

# THE WIGNER DISTRIBUTION AS A TOOL FOR SPECTRAL ANALYSIS OF INSTATIONARY SIGNALS

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## Abstract

The description of non-stationary signals causes problems which are hard to solve with conventional methods for spectral analysis. Examples for rather instationary signals are starting cycles of machines, transients or measuring of flow speed. If the spectral content of a signal changes in time, the periodogram (spectrogram) does not give any possibility to connect the presence of certain frequencies properly to a certain point of time. Methods such as the Wigner-distribution (WD) that display the signal on a time-frequency plane are far better for this purpose. The WD may be interpreted as the power-density spectrum (PDS) of the signal, even though it is not strictly positive for all signals. An interesting property of the WD besides its high time and frequency resolution is, that the first moment of the WD with respect to the frequency is equal to the derivative of the phase, which is to say the instantaneous frequency of the signal. An algorithm to compute the WD in real-time was developed and implemented on a system of four digital signal processors. This system is used for the measurement of blood velocity by ultrasound.

*Keywords:* time-variant signals, Wigner-distribution, instantaneous frequency.

## 1. Introduction

Non-stationary time signals appear in many fields of digital signal processing. If it is possible to describe the signals as nearly stationary in a sufficiently long time-interval, the conventional methods of frequency analysis (spectrogram, parametric models) can be used.

If the spectral content of the signals changes rapidly in time, these methods fail. In this case, methods which transform the signal into the time-frequency plane can help on. The WD (WIGNER, 1932) is such a transformation and can be interpreted as the time-varying PDS even though it is not positive for all times and frequencies. The definition and properties of the WD are known from literature (CLAASEN, MECKLENBRÄUKER, 1980) and are shortly summarized and demonstrated using a simulated signal.

A possibility to suppress the cross-terms appearing at Wigner-analysis is demonstrated.

Besides frequency analysis, signals can be described by their moments. For non-stationary signals their moments vary in time as well. It is possible to develop them from the WD efficient algorithms for on-line estimation of the moments in the time domain. As an example, the first moment of the PDS, the mean frequency, is considered.

In another example, the measurement of blood velocity with ultrasound, the power spectra achieved with conventional methods are compared to the results of Wigner-analysis in the case of real data. The data were deduced at the human arteria brachialis (HUCKER, 1992). The cross-sectional average of the velocity corresponds to the first moment of the PDS.

Here as well the algorithm deduced from the WD is compared to other methods for estimating the first moment of the PDS. All in all, the WD is a valuable for analyzing the time-varying velocity of human blood flow.

## 2. Conventional Methods for Power-Density-Spectrum Estimation

For the estimation of PDS different methods are developed. In general, it is not possible to calculate the true PDS of a real signal, because a real signal is time limited, disturbed, and in general, varying with time. The conventional estimators for PDS are well known and well investigated in literature (KAMMEYER, KROSCHER, 1989, KRONMÜLLER, 1991). Here, they are presented in a short summary to indicate their disadvantages in the case of time-variant signals and to compare their results with the results given by the WD.

### 2.1 The Periodogram

The base of PDS-estimation by the periodogram is the theorem of Wiener-Khinchine (WIENER, 1930, KHINCHINE, 1934). According to this theorem, the PDS  $L(f)$  of an, in general, complex-valued but time-invariant signal  $x(t)$  is the Fourier transform of the auto correlation function (ACF)  $v(\tau)$  of the signal. With the ACF

$$v(\tau) = \mathcal{E}\{x^*(t)x(t + \tau)\} \quad (1)$$

( $\mathcal{E}\{\}$  means the expected value,  $x^*(t)$  denotes the complex conjugate of  $x(t)$ ) the PDS is as follows:

$$L(f) = \int_{-\infty}^{\infty} v(\tau)e^{-j2\pi f\tau} d\tau \quad (2)$$

$$= \lim_{T_0 \rightarrow \infty} \mathcal{E} \left\{ \left| \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi ft} dt \right|^2 \right\}. \quad (3)$$

*Eqs. (2) and (3) give the two possible ways for calculation:*

1. In *Eq. (2)*, the ACF is to be determined first, then the PDS is the Fourier transform of the ACF (TUKEY, BLACKMAN, 1959).
2. In *Eq. (3)*, the Fourier transform of the signal is to be determined first, then the PDS follows from the given expected value (Spectrogram, Periodogram (KAMMEYER, KROSCHER, 1989, KRONMÜLLER, 1991)).

These two equations show the problems in the case of realization. In general, the probability density of the signal is unknown, therefore it is not possible to calculate the expected value. On the other hand, it is necessary to observe the whole signal. Real signals, however, are time-limited and, in the case of time-variant signals, they can only be considered as time-invariant in a sufficiently short interval.

The Blackman–Tukey Method is not longer considered, because the calculation of the ACF takes much time, and therefore this method is unsuitable for on-line applications. On the other hand, the solution of *Eq. (3)* can be approximated in an effective manner by the use of the Fast Fourier Transform (FFT). To take the time-variance into account, the signal  $x(t)$  is multiplied by a time-variant window function:

$$w(t - \tau) = 0 \quad \text{for} \quad |t - \tau| > \frac{T_0}{2} \quad (4)$$

( $T_0$  is the observation time). By this, the treated signal is:

$$x_w(\tau, t) = x(\tau)w(t - \tau), \quad (5)$$

and the estimated PDS:

$$\hat{L}_{\text{FFT}}(f, t) = X_w(f, t)X_w^*(f, t), \quad (6)$$

$$X_w(f, t) = \int_{-\infty}^{\infty} x_w(\tau, t) e^{-j2\pi f\tau} d\tau = e^{-j2\pi ft} \int_{-T_0/2}^{T_0/2} x(\tau + t) w(\tau) e^{-j2\pi f\tau} d\tau. \quad (7)$$

Many different window functions are possible. The simplest function is the rectangular window. It gives the best frequency resolution. Other window functions such as the Bartlett, Hanning, or Dolph–Tchebychev windows

and so on suppress the sidelobes caused by the limited observation time in the estimated spectrum. The different window functions are well known from literature (KAMMEYER, KROSCHER, 1989). Here, a detailed discussion is not necessary. The simple rectangular window is sufficient and is used for the calculation of the periodogram, the parametric models and the WD, too.

At the description of the time-variance of a signal, there are two contradictory problems: The frequency resolution is inversely proportional to the observation time. High frequency-resolution requires a long observation time, high time resolution requires a short observation time. This dilemma is shown in *Figs. 3a* and *3b*.

From the point of view of statistics, the periodogram (*Eq. (6)*) is a biased and non-consistent estimator of the PDS (*Eq. (3)*), (KRONMÜLLER, 1991). If the signal is time-invariant, there is a possibility to estimate the PDS as the mean value of periodograms belonging to different time intervals. Indeed, this estimator is consistent, but, in the case of time-variant signals, averaging is not allowed.

## 2.2 Parametric Modelling

The base of PDS estimation with parametric models is the assumption that the signal  $x(t)$  can be modelled as the output of a linear time-invariant (LTI) system excited by white noise with mean value zero. Now the problem is to estimate the parameters of the LTI system in such a manner that the spectrum of the estimated output  $\hat{x}(t)$  corresponds to the spectrum of the measured signal  $x(t)$ . For complex-valued signals, the model parameters are complex-valued, too. Therefore, it is not the task to estimate different frequencies such as when using the periodogram, but to calculate the optimal model parameters. By this the number of estimated parameters is much less than by the periodogram. The signal description by parametric modelling is well suited for time-variant signals – one can say, the ACF is modelled –, and for data reduction, too.

Here, an autoregressive (AR) model is used, because parameter estimation leads to a linear problem only in the case of the AR-model. Other models like moving average or a combination of both are possible. In the case of AR-modeling, there is:

$$x(n) + a_1x(n-1) + a_2x(n-2) + \dots + a_Kx(n-K) = w(n), \quad (8)$$

with:

- $x(n - k)$ ,  $k = 0, 1, \dots, K$ , the signal,
- $a_k$ ,  $k = 1, 2, \dots, K$ , the (in general, complex-valued) parameters,
- $w(n)$ , the input, Gaussian white

and in the  $z$  domain:

$$X(z) = \frac{1}{A(z)}W(z), \quad A(z) = 1 + a_1z^{-1} + a_2z^{-2} + \dots + a_Kz^{-K}, \quad (9)$$

If  $\sigma^2$  is the variance of noise at the input, then for the estimated PDS follows:

$$\hat{L}_{AR}(f) = \frac{\sigma^2}{A(z)A^*(z)} \Big|_{z=e^{j2\pi fT}}, \quad T : \text{the sampling time.} \quad (10)$$

Different procedures for calculating the optimal parameters from the measured signal are well known in literature (KAMMEYER, KROSCHER, 1989, KRONMÜLLER, 1991), and efficient algorithms exist for implementation (Levinson–Durbin, Lattice-filters). Here only the suitability of parametric models for PDS-estimation of time-variant signals will be discussed.

If it is possible to describe the signal by a model of order  $K$ , the first  $K$  values of the ACF are needed for optimal estimation. As the statistics of the signal are, in general, not known, these values of the ACF must be estimated, too. This is only possible with a finite observation time of the signal. If the signal is time-variant, one will choose a time dependent observation window. Doing this, the optimal model parameters and by this the estimated PDS become functions of time:

$$\hat{L}_{AR}(f, t) = \frac{\sigma^2}{A(z, t)A^*(z, t)} \Big|_{z=e^{j2\pi fT}}, \quad (11)$$

$$A(z, t) = 1 + a_1(t)z^{-1} + a_2z(t)^{-2} + \dots + a_K(t)z^{-K}. \quad (12)$$

The minimum length of the window depends on the order of the model  $K$ , this means that a higher order of the model requires a longer observation time. The time resolution decreases increasing the model order.

The necessary order for a correct description of the signal is not known a priori. But if the correct order is chosen, and if the real process is an AR process indeed, then the error of the estimation becomes white. This is the reason why recursive algorithms, handling from order  $K$  to  $K + 1$ , such as e.g. Levinson–Durbin, are preferred. There, the order is increased until the error of estimates becomes nearly white.

### 2.3 Mean Frequency of the Power Density Spectrum

The mean frequency of time-variant signals becomes a function of time, too. If the estimated PDS,  $\hat{L}(f, t)$  is known, an estimate  $\hat{f}_{\text{PDS}}(t)$  for the mean frequency  $\bar{f}(t)$  can be calculated by:

$$\hat{f}_{\text{PDS}}(t) = \frac{\int_{-\infty}^{\infty} f \hat{L}(f, t) df}{\int_{-\infty}^{\infty} \hat{L}(f, t) df}. \quad (13)$$

The time resolution of the estimated mean frequency is given by the length of the used observation window. The index 'PDS' indicates the estimation of the mean frequency from the PDS. In Sections 3 and 4 other estimators for the mean frequency are presented and tested for measured data.

## 3. The Wigner Distribution

The WD is a method to transform any complex-valued time signal into the time-frequency plane. In opposite to the periodogram, the WD does not require the calculation of expected values, it uses the *local* ACF

$$w(t, \tau) = x\left(t + \frac{\tau}{2}\right)x^*\left(t - \frac{\tau}{2}\right) \quad (14)$$

instead of the ACF. By doing this, the WD is much better suited to describe time-variant signals than the periodogram, because the local character of the periodogram has to be forced by time windowing.

### 3.1 Definition

In the continuous case, the WD is defined as the Fourier transform of the local ACF with respect to the time difference  $\tau$ :

$$W_x(t, f) = \int_{-\infty}^{\infty} w(t, \tau) e^{-j2\pi F\tau} d\tau. \quad (15)$$

Because of this definition, the local ACF is often referred to as Wigner kernel.

In the case of equidistantly sampled signals, two possibilities exist for each sample to construct the Wigner-kernel, which are the time of

sampling itself and the middle between two samples. The discrete Wigner-distribution (DWD) appears thus sampled twice as fast as the original signal:

$$DW_x(i, F) = \begin{cases} \frac{1}{2T} \sum_{k=-\infty}^{\infty} x(n+k)x^*(n-k)e^{-j4\pi Fk} & \text{for } i = 2n \\ \frac{1}{2T} e^{-j2\pi F} \sum_{k=-\infty}^{\infty} x(n+k+1)x^*(n-k)e^{-j4\pi Fk} & \text{for } i = 2n+1. \end{cases} \quad (16)$$

$F$  is the frequency normalized with the sampling time,  $F = fT$ .

The DWD is obviously periodic for  $F = 1/2$ . This means that the calculation of the DWD requires oversampling of at least twice the Nyquist rate to avoid aliasing.

### 3.2 Properties of the Wigner Distribution

The WD has some properties, which are inviting to interpret it as a kind of PDS in the time-frequency plane.

The WD is real-valued for arbitrary complex signals, as follows from the conjugate symmetry of the Wigner kernel.

Integration of the WD with respect to frequency for a certain time  $t$  equals the instantaneous power of the signal, and integration with respect to time leads to the known PDS (Eq. (2)). However, the WD is not positive for all frequencies, so that it cannot be, strictly speaking, a power density.

Another property of the WD is rather disturbing for the description of signals; it is the appearance of cross-terms for signals that contain more than one frequency component at a time. For every two frequencies, which are contained additively in the signal, a cross-term appears at the arithmetic mean of the two frequencies oscillating in time with the difference of the two invoking frequencies. The influence of the cross-terms can be reduced by smoothing the WD in time-direction. The resolution in time decreases as a consequence, but can be easily tolerated as demonstrated in Fig. 1. The smoothing is performed using MA-filters.

If one calculates the first moment of a WD with respect to frequency for a certain instant  $t$ , one obtains an estimated value for the instantaneous frequency of the signal. It can be shown (OSTERTAG, 1991) that this first moment of a WD corresponds to the derivative of the phase and thus can be calculated using differentiating systems without having to calculate the WD itself:

$$\bar{f}(t) = \frac{1}{2\pi} \operatorname{Im} \left\{ \frac{\dot{x}(t)}{x(t)} \right\}. \quad (17)$$

For example, the use of the Stirling algorithm leads to:

$$\hat{f}_{\text{WD}}(nT) = \frac{1}{24\pi T} \text{Im} \left\{ \frac{x((n-2)T) - 8x((n-1)T) + 8x((n+1)T) - x((n+2)T)}{x(nT)} \right\}. \quad (18)$$

### 3.3 The Analytic Signal

Processing of real-valued data causes problems for two reasons. The spectrum of such data is symmetric, which leads to strong cross-terms around the frequency zero. The second reason is that all the information about the signal is contained twice in its spectrum, so that the investigation of positive frequencies is sufficient. This results in reducing the bandwidth of the signal by a factor of two. For these reasons, it is advantageous to create the analytic signal connected to the real-valued signal before performing a WD. The analytic signal is connected to the real-valued one by its Fourier spectrum:

$$X_a(f) = \begin{cases} 2X_r(f) & \text{for } f > 0 \\ X_r(f) & \text{for } f = 0 \\ 0 & \text{for } f < 0, \end{cases} \quad (19)$$

where  $X_a(f)$  is the Fourier spectrum of the analytic signal  $x_a(t)$  and  $X_r(f)$  is the Fourier spectrum of the real-valued signal  $x_r(t)$ . There are several ways to obtain the analytic signal (KAMMEYER, KROSCHER, 1989), usually FIR-filtering is used.

### 3.4 The Pseudo Wigner Distribution

To calculate the Wigner kernel according to Eq. (14) and Eq. (16), one has to know the signal for all times. As this is impossible in reality, only a certain period of time is used to calculate the kernel (and the integration as well). This time window has the length  $T_0$  and is usually chosen symmetric to the time  $t$ , which means that it slides along the time axis as one calculates the WD. The sliding window has the effect of smoothing the WD in the frequency-direction, the distribution obtained this way is the Pseudo Wigner Distribution (CLAASEN, MECKLENBRÄUKER, 1980).

### 3.5 Calculation of the Discrete Wigner-Distribution

To calculate the DWD of equidistantly sampled signals, several algorithms can be developed, all of them being based on the FFT algorithm (OS-TERTAG, 1991). One of these algorithms allows the calculation of two DWDs of the same length using only one complex FFT-calculation. This algorithm, which is presented in the following, was implemented to build an on-line Wigner-analyzer.

To take advantage of the  $N$ -point FFT algorithm, with  $N = 2L$ ,  $N - 1$  samples of the time-signal have to be used for calculation. Sampling the normalized frequency  $F$  with a sampling distance of  $\Delta F = (2N)^{-1}$ , Eq. (16) for even index  $i$  leads to:

$$DW_L(n, m) = \frac{1}{2T} \sum_{k=-L+1}^{L-1} x(n+k)x^*(n-k)e^{-j2\pi\frac{km}{N}} \quad (20)$$

To obtain the standard form for a FFT, the Wigner-kernel  $w(n, k) = x(n+k)x^*(n-k)$  has to be regarded as a function that is periodic in  $k$  with period  $N$ . Originally, the kernel is constructed for  $|k| < L$ . The element of the Wigner-kernel for  $k = L$  is set to zero. Changing the summation boundaries leads to the following equation for the calculation of  $DW_L(n, m)$ :

$$DW_L(n, m) = \frac{1}{2T} \sum_{k=0}^{N-1} w(n, k)e^{-j2\pi\frac{km}{N}} \quad (21)$$

The Wigner-kernel is conjugate symmetric with respect to  $k$ , which means that it is not necessary to calculate the whole kernel but only the elements with positive  $k$ .

Further reduction of the number of calculations follows from the fact that the WD of any signal is real-valued. So it is possible to calculate two  $N$ -point DWDs using one  $N$ -point FFT by forming one new combined kernel from two kernels for different time-samples  $n$ :

$$w_{\text{comb}}(k) = w(n_1, k) + jw(n_2, k) \quad (22)$$

Calculating the  $N$ -point FFT of  $w_{\text{comb}}(k)$ ,  $w_{\text{comb}}(m)$ , it is possible to separate the DWDs of the original kernels according to:

$$w(n_1, m) = \mathbf{Re}\{w_{\text{comb}}(m)\}, \quad w(n_2, m) = \mathbf{Im}\{w_{\text{comb}}(m)\} \quad (23)$$

The algorithm outlined above was implemented using a PC-board with four ADSP2101 DSPs and a conventional IBM-AT compatible PC. The signals

were available as real and imaginary part of a complex-valued signal (see subsection 4.2) due to orthogonal demodulation of an originally real-valued ultrasonic signal. The 4 DSPs work in a cyclic scheme, synchronisation is achieved by interrupt-control.

The preparation of the data for 2-dimensional graphical output is also done by the DSPs. It is possible to calculate a 256-point DWD in about 2 milliseconds this way.

## 4. Examples for Time-Variant Signals

### 4.1 Simulated Data

To demonstrate the characteristics and the differences of the various methods for PDS estimation, the complex-valued and undisturbed signal

$$x(t) = e^{j\alpha t^2} + e^{j(\beta t^2 + \gamma \sin 2\pi f_0 t)} \quad (24)$$

is investigated. This signal consists of a linearly and a harmoniously modulated part. The results of PDS estimation are shown in *Fig. 1*.

*Fig. 1a* shows the spectrogram. The weakly instationary, linearly modulated part of the signal is well represented. The influence of leakage can be clearly seen in this part of the signal. Caused by the bad time resolution of the spectrogram, it is not possible to recognize the second part. To improve the time resolution, a shorter observation window was chosen in *Fig. 1b*. With zero padding for interpolation in the frequency domain, a better time resolution is achieved. But the gain in time resolution results in a loss of frequency resolution.

In *Fig. 1c* the results of PDS estimation by AR-modelling (order  $K = 8$ ) are given. The result is not better than the one achieved with the spectrogram.

The WD reproduces both signal parts in a correct manner, but cross-terms appear, too (*Fig. 1d*). To suppress the cross-terms, the WD was smoothed using a MA filter of length 15 (*Fig. 1e*).

### 4.2 Measured Data

The data are derived from the human arteria brachialis with a continuous wave ultrasonic Doppler velocity meter. A high frequency ultrasound beam, reflected by the blood particles flowing with different velocities, is shifted in frequency caused by the Doppler effect. After an orthogonal demodulation, the complex-valued received signal contains a whole spectrum of frequencies

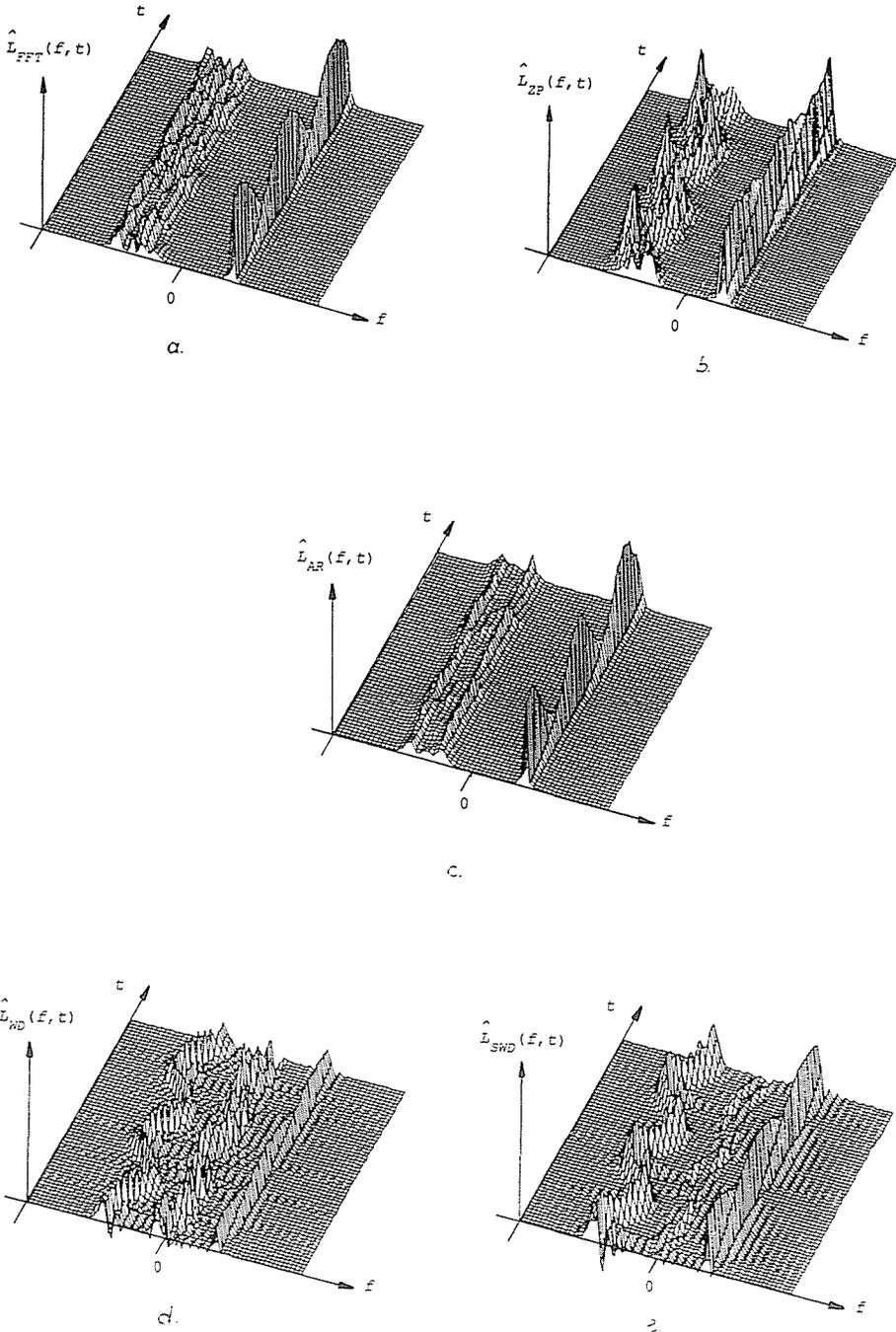


Fig. 1. PDS-estimates of a simulated, undisturbed signal

proportional to the velocity distribution inside the illuminated volume. This signal is time-variant, it is pulsating with the heart cycle (HUCKER, 1992).

*Fig. 2* shows the results of PDS estimations over the time-frequency plane. A part of one heart cycle is presented, starting with the systolic pulse.

The results of the different methods look very similar in this kind of 3-dimensional presentation. Spectrogram and WD give very noisy estimates. In the case of AR-modelling, the noise is suppressed because of the limited model order.

To illustrate the high time and frequency resolution of the WD, the nonzero values of the estimated PDSs are plotted in a 2-dimensional presentation (*Fig. 3*).

*Fig. 3a* and *3b* show the results for the spectrogram without and with zero padding. Clearly can be seen that the bad time resolution of the spectrogram without zero-padding (*Fig. 3a*) is transformed to a bad frequency-resolution of the spectrogram with zero padding (*Fig. 3b*).

The homogeneous area in the case of parametric modeling (*Fig. 3c*) is the result of the limited model order ( $K = 8$ ). Increasing the model order up to the number of samples in the observation window makes the result similar to the one achieved using the spectrogram.

The high resolution of the WD is demonstrated in *Figs. 3d* and *3e*. It is obvious that the WD is that one of the presented methods, which is least affected by the length of the used observation window. Only in the case of Wigner analysis, the narrowband increase at time of systole, caused by the flat velocity profile inside the blood vessel at this instant, becomes evident. Smoothing the WD (*Fig. 3e*) suppresses the cross-terms caused by a non-optimal signal demodulation.

The mean frequency of the PDS is proportional to the mean velocity, which means the average over the cross-sectional area of the blood vessel, and is in the case of a time-variant blood flow a function of time. *Fig. 4* shows the different estimates for the mean frequency.

Knowing the PDS, it is possible to calculate its mean frequency  $\hat{f}_{\text{PDS}}(t)$  using *Eq. (13)*. But algorithms in the time domain such as *Eq. (18)* ( $\hat{f}_{\text{WD}}(t)$ ) are better suited for on-line applications. Another possibility is the correlation based approach (HUCKER, 1992)

$$\hat{f}_{\text{ACF}}(t) = \frac{1}{2\pi T} \arctan \frac{V_{x_1 x_2}(T)}{V_{x_1}(T)} \quad (25)$$

with  $V_{x_1 x_2}(T)$  the cross-correlation between the real and imaginary parts of the signal  $x(t) = x_1(t) + jx_2(t)$  and  $V_{x_1}(T)$  the auto-correlation of the real (or imaginary) part, at time  $T$ , respectively.

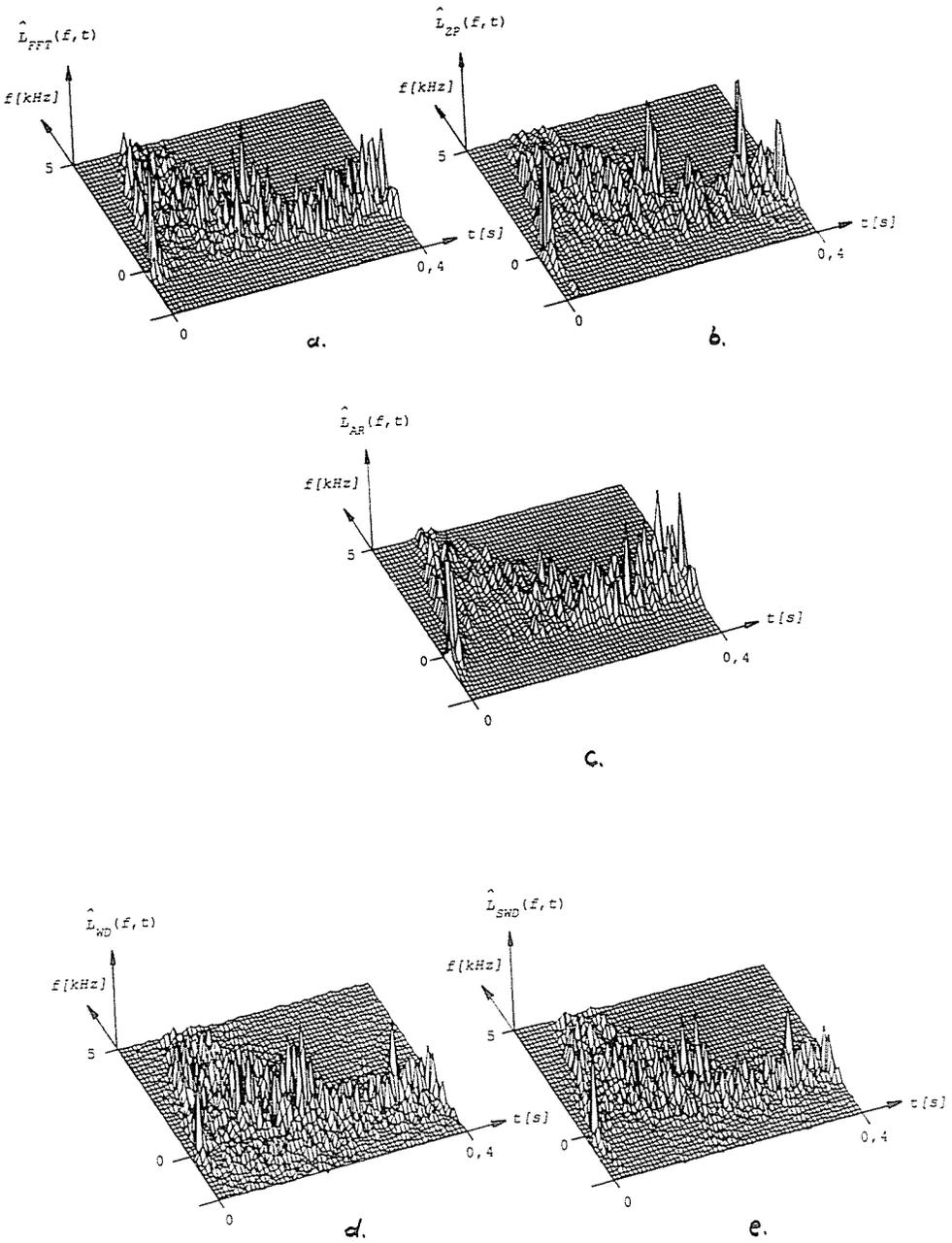


Fig. 2. PDS-estimates of measured data (human blood flow)

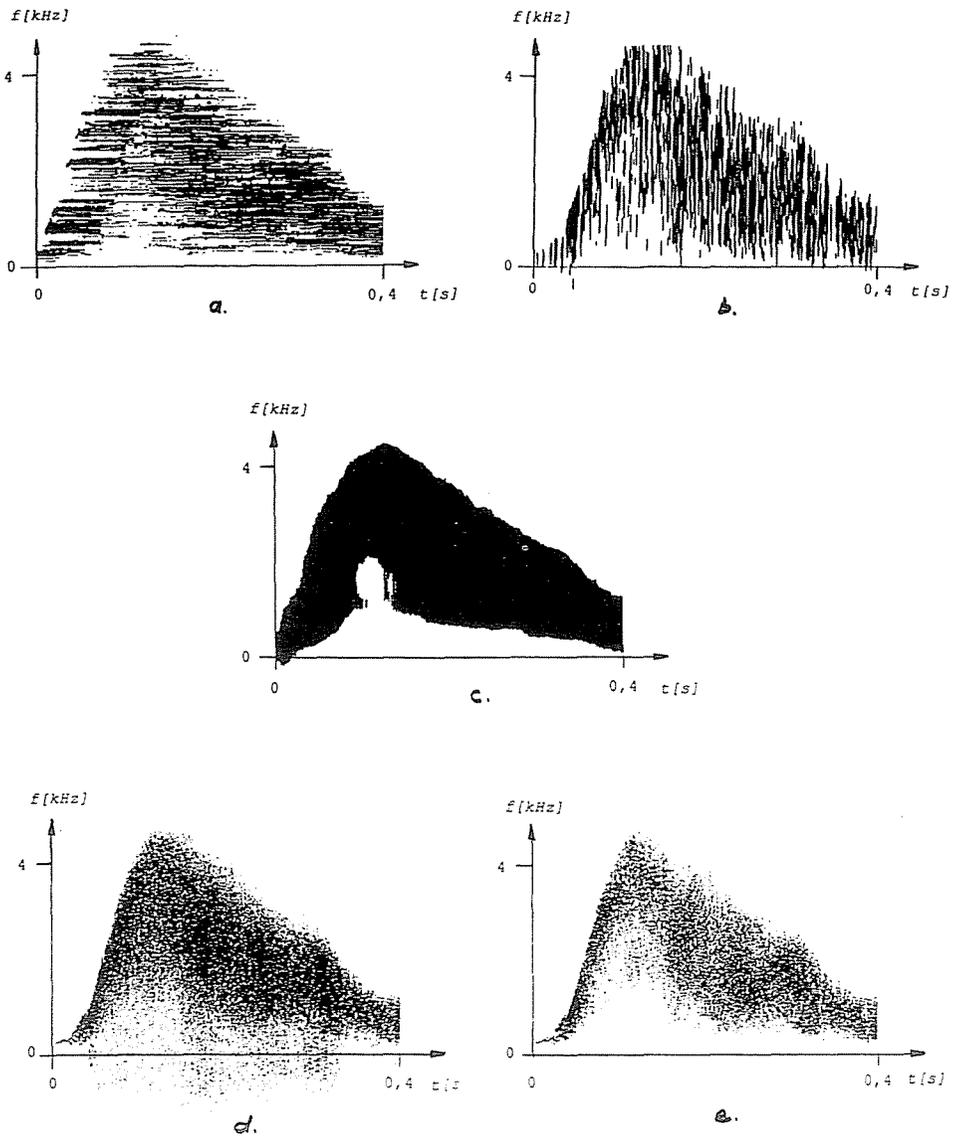


Fig. 3. A 2-dimensional plot of Fig. 2

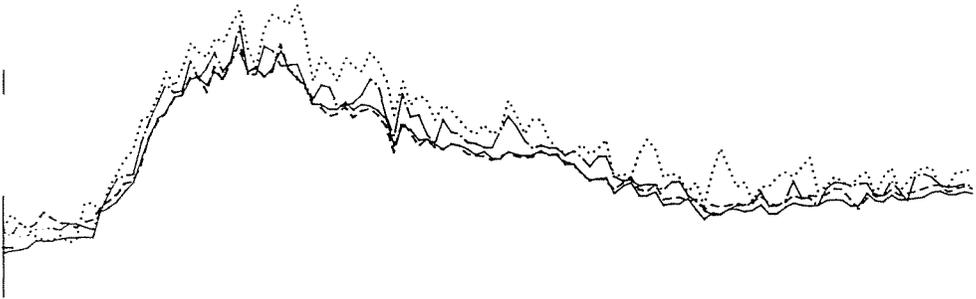


Fig. 4. Mean frequency

The most common method for estimating the mean frequency is the zero crossing detector. One can show that, in general, the zero crossing detector is a biased estimator for the mean frequency, the estimated value is too high ( $\hat{f}_{ZCD}(t)$ ).

## 5. Conclusions

The description of time-variant signals with the conventional methods for frequency analysis is only possible for signals which are only 'varying slowly' in time. Otherwise the WD is a suitable tool for time-frequency analysis.

Not only in the medical field, as presented above, but in many technical applications, too, the WD is a reasonable alternative for system and signal description.

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