. They include no statistical averaging. In the case of random signals, however, some sort of statistical averaging cannot be avoided in order to get a stable estimate.

This conflicting situation can indeed be resolved for an important type of nonstationarity, characterized by periodically time varying moments, called cyclostationarity. The resulting theoretical framework of cyclostationary random processes constitutes a direct extension of common correlation theory of stationary processes. It is based on two-dimensional cyclic auto- and cross-correlation and spectral density functions that contain the ordinary functions as special cases. It is emphasized that the resulting averaging method is well distinguished from sliding time averaging that is useful only for quasistationary signals with slowly time dependent parameters. The frequency of the periodic time variation discussed here may be in the order of the signal bandwidth.

Correlation Theory of Cyclostationary Processes

Second-order periodicity. A wide sense cyclostationary random process is characterized by periodic second-order moments:

$$\psi_x(t+t_{p2},\tau)=\psi_x(t,\tau)=E\left\{x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)\right\} \tag{1}$$

with period t_{p2}. This second-order periodicity, in general, is not coupled to the existence of a superimposed periodic component in $x(t)$ (called first-order periodicity). It can therefore not be detected as a spectral line in the usual way by filtering or Fourier transforming $x(t)$. Therefore, estimation of second-order periodicity is a far more subtle problem than the estimation of first-order periodicity.

Cyclic correlation. The dependence on two time variables (t and τ) offers the possibility to perform two different Fourier transforms. The transform with regard of the time lag τ produces the instantaneous power spectral density which is again periodically time dependent:

$$\Psi_x(t+t_{p2},f)=\Psi_x(t,f)=\int_{-\infty}^{\infty}\psi_x(t,\tau)e^{-j2\pi f\tau}\,d\tau. \tag{2}$$

The Fourier transform using the other time variable t would indicate the involved second-order periodicity as spectral lines at discrete frequencies (the cycle spectrum) or, more precisely, as discrete spectral planes since there is still the second dimension τ. Because $x(t)$ is a finite-power random process, it seems to be more practical, however, to use a modified notation:

$$\psi_{(C)x}(\alpha,\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T \psi_x(t,\tau) e^{-j2\pi\alpha t} \mathrm{d}t. \tag{3}$$

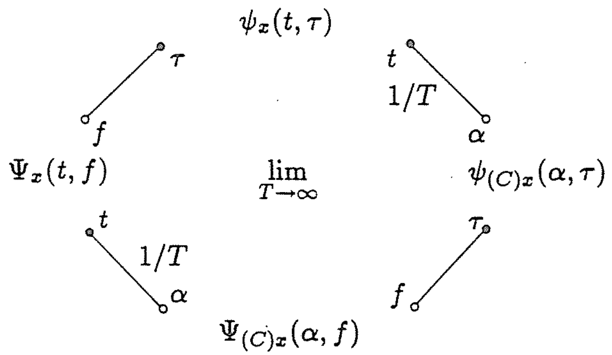
This is easily identified as a generalized Fourier coefficient relation that treats multiple (incommensurable) periodicities. A further advantage of using (3) instead of a Fourier transform is that the physical dimension of $\psi_x(t,\tau)$ is retained and the result can be interpreted as a set (indexed by α) of modified time averaged correlation functions. If the members of this set do not disappear for discrete frequencies $\alpha = \nu\alpha_2 = \nu/t_{p2}$, $\nu \neq 0$, a cyclostationary component with the fundamental second-order periodicity t_{p2} is present. For $\alpha = 0$ the stationary (simply time averaged) correlation arises. $\psi_{(C)x}(\alpha,\tau)$ is called the cyclic autocorrelation function.

Proceeding from any of the two results (2), (3) the remaining time variable can be eliminated by a second (modified) Fourier transform. The result is the bispectral cyclic spectrum:

$$\begin{aligned}\Psi_{(C)x}(\alpha,f) &= \int_{-\infty}^{\infty} \psi_{(C)x}(\alpha,\tau) e^{-j2\pi f\tau} \mathrm{d}\tau \\ &= \lim_{T\to\infty} \frac{1}{T} \int_T \Psi_x(t,f) e^{-j2\pi\alpha t} \mathrm{d}t,\end{aligned} \tag{4}$$

The relation between the cyclic auto-correlation and the cyclic spectrum is called the *cyclic Wiener-Khinchine relation.*

The introduced fourfold Fourier transform relationship is summarized as:



For the case of one single second-order periodicity, the support region of $\psi_{C)x}(\alpha,\tau)$ (and similarly that of $\Psi_{(C)x}(\alpha,f)$) is given by equidistant lines as indicated in *Fig. 1.* On these lines the set of correlation functions (or of power spectral densities, respectively) is built up. The individual members of this set correspond to the different harmonic frequencies of the second-order periodicity. Additional nonharmonic lines would arise if there were multiple (incommensurable) second-order periodicities.

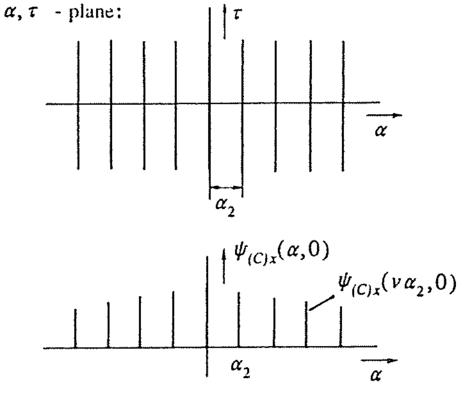


Fig. 1. A: Support lines of $\psi_{(C)}(\alpha,\tau)$ in the α, τ-plane for a single second-order periodicity t_{p2}
B: $\tau = 0$ cross-section of $\psi_{(C)}(\alpha,\tau)$

The unlimited averaging operation in (3) allows the expectation operation contained in $\psi_x(t,\tau)$ to be omitted if the process $x(t)$ can bë characterized as *cycloergodic* (GARDNER, 1987). Now the empirical cyclic autocorrelation is given as a modified time average:

$$\psi_{(C)x}(\alpha,\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi\alpha t}\mathrm{d}t. \qquad (6)$$

Obviously, this relation is an extension of common empirical autocorrelation that is contained in (6) for $\alpha = 0$. A further interpretation is that second-order periodicities can be detected as spectral lines in the lag-shifted product $x(t+\tau/2)x^*(t-\tau/2)$. Because of the twiddle factor term involved, (6) is sometimes called *cyclic averaging*.

Spectral correlation. (6) is further modified by splitting up the twiddle factor operation into two parts. With

$$x_{\alpha/2}(t) = x(t)e^{-j\pi\alpha t} \quad \text{and} \quad x_{-\alpha/2}(t) = x(t)e^{j\pi\alpha t}, \qquad (7)$$

the empirical cyclic autocorrelation becomes a cross-correlation between oppositely frequency shifted images of the same process:

$$\psi_{(C)x}(\alpha,\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x_{\alpha/2}\left(t+\frac{\tau}{2}\right) x^*_{-\alpha/2}\left(t-\frac{\tau}{2}\right) \mathrm{d}t. \tag{8}$$

Now the well-known framework of PSD estimation can be applied. The cyclic spectrum is given directly in the frequency domain as the cross power spectral density of the two frequency shifted short time spectra:

$$\Psi_{(C)x}(\alpha,f) = \lim_{T\to\infty} E\left\{\frac{1}{T} X_T\left(f+\frac{\alpha}{2}\right) X_T^*\left(f-\frac{\alpha}{2}\right)\right\} \tag{9}$$

with

$$X_T(f) = \int_T x(t)e^{-\mathrm{j}2\pi f t}\mathrm{d}t.$$

The usage of finite time-limited process records is necessary in order to ensure that the Fourier transform exists. Moreover, it meets practical purposes (finite record length). The limiting operation eliminates the bias of the estimate but it would not reduce the variance. Therefore, the expectation operation is now necessary again.

The symmetry in notation between (9) and (1) that describes a (time dependent) auto-correlation stimulates further the interpretation of $\Psi_{(C)x}(\alpha, f)$ as a (frequency dependent) *spectral correlation function* (SCF) (GARDNER, 1987, 1991). It can be concluded that a cyclostationary random process contains a second-order periodicity with frequency α_1 if there is a correlation of spectral components that are α_1 apart in frequency. A purely stationary process contains no such correlation.

Cyclic cross-correlation. So far the cyclic auto-correlation has been considered. It is obvious that an extension to cyclic cross-correlation of two random processes $x(t)$ and $y(t)$ can easily be given. Therefore, only the final result, the cross spectral correlation density function, shall be mentioned:

$$\Psi_{(C)xy}(\alpha,f) = \lim_{T\to\infty} E\left\{\frac{1}{T} X_T\left(f+\frac{\alpha}{2}\right) Y_T^*\left(f-\frac{\alpha}{2}\right)\right\} \tag{10}$$

with

$$X_T(f) = \int_T x(t)e^{-j2\pi f t}\mathrm{d}t.$$

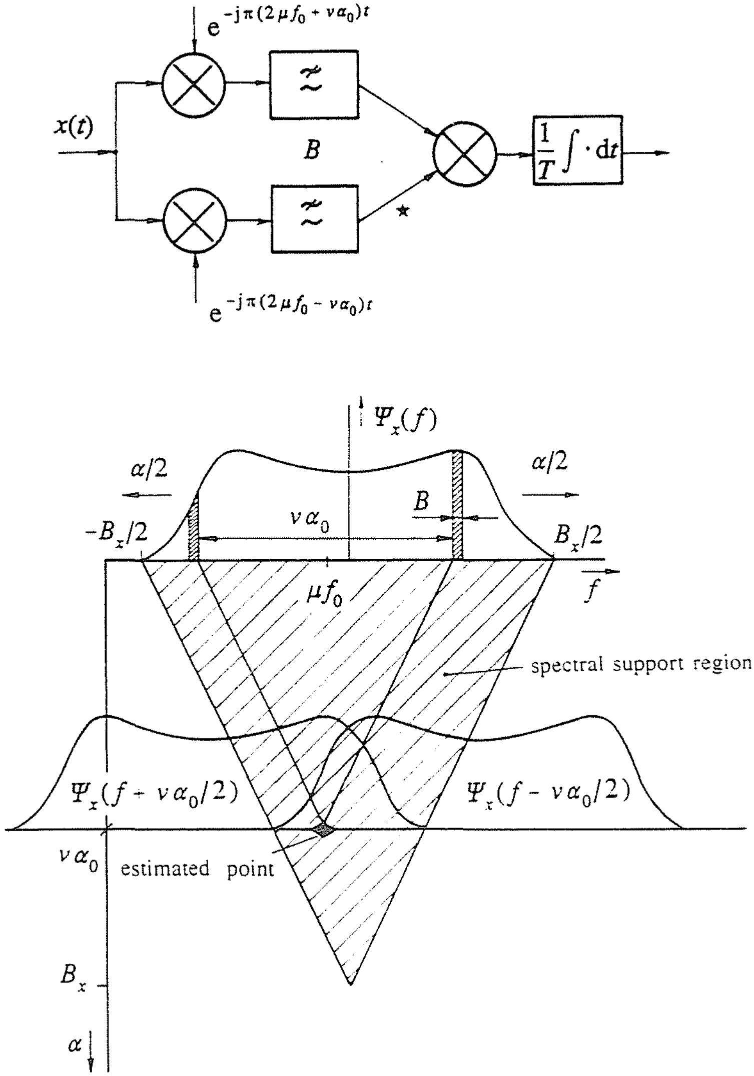


Fig. 2. Estimation of spectral correlation functions

SCF Estimation

The basic structure for estimating the spectral correlation function at one point νx_0, μf_0, in the α, f-plane is given in *Fig. 2.* It is based on averaging oppositely frequency shifted, complex demodulated and lowpass filtered images of the same process. The whole bifrequency plane α, f can be covered by successively tuning the midfrequency μf_0 and the frequency distance $\nu\alpha_0$. For a bandlimited process $x(t)$ a rhomboid cyclic spectral support region arises, because of the reduced spectral overlap with increasing α (only the positive α-plane is sketched in the figure). But if the process is real, or if only the magnitude of the spectral correlation function is of interest, one quadrant suffices. In the context of digital signal processing the estimation procedure results as averaging of modified cyclic (α-shifted) periodograms. However, as compared to stationary-signal PSD estimation, special requirements exist, concerning window function design and window function overlap in order to avoid cycle leakage and cycle aliasing.

Wigner Distribution and spectrogram. (2) reveals the strong relation to the Wigner distribution theory. Commonly the Wigner distribution (WD) is related to deterministic signals. Hence no statistical averaging is included, and the Fourier transform of the Wigner distribution is considered to be a deterministic energy density spectral correlation function. But if the concept of Wigner distribution is to be extended to cover finite power random signals, an expectation operation has to be applied. The result is just the instantaneous spectrum given by (2). Since expectation cannot be approximated in most cases by ensemble averaging, time smoothing is usually applied. But the temporal resolution is reduced as a consequence (more and more with an increasing averaging time). Cyclic averaging or Fourier transforming of a pseudo Wigner distribution (PWD), however, can effectively be used for SCF estimation, since the periodic time dependence of a cyclostationary process is completely resolved in the cyclic frequency domain and the better the resolution the longer the averaging time is. In the WD context, spectral correlation is indicated as periodic cross-terms in the expected WD (*statistical cross-terms*, see THOMÄ, 1993). Concerning the auto-spectrogram it turns out to be a time and frequency smoothed WD, and therefore it limits cycle frequencies to the low-pass region. Only if cross-spectrograms of frequency shifted images are calculated, is the whole cycle frequency plane covered.

Influence of Signal Processing Operations

Time-invariant linear systems. For convolution-based linear systems

$$y(t) = g(t) \star x(t) = \int_{-\infty}^{\infty} g(t-\tau)x(t)\ \mathrm{d}t, \tag{11}$$

the auto-SCF of the output signal follows as

$$\Psi_{(C)y}(\alpha, f) = G\left(f + \frac{\alpha}{2}\right) G^*\left(f - \frac{\alpha}{2}\right) \Psi_{(C)x}(\alpha, f). \tag{12}$$

The system impulse response function $g(t)$ appears as its Fourier transformed Wigner distribution or deterministic energy density spectral correlation function. For $\alpha = 0$ (12) reduces to the well-known input/output relation for stationary signals in the PSD domain. The cross-SCF between input and output results as

$$\Psi_{(C)yx}(\alpha, f) = G\left(f + \frac{\alpha}{2}\right)\Psi_{(C)x}(\alpha, f). \tag{13}$$

Frequency-invariant linear systems. This class of linear systems covers memoryless time-variant systems described by product modulation

$$y(t) = x(t)\ m(t). \tag{14}$$

The output auto-SCF results from the two-dimensional convolution of the SCFs of the input $x(t)$ and the modulating function $m(t)$ ($m(t)$ and $x(t)$ are independent):

$$\begin{aligned}\Psi_{(C)y}(\alpha, f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{(C)x}(\beta, \phi)\ \Psi_{(C)m}(\alpha - \beta, f - \varphi)\mathrm{d}\beta\mathrm{d}\varphi \\ &= \Psi_{(C)x}(\alpha, f)\ \star_{\alpha,f}\ \Psi_{(C)m}(\alpha, f).\end{aligned} \tag{15}$$

Frequency- and time-variant linear systems can often by described by chaining (12) and (15). Periodically time-variant linear systems are of special importance (e. g. modulator and filter; filter and sampler; digital filter and decimator). Then the modulating functions are periodic signals (typically complex exponentials, sinusoids, pulse trains, etc.) with discrete SCF. Generally, these modulating functions shift the original SCF in the α, f-plane. In most cases new cyclostationarities are introduced and spectral correlated signal components are superimposed (if there are any in the input signal). Simple analysis of such systems on the basis of common stationary correlation analysis would neglect these effects and may therefore produce wrong results.

Signal superposition. If the analyzed signal is represented as an additive superposition of individual terms, e. g.:

$$x(t) = x_1(t) + x_2(t), \tag{16}$$

the resulting SCF follows as the sum of the single component auto-SCFs and all possible cross-SCFs:

$$\Psi_{(C)x}(\alpha, f) = \Psi_{(C)x_1}(\alpha, f) + \Psi_{(C)x_2}(\alpha, f) + \Psi_{(C)x_1x_2}(\alpha, f) + \Psi_{(C)x_2x_1}(\alpha, f). \tag{17}$$

(17) is called the quadratic superposition principle.

Examples

Product modulation is performed by multiplying a given stationary process $x(t)$ and a suitable modulating function $m(t)$, e.g. in the case of a sinusoidal function: $y(t) = x(t) \; m(t) = x(t)\cos(2\pi f_1 t + \varphi_1)$. From

$$\Psi_{(C)m}(\alpha, f) = \begin{cases} \frac{1}{4}\delta(f - f_1) + \frac{1}{4}\delta(f + f_1) & \text{for } \alpha = 0 \\ \frac{1}{4}e^{\pm j2\varphi_1}\delta(f) & \text{for } \alpha = \pm 2f_1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

and

$$\Psi_{(C)x}(\alpha, f) = \begin{cases} \Psi_x(f) & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha \neq 0 \end{cases} \tag{19}$$

using (15):

$$\Psi_{(C)y}(\alpha, f) = \begin{cases} \frac{1}{4}\Psi_x(f + f_1) + \frac{1}{4}\Psi_x(f - f_1) & \text{for } \alpha = 0 \\ \frac{1}{4}e^{\pm j2\varphi_1}\Psi_x(f) & \text{for } \alpha = \pm 2f_1 \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

The upper part of *Fig. 3* shows the SCF of $m(t)$ (positive α-half only). Because of the periodic deterministic nature of $m(t)$ it consists of spectral lines with 100% spectral correlation. Since $x(t)$ is stationary, it shows no spectral correlation, but the modulated signal reveals spectral correlation. A signal with the same PSD can be generated, if a white stationary noise is appropriately bandpass filtered. Certainly, this signal is again stationary and will therefore show no spectral correlation. A conventional FFT analyzer could not distinguish between the two cases, since it measures the averaged PSD only that maps to the $\alpha = 0$ intersection of the SCF. Moreover, if there were possibly frequency overlapping signals with different modulation frequencies, they could be well identified in the SCF domain.

Identification of periodically time variant systems. Such systems occur e.g. in digital signal processing as multirate systems. Examples are decimating and interpolating filters, halfband and QMF filters. Since the involved bandlimiting operations will usually not perform in an ideal way, some aliasing occurs. If the system transfer function is measured using the conventional cross-spectral averaging approach, the aliased components are superimposed and can therefore not be identified separately. But spectral correlation analysis allows the transfer function to be resolved

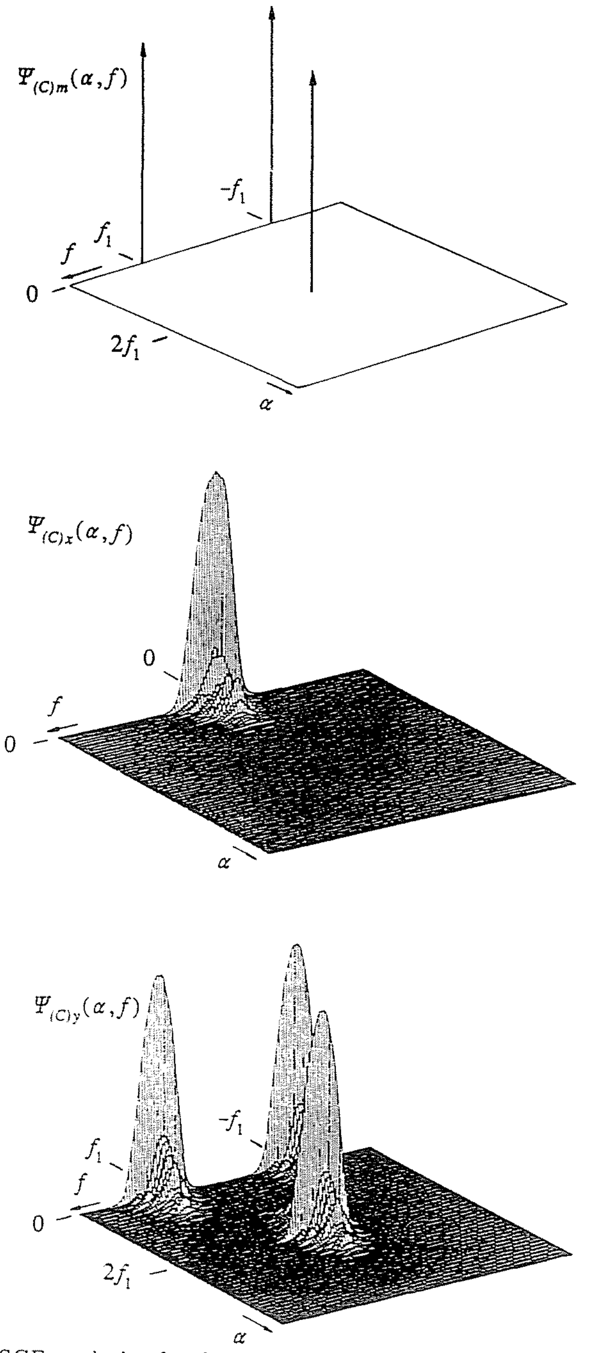


Fig. 3. SCF analysis of a sinusoidal product modulated process

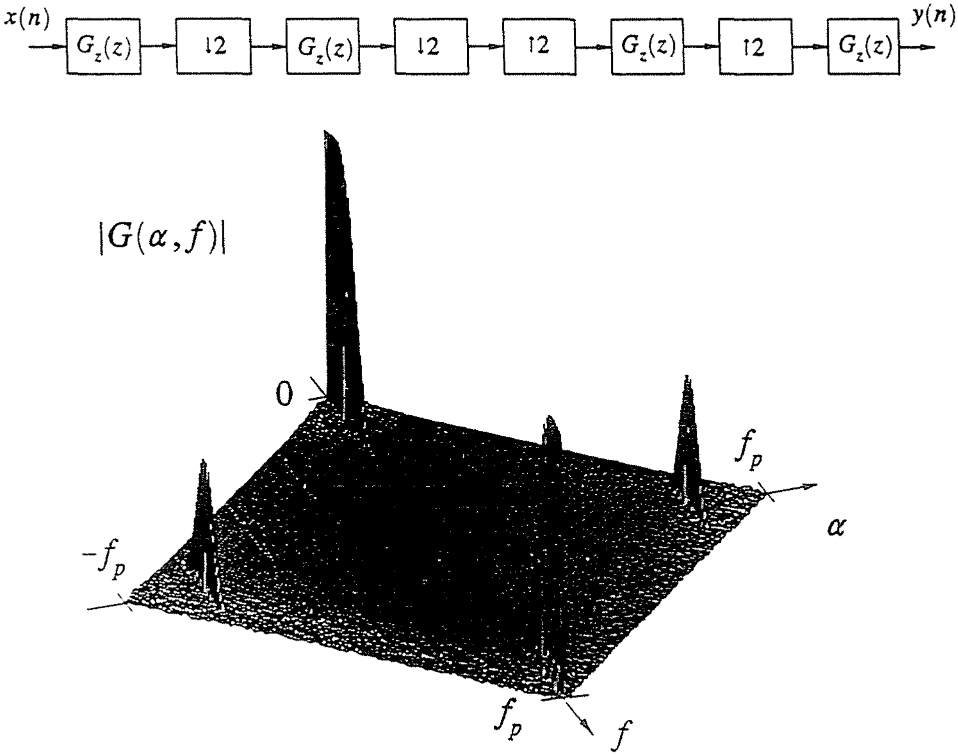


Fig. 4. Bifrequent cyclic transfer function estimate of a two stage QMF filter branch

into the aliased components. A bifrequent cyclic system transfer function is introduced for that purpose as

$$G(\alpha, f) = \frac{\Psi_{(C)yx}(\alpha, f)}{\Psi_x(f)}. \tag{21}$$

The input signal $x(t)$ is chosen as a stationary signal. If a random-phase multisinus signal is used for system excitation no special window function is applied and there are no problems concerning cycle frequency leakage and cycle frequency aliasing. The estimate shows spectral correlation (for $\alpha \neq 0$) only if there are residual aliasing components. For $\alpha = 0$ (21) reduces to the conventional transfer function estimate. As an example, a single branch of a two stage QMF filter bank as given in *Fig. 4* was analyzed. The estimated $|G(\alpha, f)|$ clearly shows the residual aliasing. The main diagonal of the matrix represents the conventional stationary frequency response. If the decimating and interpolating factors are known in advance it is sufficient to estimate only the interesting cross-sections of $|G(\alpha, f)|$.

Conclusions

Cyclic spectral analysis is especially useful if random signals are given that are modified by some periodic operation such as periodic sampling, multirate filtering, sinusoidal modulation or demodulation etc. Different sources of cyclostationary can be identified and distinguished from each other and from stationary parts that could not be resolved by standard spectrum analysis methods. Areas of application for example are detection and identification of modulated signals (even in the case of spread spectrum signals) and time delay of arrival analysis of signals (resolution of signals that join common band and direction but offer different modulation frequency and type). In system analysis spectral correlation can be used to identify periodically time varying systems. The well directed usage of cyclostationary test signals allows the identification of nonlinear systems in terms of Volterra kernels. Even the standard identification problem of linear time invariant systems can be refined: using (13) stationary input and output interference signals can be separated and bias error is therefore avoided if cyclostationary test signals are used.

Concerning measurement and estimation of spectral correlation, different concepts can be adopted and extended from traditional spectrum analysis. But because of its two-dimensional nature, spectral correlation is an expensive procedure and acceptable computation times can only be achieved if fast DSP computing is used.

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