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A NEW METHOD TO PARAMETRIZE THE MINIMUM-TIME DEAD-BEAT CONTROL SEQUENCE

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Abstract

Supposing time-invariant, discrete-time multivariable systems a simple method is presented to determine a parametrized form of the feedback matrix which ensures the minimum-time dead-beat operation. The suggested method is based on well-known principles. In respect of applications a useful result is that restrictions which can be imposed on the control sequence or on the characteristics of the trajectory can be given by choosing proper parameter values.

Keywords: Luenberger's second canonical form, Brunovsky's canonical form, parametrized minimum-time dead-beat control.

Introduction

Let us consider a linear time-invariant multivariable discrete-time controllable system and a linear state feedback in the following form:

$$x(k+1) = \mathbf{A}x(k) + \mathbf{B}u(k), \quad k = 0, 1, ... \tag{1}$$

$$u(k) = \mathbf{K}x(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the values of the state vector and the control vector at time k, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{B} \in \mathbb{R}^{n\times m}$ the input matrix and $\mathbf{K} \in \mathbb{R}^{m\times n}$ the feedback matrix. $\mathbb{R}^n$ is the state space, $\mathbb{R}^m$ is the control space with dimension $m(\leq n)$, respectively. It is well known that the minimum-time dead-beat (MTDB) control is interpreted in terms of the $\nu(\leq n)$ controllability index of the system (1). If $x(\nu) = (\mathbf{A}+\mathbf{BK})^\nu x_0 = 0$ for every $x(0) = x_0$ initial state, then consequently (2) generates an MTDB control sequence for $k = 0,\ 1, \dots, \nu - 1$. This requirement includes that the matrix $(\mathbf{A}+\mathbf{BK})$ has to be nilpotent according to ν that is, $(\mathbf{A}+\mathbf{BK})^\nu = 0$. The feedback matrix $\mathbf{K}$ which satisfies this condition is usually not exclusive, see for instance FAHMY and O'REILLY (1983a, 1983b) furthermore SCHLEGEL (1982). Construction of the $\mathbf{K}$ matrix in non-exclusive form is called parametrization in the literature.

Here we prescribe for system $S(\mathbf{A}, \mathbf{B})$ given by (1) the following criteria

$$S(\mathbf{A},\mathbf{B}) = \{(\mathbf{A},\mathbf{B}) \text{ controllable pair; rank } (\mathbf{B}) = m \text{ and } n_1 \geq n_2 \geq \dots \geq n_m \geq 1\}\,. \tag{3}$$

Consequently, we suppose that the full rank of the controllability matrix

$$\mathbf{C} = \left[\mathbf{B}, \mathbf{AB}, \dots \mathbf{A}^{n-1}\mathbf{B}\right] \tag{4}$$

is (n), the input matrix $\mathbf{B} \in I\!R^{n\times m}$, has a full rank of (m), furthermore for the $n_i \geq 1$, $i = 1, \dots, m$ Kronecker indices (where $n_1 + n_2 + \dots + n_m = n$), determined by the $(\mathbf{A}, \mathbf{B})$ pair, the above arrangement exists. Note that this kind of ordered set can always be created by suitable rearrangement of the b_i, $i = 1, \dots, m$ column vectors of the $\mathbf{B}$ matrix, see e.g. ACKERMANN (1977).

To determine the Kronecker indexes, n linearly independent column vectors have to be chosen from the $\mathbf{C}$ matrix (with full rank n) in the following form

$$\mathrm{Im}\,(\mathbf{A}^{n_i} b_i) \subset Im \Big(b_1, b_2, \dots, b_m \mid \mathbf{A}b_1, \mathbf{A}^2 b_1, \dots$$

$$\mathbf{A}^{n_1-1} b_1 \mid \dots \mid \mathbf{A}b_i, \mathbf{A}^2 b_i, \dots, \mathbf{A}^{n_i-1} b_i, \Big) \tag{5}$$

$$i = 1, 2, \dots, m,$$

where for $i = 1, 2, \dots, m$ the $(m - i + n_1 + \dots + n_i)$ vectors have to be independent of each other. The latter vectors will be chosen. Then $n_i \geq 1$ will be the smallest positive integer for which the linear dependence described in (5) will exist. Finally, we have n linearly independent vectors which can be arranged into the following matrix:

$$\mathbf{L} = \left[\mathbf{b}_1, \mathbf{Ab}_1, \dots, \mathbf{A}^{n_1-1} b_i \mid \dots \mid b_m, \mathbf{Ab}_m, \dots, \mathbf{A}^{n_m-1} b_m\right], \tag{6}$$

where $n_1 + n_2 + \dots + n_m = n$ and the controllability index is max $n_i = \nu$ see e.g. LUENBERGER (1967). In (5) we supposed that the $\mathbf{B}$ matrix has a full rank of m.

Here we take Brunovsky's $(\mathbf{A}_c, \mathbf{B}_c)$ canonical form of the $(\mathbf{A}, \mathbf{B})$ pair as our starting point, then using results of WANG and DAVISON (1976) which refer to this, we derive the parametrized form of the $\mathbf{K}$ feedback matrix to generate the parametrized form of the MTDB control sequence.

Feedback Matrix in Parametrized Form

We can arrive at the non-exclusive $\mathbf{K} \in \mathbb{R}^{m\times n}$ feedback matrix through three steps. Here we only emphasize the main relationships and for details we will refer to the literature.

a.) LUENBERGER (1967) showed that for system (1) the following linear coordinate transformations exist in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively

$$y(k) = \mathbf{S}x(k), \quad v(k) = \mathbf{C}u(k), \tag{7}$$

so that in this new coordinate system the $(\mathbf{A}, \mathbf{B}_c)$ pair has a special form and we can write

$$(\mathbf{A}, \mathbf{B}) \longrightarrow \left(\tilde{\mathbf{A}} = \mathbf{SAS}^{-1}, \mathbf{B}_c = \mathbf{SBC}^{-1}\right). \tag{8}$$

The $(\tilde{\mathbf{A}}, \mathbf{B}_c)$ pair is the so-called Luenberger's canonical form.

b.) If the linear state feedback is given by the following equation

$$v(k) = (\hat{\mathbf{K}} + \mathbf{F})y(k) + \mathbf{G}r(k), \tag{9}$$

and the $\hat{\mathbf{K}} \in \mathbb{R}^{m\times n}$ matrix is given by using the $\sigma_i^{\text{th}} (= n_1 + n_2 + \cdots + n_i)$ row vector of the $\hat{\mathbf{A}}$ matrix in the form

$$\hat{\mathbf{K}} = \begin{bmatrix} -e_1^T & \mathbf{A}^{n_1} \\ \vdots & \\ -e_m^T & \mathbf{A}^{n_m} \end{bmatrix} \mathbf{S}^{-1}, \tag{10}$$

where $e_i^T (i = 1, \ldots, m)$ is the σ_i^{th} row vector of the inverse of the $\mathbb{L} \in \mathbb{R}^{n\times n}$ matrix given by (6). Then using (8) we obtain

$$\left(\tilde{\mathbf{A}}, \mathbf{B}_c\right) \longrightarrow ([\mathbf{A}_c + \mathbf{B}_c\mathbf{F}], \mathbf{B}_c\mathbf{G}), \tag{11}$$

and the matrix

$$\mathbf{A}_c = \mathbf{SAS}^{-1} + \mathbf{B}_c\hat{\mathbf{K}} = \mathbf{S}(\mathbf{A} + \mathbf{BC}^{-1}\hat{\mathbf{K}}\mathbf{S})\mathbf{S}^{-1}, \tag{12}$$

which is nilpotent concerning the ν controllability index hence $\mathbf{A}_c^\nu = 0$ can be written. Note that the $(\mathbf{A}_c, \mathbf{B}_c)$ pair regarding the $n_1 \geq n_2 \geq \cdots \geq n_m$ ordered set of Kronecker's indices mentioned in (3) is called Brunovsky's canonical form (see e.g. WANG and DAVISON (1976)). The structures of the matrices are

$$\mathbf{A}_c = \text{block diag}\,[\mathbf{A}_{c1}, \mathbf{A}_{c2}, \ldots, \mathbf{A}_{cm}],$$
$$\mathbf{B}_c = \text{block diag}\,[b_{c1}, b_{c2}, \ldots, b_{cm}],$$

where

$$\mathbf{A}_{ci} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad b_{ci} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

which have the size of $n_i \times n_i$ and $n_i \times 1$, respectively.

c.) Let

$$z(k) = \mathbf{T}y(k) \tag{13}$$

be a new basis in $I\!R^n$. Now if $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$, where $n_1 + n_2 + \cdots + n_m = n$ and if the non-singular matrix $\mathbf{T}$ and the $\mathbf{F} \in I\!R^{m\times n}$ and $\mathbf{G} \in I\!R^{m\times m}$ matrices in (9) are given by WANG and DAVISON (1976) then from (11)

$$([\mathbf{A}_c + \mathbf{B}_c\mathbf{F}], \mathbf{B}_c\mathbf{G}) \longrightarrow (\mathbf{A}_c, \mathbf{B}_c)\,, \tag{14}$$

that is,

$$\mathbf{A}_c = \mathbf{T}\,[\mathbf{A}_c + \mathbf{B}_c\mathbf{F}]\,\mathbf{T}^{-1}, \tag{15}$$

$$\mathbf{B}_c = \mathbf{T}\ \mathbf{B}_c\mathbf{G}, \tag{16}$$

and the three matrices ($\mathbf{T}$, $\mathbf{F}$, $\mathbf{G}$) are usually non-exclusive (see WANG and DAVISON (1976)).

The non-exclusivity of ($\mathbf{T}$, $\mathbf{F}$, $\mathbf{G}$) gives the possibility to get the parametrized form of the feedback matrix $\mathbf{K} \in I\!R^{m\times n}$ which ensures the MTDB control. Using (11) and (8) for (15) and (16)

$$\mathbf{A}_c = \mathbf{TS}[\mathbf{A} + \mathbf{B}(\mathbf{C}^{-1}\hat{\mathbf{K}}\mathbf{S} + \mathbf{C}^{-1}\mathbf{FS})]\mathbf{S}^{-1}\mathbf{T}^{-1}, \tag{17}$$

$$\mathbf{B}_c = \mathbf{TSBC}^{-1}\mathbf{G} \tag{18}$$

can be written. Since $\mathbf{A}^{\nu} = \mathbf{0}$ thus in (17) the following relationship has to be valid

$$\left[\mathbf{A} + \mathbf{B}(\mathbf{C}^{-1}\hat{\mathbf{K}}\mathbf{S} + \mathbf{C}^{-1}\mathbf{FS})\right]^{\nu} = \mathbf{0}. \tag{19}$$

Consequently, from (9) for $\mathbf{r}(k) = 0$ using (7) we have the required MTDB control law

$$u(k) = \mathbf{K}x(k), \quad k = 0, 1, \ldots, \nu - 1 \tag{20}$$

since the feedback matrix

$$\mathbf{K} = \mathbf{C}^{-1}\hat{\mathbf{K}}\mathbf{S} + \mathbf{C}^{-1}\mathbf{F}\mathbf{S} \tag{21}$$

gives just the solution of (19). Since the matrix $\mathbf{F} \in \mathbb{R}^{m \times n}$ is usually non-exclusive, the matrix $\mathbf{K}$ (above) is not exclusive, either. Considering (10), equation (21) can also be given in the following form

$$\mathbf{K} = \mathbf{C}^{-1}\begin{bmatrix} -e_1^T & \mathbf{A}^{n_1} \\ \vdots & \\ -e_m^T & \mathbf{A}^{n_m} \end{bmatrix} + \mathbf{C}^{-1}\mathbf{F}\mathbf{S}. \tag{22}$$

Now we arrived at the simple expression of the parametrized form of the feedback matrix. For completeness' sake we give a few supplementary notes for the numerical calculation of (22).

Notes

In this Section we summarize the further relationships which are necessary to calculate the matrix $\mathbf{K}$ given by (22). The structure of the transformation matrix $\mathbf{S}$ is

$$\mathbf{S} = \left[\mathbf{S}_1^T, \mathbf{S}_2^T, \ldots, \mathbf{S}_m^T\right]^T, \tag{23}$$

where the partial matrices $\mathbf{S}_i \in \mathbb{R}^{n_i \times n}$ have the following structure

$$\mathbf{S}_i = \begin{bmatrix} e_i^T & \\ e_i^T & \mathbf{A} \\ & \vdots \\ e_i^T & \mathbf{A}^{n_i - 1} \end{bmatrix}, \quad i = 1, \ldots, m \tag{24}$$

(see e.g. LUENBERGER, (1967). The superscript T used above denotes the transposed of the matrix (or the vector). We mention that the non-singular upper triangular matrix $\mathbf{C} \in \mathbb{R}^{m \times m}$ introduced by Luenberger can be calculated in the following form

$$\mathbf{C} = \begin{bmatrix} e_1^T & \mathbf{A}^{n_1 - 1} \\ & \vdots \\ e_m^T & \mathbf{A}^{n_m - 1} \end{bmatrix}\mathbf{B}. \tag{25}$$

To prove this we have to see that

$$\begin{bmatrix} e_1^T & & \mathbf{A}^{n_1-1} \\ & \vdots & \\ e_m^T & & \mathbf{A}^{n_m-1} \end{bmatrix} \mathbf{S}^{-1}\mathbf{B}_c = \mathbf{I}_m,$$

where $\mathbf{I}_m$ is the unit matrix of size $m \times m$. (25) directly follows from this relation, because according to (8) we have the relation $\mathbf{S}^{-1}\mathbf{B}_c = \mathbf{B}\mathbf{C}^{-1}$. In accordance with (22), calculation of $\mathbf{K}$ does not require the transformation matrix $\mathbf{T}$ given in (13). Thus it is enough to touch upon the explanation of the matrix $\mathbf{F} \in I\!R^{m\times n}$ .WANG and DAVISON (1976) showed that matrix $\mathbf{T}$ is not singular if the matrix

$$\mathbf{V} = \left[t_{ij}^{(n_j-n_i+1)}\right] \in I\!R^{m\times m}, \quad i,j = 1,2,\ldots,m \tag{26}$$

is not singular where the $t_{ij}^{(\,)}$ elements can be freely chosen for every $i,j = 1,2,\ldots,m$ with the restriction det $(\mathbf{V}) \neq 0$. If

$$\mathbf{G} = \mathbf{V}^{-1}, \tag{27}$$

then the matrix $\mathbf{F} \in I\!R^{m\times n}$ matrix in (21) is

$$\mathbf{F} = -\mathbf{G}\overline{\mathbf{F}}, \tag{28}$$

where

$$\overline{\mathbf{F}} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ \vdots & & & \\ f_{m1} & f_{m2} & \cdots & f_{mm} \end{bmatrix} \in I\!R^{m\times n}, \tag{29}$$

in which the row vector elements with size $1 \times n_j$ have the following form;

$$f_{ij} = \begin{bmatrix} 0, \ldots, 0, & t_{ij}^1, t_{ij}^2, \ldots, t_{ij}^{(n_j-n_i)} \end{bmatrix}, \qquad |\leftarrow n_i \rightarrow | \longleftarrow n_j - n_i \longrightarrow | \qquad i,j = 1,\ldots,m, \tag{30}$$

where the $t_{ij}^{(\,)}$ scalar elements can be considered as freely chosen parameters. We should note that for $i \leq j$ the vector $f_{ij} = 0$ because $n_1 \geq n_2 \geq \cdots \geq n_m$. Consequently in the matrix $\overline{\mathbf{F}}$, given by (29), the number of parameters is

$$Z = \sum_{i=2}^{m}\sum_{j=1}^{i-1}(n_j - n_i), \quad n_j > n_i; \quad j < i. \tag{31}$$

Note the otherwise obvious fact that if $n_1 = n_2 = \cdots = n_m$ since $\overline{\mathbf{F}} = 0$ the feedback matrix becomes exclusive. Thus it is in accordance with the statement of (FARISON and FU (1970)).

III. Control to Non-zero Final State

Let $\mathbf{r}(k) = \mathbf{E}\mathbf{x}^F$ in (9) where the matrix $\mathbf{E} \in I\!R^{m\times n}$ is unknown for the time being. By reason of (9) and (32) using (7) for the system described by (1), we have the control law

$$u(k) = \mathbf{K}x(k) + \mathbf{L}x^F, \quad k = 0, 1, \ldots, \nu - 1 \tag{32}$$

where

$$\mathbf{L} = \mathbf{C}^{-1}\mathbf{G}\mathbf{E}. \tag{33}$$

We assert that a subspace $\mathcal{F} \subset I\!R^n$ of $x^F \neq 0$ non-zero final states exists where an $x^F \in \mathcal{F}$ final state can be reached in the form of $x(\nu) = x^F$ from every $x_0 \in I\!R^n$ initial state by the control sequence given by (32), where ν is the controllability index. Considering (1) and (32), the state equation of the closed system is as follows:

$$x(k+1) = \mathbf{W}x(k) + \mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}x^F, \tag{34}$$

where

$$\mathbf{W} \triangleq [\mathbf{A} + \mathbf{B}\mathbf{K}], \tag{35}$$

from which, according to (19), we can write $\mathbf{W}^\nu = \mathbf{0}$. The trajectory of the closed system is as follows:

$$x(k) = \mathbf{W}^k x_0 + (\mathbf{W}^{k-1} + \cdots + \mathbf{W} + \mathbf{I})\mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}x^F, \tag{36}$$

$$k = 1, 2, \ldots \nu.$$

The requirement, $x(\nu) = x^F$, considering that $\mathbf{W}^\nu = \mathbf{0}$, requires the following condition

$$x^F = (\mathbf{W}^{\nu-1} + \cdots + \mathbf{W} + \mathbf{I})\mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}x^F, \tag{37}$$

where similarly as above

$$\text{rank}\,(\mathbf{B}\mathbf{C}^{-1}\mathbf{G}) = m. \tag{38}$$

We can transform (37) into the following simplified form

$$x^F = \mathbf{H}\mathbf{E}x^F, \tag{39}$$

where

$$\mathbf{H} = (\mathbf{W}^{\nu-1} + \cdots + \mathbf{W} + \mathbf{I})\mathbf{B}\mathbf{C}^{-1}\mathbf{G}. \tag{40}$$

To determine the image space of the matrix $\mathbf{H} \in \mathbb{R}^{n\times m}$ we need an additional consideration. In (17) let

$$\mathbf{S}^{-1}\mathbf{T}^{-1} = \mathbf{M} = [\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_m], \tag{41}$$

where the partial matrix $\mathbf{M}_i \in \mathbb{R}^{n\times n_i}$ is given by its column vectors

$$\mathbf{M}_i = [m_{i,0}, m_{i,1}, \ldots, m_{i,n_i-1}], \qquad i = 1, 2, \ldots, m. \tag{42}$$

From (18), using (41), we have

$$\mathbf{M}\mathbf{B}_c = \mathbf{B}\mathbf{C}^{-1}\mathbf{G}. \tag{43}$$

Considering the structure of matrix $\mathbf{B}_c$ and (38) we can see that

$$\mathbf{B}\mathbf{C}^{-1}\mathbf{G} = [m_{1,n_1-1}, m_{2,n_2-1}, \ldots, m_{m,n_m-1}], \tag{44}$$

and thus the last column vector m_{i,n_i-1}, $i = 1, \ldots, m$ in (42) is known. From (17) using the notation of (35)

$$\mathbf{W}\ \mathbf{M} = \mathbf{M}\ \mathbf{A}_c \tag{45}$$

can be written which contains the following matrix equations

$$\mathbf{W}\ \mathbf{M}_i = \mathbf{M}_i\mathbf{A}_{ci}. \qquad i = 1, 2, \ldots, m. \tag{46}$$

From (46) the vector equations

$$\left.\begin{array}{l} \mathbf{W}m_{1,0} = 0 \\ \mathbf{W}m_{i,j} = m_{i,j-1} \end{array}\right\} \qquad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n_i - 1 \end{array} \tag{47.a, b}$$

arise. Since m_{i,n_i-1} is known, (47.b) gives the further column vectors of the partial matrix $\mathbf{M}_i$, we can write

$$m_{i,j-1} = \mathbf{W}^{n_i-j} m_{i,n_i-1}, \qquad \begin{array}{l} i = 1, \ldots, m, \\ j = 1, \ldots, n_i - 1. \end{array} \tag{48}$$

Note that since rank $\mathbf{TS} = n$ the vectors $m_{i,0}, m_{i,1}, \ldots, m_{i,n_i-1}$, $i = 1, \ldots, m$ are linearly independent so they form a basis in $\mathbb{R}^n$. Consequently, in accordance with (48), an exact definition can be given for the vectors which span the image space of the $\mathbf{H} \in \mathbb{R}^{n\times m}$ matrix. On the other hand it arises that

$$\text{rank}\ \mathbf{H} = m. \tag{49}$$

Let $\mathbf{Q} \in I\!R^{n\times n}$ matrix with the following characteristic

$$\text{rank } \mathbf{Q} = m, \tag{50.a}$$

$$\text{Im } \mathbf{QH} = \text{Im } \mathbf{H}. \tag{50.b}$$

If we multiply (39) by the matrix $\mathbf{Q}$ from the left side, concerning the conditions in (50) we can obtain the matrix $\mathbf{E} \in I\!R^{m\times n}$ by solving the following equation

$$\mathbf{Q} = \mathbf{QHE}, \tag{51}$$

and obviously rank $\mathbf{E}= m$. The $\mathcal{F} \subset I\!R^n$ subspace of x^F final states from the above equation follows as

$$\mathcal{F} = \text{Im } \mathbf{Q}, \tag{52}$$

which has the dimension $d(\mathcal{F}) = m$. We note that $\mathbf{Q}$ in (50.b) is exclusive for one scalar multiplicator. Let $k = \nu$ and $x(\nu) = x^F \in \mathcal{F}$ in (34). Then using (37) we can write

$$\begin{aligned} x(\nu+1) &= (\mathbf{W}^{\nu} + \cdots + \mathbf{W})\mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}\, x^F + \mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}\, x^F = \\ &= (\mathbf{W}^{\nu-1} + \cdots + \mathbf{W} + \mathbf{I})\mathbf{B}\mathbf{C}^{-1}\mathbf{G}\mathbf{E}\, \mathrm{x}^F, \end{aligned} \tag{53}$$

where we used the fact that $\mathbf{W}^{\nu} = \mathbf{0}$. If the ε set of x^E equilibrium states of the system given by (1) is not empty and $x^F = x^E \in \mathcal{F}$, from (53) it follows that $x(\nu+1) = x^E$. Consequently the control law in (32) stabilizes every $x^E \in \mathcal{F}$ equilibrium final state independently of the parameters included in the $(\mathbf{K}, \mathbf{L})$ pair. We note here that considering (17) and (18) the parameters in the $(\mathbf{K}, \mathbf{L})$ pair can be chosen freely at any discrete time k. Consequently, a trajectory which starts out of an $x_0 \in I\!R^n$ initial state and goes to an $x^F \in \mathcal{F}$ final state can be modified using proper parameters. If there is no restriction for the characteristics of the trajectory then the parameters can be used to satisfy restrictions for $u(k)$, $k = 0, \ldots \nu - 1$ control sequence. Note that concerning (17) for the matrix $\mathbf{W}$ given by (35) follows as rank $\mathbf{W} = n - m$.

An Example

Taking the case of $n = 4$ and $m = 2$ let the controllable $(\mathbf{A}, \mathbf{B})$ pair be given by the following state matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which were pondered by SCHLEGEL (1982). In accordance with (5) the Kronecker indexes are $n_1 = 3$ and $n_2 = 1$. The controllability index is $n_1 = \nu = 3$. Since $n_1 > n_2$ hence system (1) characterized by the $(\mathbf{A}, \mathbf{B})$ pair belongs to the class given by (3). According to (26) we can write

$$\mathbf{V} = \begin{bmatrix} t_{11}^1 & 0 \\ t_{21}^3 & t_{22}^1 \end{bmatrix}.$$

Let us choose $\mathbf{V}$ in the form of a unity matrix $\mathbf{V} = \mathbf{I}_m$ that is $t_{11}^1 = t_{22}^1 = 1$ and $t_{21}^3 = 0$. In accordance with (30), for (29) we can write

$$\overline{\mathbf{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{21}^1 & t_{21}^2 & 0 \end{bmatrix}.$$

Since

$$e_1^T = [0, \quad 1, \quad 0, \quad 1,],$$

$$e_2^T = [1, \quad 0, \quad 1, \quad -1,],$$

hence the $\mathbf{S}$ transformation matrix from (24) is

$$\mathbf{S} = \begin{bmatrix} e_1^T & & \\ e_1^T & \mathbf{A} & \\ & e_1^T & \mathbf{A}^2 \\ & e_2^T & \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Now for (25) we have $\mathbf{C} = \mathbf{I}_m$. Since we chose $\mathbf{V} = \mathbf{I}_m$ $\mathbf{F} = -\mathbf{V}^{-1}\overline{\mathbf{F}} = -\overline{\mathbf{F}}$ according to (27) and (28). Hereupon in accordance with (22)

$$\mathbf{K} = \mathbf{C}^{-1} \begin{bmatrix} -e_1^T & \mathbf{A}^3 \\ -e_2^T & \mathbf{A} \end{bmatrix} + \mathbf{C}^{-1}\mathbf{F}\mathbf{S} =$$

$$\begin{bmatrix} 0 & -1 & -1 & -2 \\ t_{21}^1 + t_{21}^2 & -1 - t_{21}^1 - t_{21}^2 & -t_{21}^1 - t_{21}^2 & 1 - t_{21}^2 \end{bmatrix}, \qquad \text{(E - 1)}$$

where t_{21}^1 and t_{21}^2 are real scalar parameters which can be freely chosen. The state matrix of the closed system is

$$\mathbf{W} = \mathbf{A} + \mathbf{BK} =$$
$$\begin{bmatrix} 1 + t_{21}^1 + t_{21}^2 & -1 - t_{21}^1 - t_{21}^2 & -1 - t_{21}^1 - t_{21}^2 & -1 - t_{21}^2 \\ 0 & 0 & 0 & -1 \\ t_{21}^1 + t_{21}^2 & -t_{21}^1 - t_{21}^2 & -t_{21}^1 - t_{21}^2 & 1 - t_{21}^2 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \quad (\mathrm{E}-2)$$

where $\mathbf{W}^3 = 0$ because $\nu = 3$. Since $\mathbf{G} = \mathbf{V}^{-1} = \mathbf{I}_m$ and $\mathbf{C} = \mathbf{I}_m$ from (44) it follows that

$$\mathbf{BC}^{-1}\mathbf{G} = [m_{1,2}, m_{2,0}] = [b_1, b_2]\,.$$

Using matrix $\mathbf{W}$ given by (E – 2) and using (45) for (40) we can write

$$\mathbf{H} = (\mathbf{W}^2 + \mathbf{W} + \mathbf{I})\mathbf{BC}^{-1}\mathbf{G} =$$

$$= [m_{1,0} + m_{1,1} + m_{1,2} \mid m_{2,0}] = \begin{bmatrix} -t_{21}^1 - t_{21}^2 & 1 \\ 1 & 0 \\ -t_{21}^1 - t_{21}^2 & 1 \\ 0 & 0 \end{bmatrix},$$

and rank $(\mathbf{H}) = m = 2$. In our case the matrix $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\mathrm{E}-3)$$

We can see that $\mathbf{QH} = \mathbf{H}$. The solution of the matrix equation $\mathbf{Q} = \mathbf{QHE}$ given by (50) is as follows:

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & t_{21}^1 + t_{21}^2 & 0 & 0 \end{bmatrix}. \quad (\mathrm{E}-4)$$

Consequently, in the control law given by (32), $\mathbf{L} = \mathbf{C}^{-1}\mathbf{GE} = \mathbf{E}$. The subspace of possible x^F final states in accordance with (52) is

$$\mathcal{F} = \mathrm{Im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that all the x^E equilibrium states of the system (1), characterized by the $(\mathbf{A},\ \mathbf{B})$ pair, can be found in this $\mathcal{F}$ subspace. Now let

$$x_0 = [1, \quad 1, \quad 0, \quad 1]^T,$$

$$x^F = [-1, \quad -2, \quad -1, \quad 0]^T,$$

and suppose that there are restrictions for the characteristics of the trajectory given by

$$x(k+1) = \mathbf{W}x(k) + \mathbf{BL}x^F.$$

In the first step $k = 0$ the following state will arise

$$x(1) = [-4 - 2t_{21}^1 - 3t_{21}^2, \quad -3, \quad -2t_{21}^1 - 3t_{21}^2, \quad -3]^T$$

and we can see that only one parameter is effective. If the restriction is e. g. $\mathbf{x}_1(1) = -2$ then it can be satisfied by choosing for instance $t_{21}^2 = 0$ and $t_{21}^1 = -1$. Consequently, the third coordinate becomes fixed: $x_3(1) = 2$. Then for $k = 1$ we have the following state which is independent of the parameter t_{21}^2

$$x(2) = [-1 - 3t_{21}^1, \quad 1, \quad -4 - 3t_{21}^1, \quad 0]^T.$$

If the requirement is e.g. $x_3(2) = -1$ then we have $t_{21}^1 = -1$ and consequently $x_1(2) = 2$. In the last step ($k = 2$) we have the result $x(3) = x^F$. Sience x^F corresponds to a possible x^E equilibrium state, the trajectory is stuck in the $x^F = x^E$ equilibrium final state in accordance with the above mentioned cases.

V. Conclusions

We have given a very simple method to construct the feedback matrix $\mathbb{K}$ which ensures MTDB control in parametrized form. It has the definitely advantageous feature that no parameter can exist in the denominator of the elements of the matrix $\mathbb{K}$, consequently the matrix $\mathbb{K}$ always exists. The proposed method has the disadvantage that it supposes an ordered set of the Kronecker indexes $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$, and the dimension of the $\mathcal{F}$ subspace of reachable x^F final states always equals $m(\leq n)$ which is the rank of the input matrix $\mathbb{B}$. Consequently, if $m = n$, $\mathcal{F}$ is the whole $\mathbb{R}^n$. We have shown that if the ε set of the x^E equilibrium states of system (1) is not empty and if $\mathcal{F} \cap \varepsilon \neq \{0\}$, the suggested MTDB control law stabilizes all the $x^F = x^E$ equilibrium final states. Note that in the case of $x^F \neq 0$ sometimes it is also called MTDB operation, see e.g. JORDAN and KORN (1980).

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