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ROBUST SMOOTHING OF SIGNALS

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Abstract

The problem of the extraction of the useful signal from a noisy background is one of the most important areas of signal processing. Order Statistic (OS) smoothers, based on amplitude ordering of signal samples, have been shown to offer an effective alternative to linear smoothers. It is the case particularly when there is uncertainty concerning noise statistics, or when the useful signal possesses local features such as sharp edges. In this paper we consider some linear and nonlinear (OS) smoothers, and propose a new smoothing algorithm. Simulation results are presented to illustrate the performance of the proposed smoother.

Keywords: Signal smoothing, robust statistics, order statistics.

Introduction

The problem of the extraction of the useful signal from a noisy discrete time signal is one of the most important areas of signal processing. The contaminating noise can be impulsive, and the useful signal may possess local features such as sharp edges, pulses, or trends, which are important carriers of information. The ability to suppress unwanted components while preserving local signal features is crucial in many applications.

The first techniques used for noise suppression were linear. Linear techniques are analytically very well understood due to the nice properties, such as superposition, and are very well suited for frequency domain interpretation. Due to this fact, linear systems are generally easy to describe analytically, and can be characterized uniquely by a transfer function, independently from the input. Unfortunately, despite this analytical simplicity, some problems of signal smoothing have not been satisfactorily addressed by using linear smoothers. Linear smoothers smear edges, and do not perform well in the presence of impulsive noise.

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In the early signal smoothing techniques, the emphasis was on the derivation of optimum schemes based on a priori assumptions about signal and noise models. A typical example of this is the running mean (RM) smoother which is optimal for a slowly varying signal in Gaussian noise. An important question is: what will be the performance of such optimum scheme if the signal and noise characteristics deviate from those for which the scheme is designed? This is an important question, because in practice one rarely has perfect knowledge of the noise characteristics. The a priori assumptions are often only mathematically convenient formulations of an uncertain knowledge. For example, the RM smoother can suffer a drastic degradation in performance if the measurement data contain largely outlying observations (impulses). This has led to the need for robust signal smoothing techniques (KASSAM et al, 1985): techniques which can tolerate small deviations from ideal a priori assumptions.

The above problems have stimulated considerable interest and research work, during the last two decades, to search for new approaches that address these problems. Various approaches have been proposed, and extensive research is still introducing different new methods. Among the most successful approaches are those based on the robust *estimation of location parameter*. In particular three classes of smoothers can be distinguished, namely Order Statistic smoothers (BOVIK et al, 1983), which are based on the linear combinations of order statistics, M-smoothers (maximum likelihood estimators, LEE et al, 1985), R-smoothers (CRINON, 1985), which are based on rank tests.

The organization of this paper is as follows. In the following two sections we consider linear and Order Statistic smoothers, respectively, and discuss their properties. In the last two sections we propose a simple and efficient smoothing algorithm, and present some simulation results comparing the performance of the smoothing algorithms discussed in the paper.

Linear Smoothing

Linear smoothing in time domain is achieved by applying a linear operation through a window W_k, which moves over the input signal. The output of the smoothing at time index k is computed by the average value of the N points inside the window W_k:

$$y(k) = \frac{\sum\limits_{i=1}^{2n+1} a_i x(i)}{\sum\limits_{i=1}^{2n+1} a_i}, \quad x(i) \in W_k . \tag{1}$$

This is called a linear shift invariant (LSI) finite impulse response (FIR) filter if the weighting coefficients (a_i) are fixed for the window W_k . The RM smoother is obtained when all (a_i) are equal to a constant.

Let us consider an input signal consisting of noise superimposed on a useful signal, which displays abrupt and sustained changes (discontinuities). Such discontinuities contain high frequency components, and cannot be distinguished from the noise component, as far as their spectral content is concerned. Thus, the RM (lowpass filter) will smear out the sharp edges in the data and suppress the noise. In many applications this is intolerable, because edges carry important information. On the other hand, a single impulsive data point at the input produces a copy of the filter impulse response at the output. When an impulse is present inside the window, the RM tends to suppress it by distributing its energy to the neighbouring points, while an ideal solution is to give aberrant data points less (or no) weight in the computation. The search for such ideal impulse suppressor has led to the development of the so called *Robust Statistical Estimators*. One of the most interesting estimators of this type is the median smoother (MS).

Order Statistic Smoothers

OS smoothers form an interesting class of nonlinear algorithms, which are useful for the robust smoothing of noisy discrete signals. The output of an OS smoother at time index k is obtained by replacing the corresponding input point by linear combination of the amplitude ordered samples in the neighbourhood of that point. The class of OS smoothers includes as special case the MS, the RM, Alpha-Trimmed-Mean (ATM), and the Ranked-Order (RO) smoothers. The output of an OS smoother of window size $2N + 1$ at time index k, for an input sequence $\{x_i\}$ is given by:

$$Y(k) = \sum_{j=1}^{2n+1} A_j X_{(j)}^k \ , \tag{2}$$

where $X_{(1)}^k$ is the smallest sample inside the window centred at k, and A_j is a set of constant weights with $\sum A_j = 1$. The optimal coefficients are chosen so that they minimize the MSE, with the constraint that the resulting estimator is unbiased (BOVIK et al, 1983). The estimation of the coefficients for an arbitrary signal is generally intractable, due to the nonlinearity. For the simple case of constant signal in additive zero mean white noise, the optimal weights for an OS smoother, minimizing the output variance are obtained in (BOVIK et al, 1983). It is interesting to note that as the tails of the noise distribution grow short the optimal smoother approaches the midrange, i.e. the weights for the extreme values of X ($X^k_{(1)}$, and $X^k_{(2N+1)}$) approach 0.5 and the rest of A_j approach zero. Conversely, as the tails of the noise distribution grow heavy, the optimal filter approaches the median, i.e. the weight in the centre ($A_{(N+1)}$) approaches unity, and the rest approaches zero. These results are expected since the midrange and the median are the respective maximum likelihood estimators for these distributions.

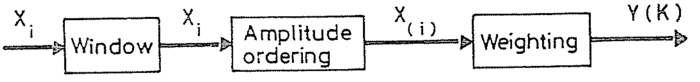

Fig. 1. Block diagram of the Order Statistic smoother

A block diagram of the OS smoother is shown in *Fig. 1*. As shown in the figure the OS operation can be decomposed into three sequential steps :

—windowing,

—amplitude ordering of data inside the running window,

—linear weighting which is identical to FIR filtering.

By proper choice of weights, OS smoothers are tuneable from the linear RM to nonlinear smoothers such as median. The median smoother is obtained by choosing the weights as follows:

$$A_j = \begin{cases} 1; & j = (N+1), \\ 0; & \text{otherwise} . \end{cases} \tag{3}$$

The RM is obtained by choosing equal weights:

$$A_j = 1/(2N+1) \; ; \qquad 1 \le j \le 2N+1 \; . \tag{4}$$

Median Smoother

The median smoother (MS) is the most interesting member of the OS smoothers. It was first proposed in 1974 by TUKEY as a time series tool for robust noise suppression. Since then the MS has received considerable attention in the signal processing literature. It has found application in many fields, e.g.: computer assisted tomography scan systems, geophysical signal processing, speech processing, and picture processing. The interest in the MS is due to its desirable properties such as impulse noise suppression and particularly its ability to smooth while preserving edges.

With median smoothing, extreme sample values (impulses, or outliers) are entirely removed by the smoothing operation. In addition, the MS also smooths the input signal; the degree of smoothing increases with the size of the window. Analytical explanation of the success of the MS in applications such as the above are available in the literature (JUSTUSSON, 1981; DAVID, 1970; GALLAGHER, 1988; TYAN, 1981).

Coefficient Censored OS Smoothers

An important class of OS smoothers is the so-called coefficient censored (CS) smoothers. Coefficient Censoring (trimming) is achieved by assigning zero weights to the extreme value samples inside the window. Coefficient censoring is widely used by statisticians for parameter estimation when the data contain unreliable or outlying samples. The number of censored samples can be either randomly determined or fixed. Examples of such filters are the Modified Trimmed Mean (MTM) smoother (KASSAM et al, 1985), and the ATM smoother (BENDAR et al, 1984). The motivation for these smoothers is to combine the nonimpulsive noise suppression ability of the RM, with the impulsive noise suppression and edge preservation ability of the MS.

In ATM smoothing, a set of $(N-T)$ samples closest to the sample median is selected from the two sets of N samples on either side of the sample median, where T is the trimming parameter. Then the average of the $2(N-T)$ selected samples and the sample median is used as the output. The MTM smoother determines the sample median M_k for the current window position, censors all samples that fall outside the range $[M_k - q,\ M_k + q]$, and averages the remaining samples. The parameter q is a preselected constant depending on the noise variance and the minimum edge height.

Quick and Simple Smoother

The MS is an effective algorithm for suppressing impulsive noise and preserving edges, however, it fails to provide sufficient smoothing of nonimpulsive noise. Therefore, when the noise consists of both impulsive and nonimpulsive components, powerful trimming algorithms such as the MTM, and DWMTM (LEE et al, 1985) are called for. These algorithms are computationally demanding, and have rather complicated structure. Sometimes one would like to have a solution which is fast, simple, and yet safe (robust),

Table 1
Relative efficiency of the FS estimator to the best possible one

Distribution	Relative Efficiency
Cauchy	.800
Laplace	.847
Logistic	.910
Normal	.781

i.e. its efficiency is guaranteed to be within acceptable limits for some class of noise distributions. As an example of such techniques, we propose a fast and simple (FS) smoother with an output defined as follows:

$$Y(k) = 0.3X(33.33) + 0.4X(50) + 0.3X(66.66) , \tag{5}$$

where $X(33.33)$, $X(50)$, and $X(66.66)$ are the $33 - {}^{1}\!/_{3}$, 50 and $66 - {}^{2}\!/_{3}$ percentiles, respectively, of the data inside the current window. This is basically an estimator proposed in the statistical literature by J. L. GASTWIRTH, but we have not found any report of its usage as a smoother in the literature.

It should be noted here that although many statistical estimators of the location parameter can be applied for signal smoothing, the signal smoothing problems impose their own distinct requirements which are not considered in statistics. In statistics the input data set is modelled as a sample from a given parent distribution, and the aim is to estimate the centre of symmetry (location parameter) of this distribution. Conversely, in signal smoothing the input is generally a random variable with varying mean. Thus the a priori assumptions of the signal model for the estimation of location parameter are not valid in the case of smoothing nonconstant signal in noise.

Therefore, when a statistical estimator is to be used for smoothing it should be checked to see if it is suited for that particular task. Unfortunately, in the case of OS estimators which, as mentioned, are analytically rather intractable, empirical approaches are usually employed.

Table 1 (taken from GASTWIRTH, 1966) shows the efficiency of the proposed estimator relative to the best estimators for the distributions considered. It is seen that the FS estimator has a minimum efficiency of approximately 80 % for noise distributions ranging from Gaussian to Cauchy. Thus, it is quite robust and yet has a very simple structure.

In the table, the logistic probability density function is defined by:

$$f(x) = e^{-x}/(1 + e^{-x})^2 , \quad -\infty < x < \infty . \tag{6}$$

Simulation Results

Fig. 2 a shows the test signal used for empirical evaluation and comparison of the performance of the proposed algorithm with the RM, MS, and the MTM smoothers. This signal is designed to contain a variety of features that we like to preserve. To this signal a noise sequence is added, which consists of nonimpulsive (Gaussian, $\sigma = 2$) noise and several impulses. The noisy signal is shown in *Fig. 2b*.

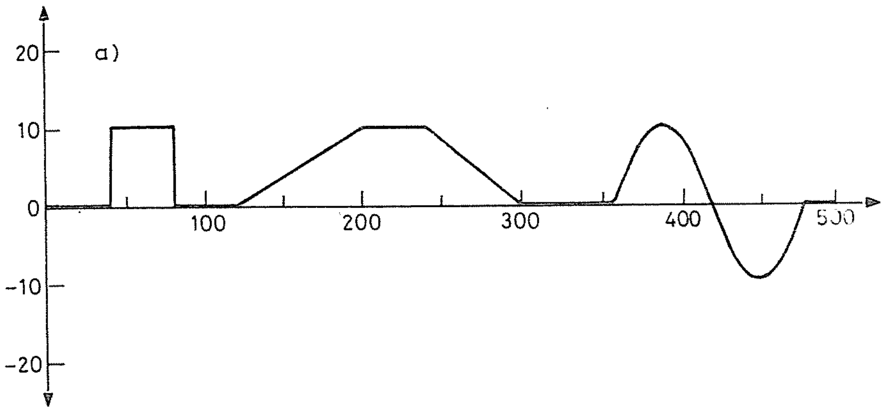

Fig. 2a. Simulation results: Test signal.

Fig. 2c shows the noisy signal smoothed by a RM with window size $W = 11$. One can see that the RM fails to suppress the impulsive noise, and smears signal edges. *Fig. 2d* shows the noisy input signal smoothed by a MS with window length $W = 11$. The MS suppresses the impulsive noise, and preserves the edges very well. However, it fails to provide sufficient smoothing of the nonimpulsive noise.

Fig. 2e shows the noisy signal smoothed by the proposed smoother. We can see that the FS smoother suppresses nonimpulsive noise better than the MS smoother, and suppresses impulsive noise and preserves edges better than the RM smoother.

Fig. 2f shows the output of MTM smoother with window size $W = 11$, and parameter $q = 3\sigma$. The MTM smoother has a good performance with respect to edge preservation, nonimpulsive noise smoothing, and impulsive noise suppression. It is slightly better than the FS smoother, but the FS smoother is much simpler and much faster than the MTM smoother.

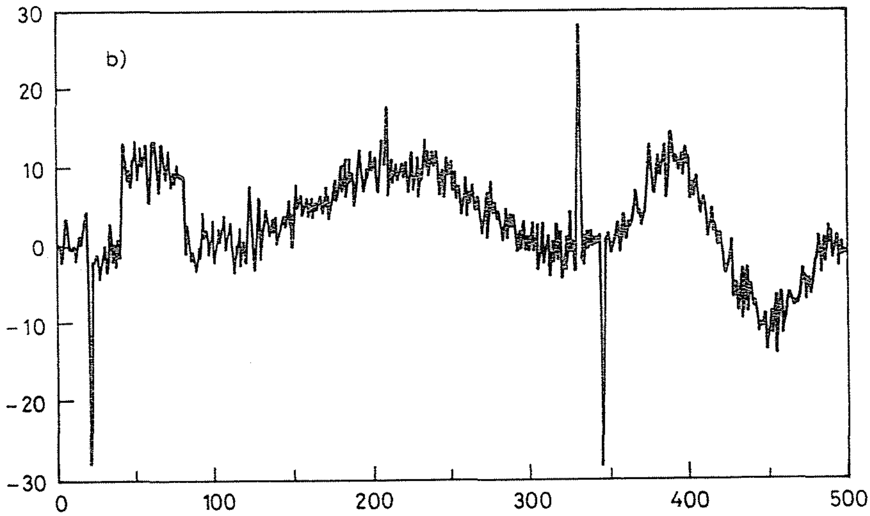

Fig. 2b. Simulation results: Noisy signal.

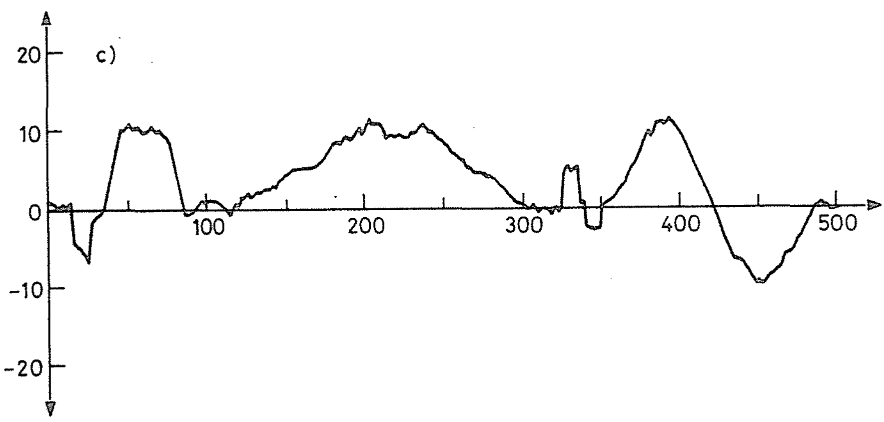

Fig. 2c. Simulation results: Output of the RM.

Conclusion

In this paper we have considered robust algorithms for signal smoothing. Robustness could be formulated in many ways, but here we consider it as the robustness with respect to uncertain a priori information about the noise probability distribution, and resistance to extreme value noisy

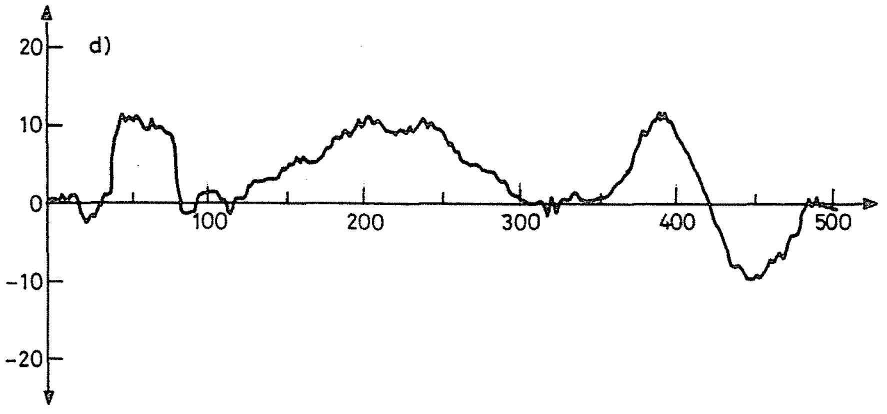

Fig. 2d. Simulation results: Output of the MS.

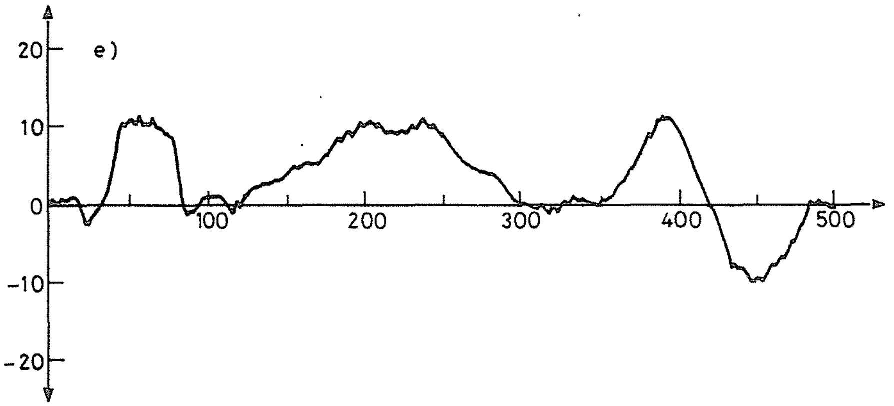

Fig. 2e. Simulation results: Output of the FS.

samples (impulses). Another important issue is the suppression of noise while preserving local signal features. Linear smoothing techniques are analytically very well understood, due to the rich and well established linear theory; however, they fail to perform well in the case of impulsive noise contamination, and distort some local signal features. Order statistic smoothers, based on amplitude ordering of signal samples, have been shown to be a promising solution to the problem. Being nonlinear, they can be mainly analyzed empirically; some examples are presented in the simulation section.

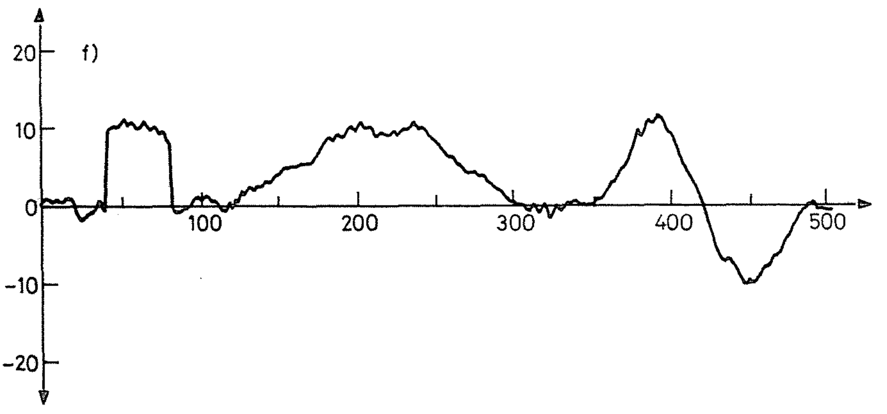

Fig. 2f. Simulation results: Output of the MTM.

We have proposed a robust nonlinear smoothing algorithm, and empirically compared it to the RM, MS, and the MTM smoothers. It is shown that the FS smoother is superior to the RM and MS, and comparable to the MTM smoother. The FS smoother has the advantage of simple structure, higher speed, and high efficiency for the classes of noise distributions ranging from Gaussian to Cauchy .

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