PERIODICA POLYTECHNICA SER. EL. ENG. VOL. 35, NO. 4, PP. 251–261 (1991)

INVESTIGATION OF THE BOOST CHOPPER CIRCUIT

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Received Nov. 5, 1991.

Abstract

It is shown by the application of the Z-transform that the dynamic properties of the boost chopper depend on the value of the duty cycle, due to the cyclic interruption of the oscillating system. A switchless equivalent circuit is proposed.

Keywords: boost chopper, transients, Z-transform.

The Circuit and Operation Principle of the DC Boost Chopper (*Fig. 1*)

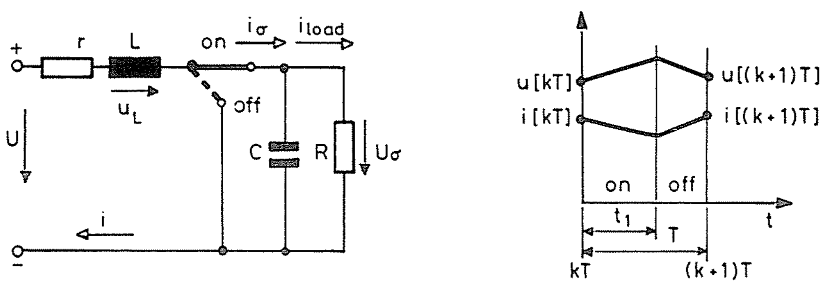

Fig. 1. The circuit of the chopper

The operation principle of chopper circuits is based on the condition that the voltage on the inductance satisfies in steady state the equation:

$$\int_0^T u_L \mathrm{d}t = 0\,.$$

Using the notations: T – switching period

$$t_1 \quad - \text{switch-on time}$$
$$D \;=\; \frac{t_1}{T} - \text{duty cycle} \tag{1}$$

and neglecting the resistance r, the steady state output voltage will be:

$$(U - U_o) \cdot D \cdot T + U(1 - D) \cdot T = 0\,,$$

$$U_o = \frac{U}{D}\,. \tag{2}$$

First the special case of the ideal circuit is analysed.

Solution of the Ideal Circuit (r=0, R=infinite)

The state variables are: i – current of the inductance
u – voltage of the capacitor.
The state equation for the switch-on time $(0 < t < t_1)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ i \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \cdot U\,. \tag{3}$$

Using the well-known relationships

$$\dot{x} = A \cdot x + B \cdot U\,, \qquad x = e^{At} \cdot x(0) + A^{-1} \cdot (e^{At} - 1) \cdot B \cdot U\,, \tag{4}$$

the u, i values at the end of the switch-on period will be:

$$i(t_1) = i(0) \cdot \cos\omega t_1 - \frac{u(0)}{L\omega} \cdot \sin\omega t_1 + \frac{U}{L\omega} \cdot \sin\omega t_1\,,$$

$$u(t_1) = \frac{i(0)}{C\omega} \cdot \sin\omega t_1 + u(0) \cdot \cos\omega t_1 + U(1 - \cos\omega t_1)\,,$$

where

$$\omega = \sqrt{\frac{1}{LC}}\,. \tag{5}$$

The values at the end of the switch-off time $(t_1 < t < T)$:

$$i(T) = i(t_1) + \frac{U}{L} \cdot (T - t_1)\,,$$

$$u(T) = u(t_1)\,.$$

The discrete values for any full switching cycle:

$$i[(k+1)T] = i[kT]\cdot\cos\omega t_1 - \frac{u[kT]}{L\omega}\cdot\sin\omega t_1 + \frac{U}{L\omega}\cdot\sin\omega t_1 + \frac{U}{L}\cdot(T-t_1)\,,$$

$$u[(k+1)T] = i[kT]\cdot\frac{\sin\omega t_1}{C\omega} + u[kT]\cdot\cos\omega t_1 + U(1-\cos\omega t_1)\,. \tag{6}$$

Changing to Z-transform (with zero initial conditions) and introducing the notations:

$$\cos\omega t_1 = c\,,$$

$$\sin\omega t_1 = s\,,$$

$$\omega(T-t_1) = a\,,$$

$$L\omega = \frac{1}{C\omega} = X\,, \tag{7}$$

$$z\cdot i(z) = i(z)\cdot c - \frac{u(z)}{X}\cdot s + \frac{z}{z-1}\cdot\frac{U}{X}\cdot s + \frac{z}{z-1}\cdot\frac{U}{X}\cdot a\,,$$

$$z\cdot u(z) = i(z)\cdot X\cdot s + u(z)\cdot c + \frac{z}{z-1}\cdot U(1-c)\,, \tag{8}$$

$$i(z) = \frac{z(z-c)}{(z-1)(z^2-2cz+1)}\cdot\frac{U}{X}\cdot(s+a) - \frac{z}{(z-1)(z^2-2cz+1)}\cdot\frac{U}{X}\cdot s(1-c)\,,$$

$$u(z) = \frac{z}{(z-1)(z^2-2cz+1)}\cdot U s(s+a) + \frac{z(z-c)}{(z-1)(z^2-2cz+1)}\cdot U(1-c)\,. \tag{9}$$

The inverse transform is obtained by using the identities:

$$\mathcal{Z}\{1(kT)\} = \frac{z}{z-1},$$

$$\mathcal{Z}\{\cos(\nu kT)\} = \frac{z(z-\cos\nu T)}{(z^2-2z\cdot\cos\nu T+1)},$$

$$\mathcal{Z}\{\sin(\nu kT)\} = \frac{z\cdot\sin\nu T}{(z^2-2z\cdot\cos\nu T+1)}. \tag{10}$$

The partial fraction expansions are shown in App. 1.

The discrete solution in time domain is:

$$i(kT) = \frac{U}{2X}\cdot\left[a\cdot[1-\cos(\nu kT)] + \left(2+\frac{sa}{1-c}\right)\cdot\sin(\nu kT)\right],$$

$$u(kT) = \frac{U}{2}\cdot\left[2+\frac{sa}{1-c} - \left(2+\frac{sa}{1-c}\right)\cdot\cos(\nu kT) - a\cdot\sin(\nu kT)\right]. \tag{11}$$

This is a sinusoidal oscillation having the frequency

$$\nu = \omega\cdot\frac{t_1}{T} = \omega\cdot D = \omega^*. \tag{12}$$

Fig. 2 shows the computed solution.

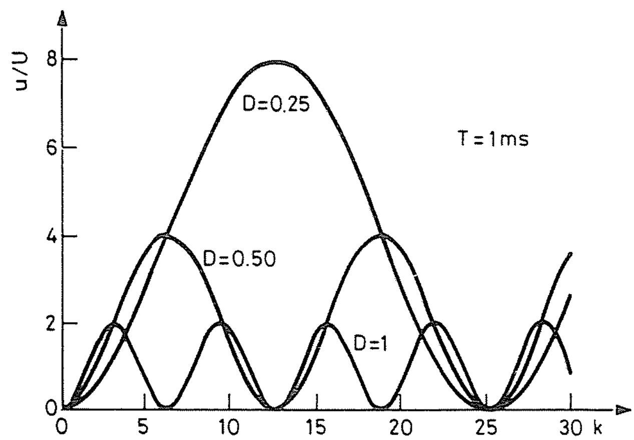

Fig. 2. The time variation of $u(kT)$ for different values of the duty cycle. (L=1 H , C=1 μF)

The cyclically interrupted oscillator has a smaller natural frequency proportional to the duty cycle. This can be interpreted as an apparent modification of the parameters. The equivalent parameters L^* and C^* must satisfy the relationship:

$$\nu^2 = \frac{1}{L^* C^*} = \frac{D^2}{LC}.$$

If we consider the chopper as a transformer having the transformation ratio $1/D$ from the input to the output, the inductance L of the input side can be reduced to the output side as $L^* = L/D^2$. This gives the suggestion to choose the equivalent parameters:

$$L^* = L/D^2 \qquad \text{and} \qquad C^* = C. \tag{13}$$

The Damped Circuit

a) Switch-on position, $0 < t < t_1$
The state space equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ u \end{bmatrix} = \begin{bmatrix} -r/L & -1/L \\ 1/C & -1/CR \end{bmatrix} \cdot \begin{bmatrix} i \\ u \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \cdot U. \tag{14}$$

The important constants can be obtained from the Laplace transform of the transition matrix:

$$e^{At} = [1 \cdot s - A]^{-1} = \frac{1}{s^2 + s \cdot \left(\frac{r}{L} + \frac{1}{CR}\right) + \frac{r}{RCL} + \frac{1}{LC}} \cdot \begin{bmatrix} s + \frac{1}{CR} & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{r}{L} \end{bmatrix}, \tag{15}$$

$$\begin{aligned} \omega_n &= \sqrt{\tfrac{1}{LC} \cdot \left(1 + \tfrac{r}{R}\right)} && \text{– reference frequency} \\ \xi &= \frac{\frac{r}{L} + \frac{1}{CR}}{2\omega_n} && \text{– damping factor} \\ \omega_d &= \omega_n \cdot \sqrt{1 - \xi^2} && \text{– damped frequency.} \end{aligned} \tag{16}$$

The solution of *Eq.* (14) is given below:

$$i(t) = e^{-\xi\omega_n t}\Bigg\{\left[\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}} \cdot \sin\omega_d t + \frac{1}{CR\omega_d} \cdot \sin\omega_d t\right] \cdot i(0)$$

$$-\frac{1}{L\omega_d} \cdot \sin\omega_d t \cdot u(0) + \frac{U}{L\omega_n^2} \cdot \left[-\frac{1}{CR} \cdot \cos\omega_d t + \frac{1}{CR} \cdot \frac{\xi}{\sqrt{1-\xi^2}} \cdot \sin\omega_d t\right.$$

$$-\frac{1}{(CR)^2\cdot\omega_d}\cdot\sin\omega_d t+\frac{1}{CR}\cdot e^{\xi\omega_n t}+\frac{1}{LC\omega_d}\cdot\sin\omega_d t\Big]\Big\}, \tag{17}$$

$$u(t)=e^{-\xi\omega_n t}\Big\{\frac{1}{C\omega_d}\cdot\sin\omega_d t\cdot i(0)+\Big[\cos\omega_d t-\frac{\xi}{\sqrt{1-\xi^2}}\cdot\sin\omega_d t$$

$$+\frac{r}{L\omega_d}\cdot\sin\omega_d t\Big]\cdot u(0)+\frac{U}{L\omega_n^2}\cdot\Big[-\frac{1}{C}\cdot\cos\omega_d t+\frac{1}{C}\cdot\frac{\xi}{\sqrt{1-\xi^2}}\cdot\sin\omega_d t$$

$$-\frac{1}{C^2R\omega_d}\cdot\sin\omega_d t+\frac{1}{C}\cdot e^{\xi\omega_n t}-\frac{r}{LC\omega_d}\cdot\sin\omega_d t\Big]\Big\}. \tag{18}$$

b) Switch-off position, $t_1<t<T$

$$i(t)=i(0)\cdot e^{-\frac{r}{L}\cdot t}+\frac{U}{r}\cdot(1-e^{-\frac{r}{L}\cdot t}), \tag{19}$$

$$u(t)=u(0)\cdot e^{-\frac{t}{CR}}. \tag{20}$$

To be able to write the following equations, some simplifying notations are necessary:

$$a_1=-\frac{\xi}{\sqrt{1-\xi^2}}+\frac{1}{CR\omega_d}, \qquad b_1=\frac{1}{CR}\cdot\frac{\xi}{\sqrt{1-\xi^2}}-\frac{1}{C^2R^2\omega_d}+\frac{1}{LC\omega_d},$$

$$a_u=-\frac{\xi}{\sqrt{1-\xi^2}}+\frac{r}{L\omega_d}, \qquad b_u=\frac{1}{C}\cdot\frac{\xi}{\sqrt{1-\xi^2}}-\frac{1}{C^2R\omega_d}+\frac{r}{LC\omega_d},$$

$$c_1=\cos\omega_d t_1, \qquad s_1=\sin\omega_d t_1, \qquad d_{12}=e^{-\xi\omega_n t_1}\cdot e^{-\frac{r}{L}\cdot(T-t_1)},$$

$$d_{13}=e^{-\xi\omega_n t_1}\cdot e^{-\frac{(T-t_1)}{CR}}. \tag{21}$$

The discrete values at the end of the k-th switching period obtained from *Eqs.* (17–20) have the form:

$$i(k+1) = d_{12}\Big\{\Big[c_1 + a_i s_1\Big] \cdot i(k) - \frac{1}{L\omega_d} \cdot s_1 u(k)$$

$$+\frac{U}{L\omega_n^2} \cdot \left[\frac{1}{CR} \cdot \left(e^{\xi\omega_d t_1} - c_1\right) + b_i s_1\right]\Big\} + \frac{U}{r} \cdot \left(1 - e^{\frac{r}{L}\cdot(T-t_1)}\right), \tag{22}$$

$$u(k+1) = d_{13}\Big\{\frac{1}{C\omega_d} \cdot s_1 i(k) + \Big[c_1 + a_u s_1\Big] \cdot u(k)$$

$$+\frac{U}{L\omega_n^2} \cdot \left[\frac{1}{C} \cdot \left(e^{\xi\omega_n t_1} - c_1\right) + s_1 b_u\right]\Big\}, \tag{23}$$

or by introducing the notations c_i, c_u, d_i, d_u:

$$i(k+1) = d_{12}c_i \cdot i(k) - d_{12} \cdot \frac{s_1}{L\omega_d} \cdot u(k) + d_i U\,, \tag{24}$$

$$u(k+1) = d_{13} \cdot \frac{s_1}{C\omega_d} \cdot i(k) + d_{13}c_u \cdot u(k) + d_u U\,. \tag{25}$$

The *Eq.* (25) was used for computation, see *Fig. 3*.

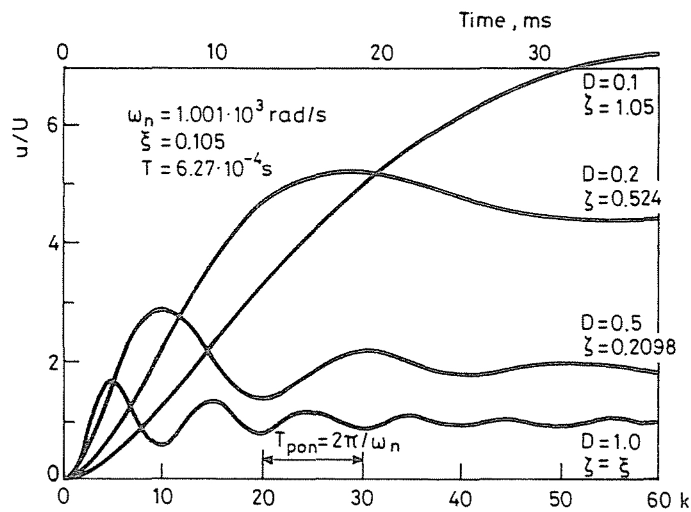

Fig. 3. The output voltage u computed from *Eq.* (25) for different values of D

The Z–transform of *Eqs.* (24–25) is

$$z \cdot i(z) = d_{12}c_i \cdot i(z) - d_{12} \cdot \frac{s_1}{L\omega_d} \cdot u(z) + \frac{z}{z-1} \cdot d_i U\,, \tag{26}$$

$$z \cdot u(z) = d_{13} \cdot \frac{s_1}{C\omega_d} \cdot i(z) + d_{13}c_u \cdot u(z) + \frac{z}{z-1} \cdot d_u U\,. \tag{27}$$

The complete solution is not necessary for our investigation, we observe only the expression of $u(z)$:

$$u(z) = \frac{z d_{13} \cdot \frac{s_1}{C\omega_d} \cdot d_i + z\Big(z - d_{12}c_i\Big) d_u}{(z-1) \cdot \Big[z^2 - z\Big(d_{12}c_i + d_{13}c_u\Big) + d_{12}d_{13} \cdot \Big(c_i c_u + \frac{s_1^2}{LC\omega^2}\Big)\Big]} \cdot U\,. \tag{28}$$

The following transform expressions are needed:

$$\mathcal{Z} \cdot \Big[e^{-\alpha kT} \cdot \sin \nu kT\Big] = \frac{z e^{-\alpha T} \cdot \sin \nu T}{z^2 - 2z e^{-\alpha T} \cdot \cos \nu T + e^{-2\alpha T}}\,, \tag{29}$$

$$\mathcal{Z} \cdot \Big[e^{-\alpha kT} \cdot \cos \nu kT\Big] = \frac{z\Big(z - e^{-\alpha T} \cdot \cos \nu T\Big)}{z^2 - 2z e^{-\alpha T} \cdot \cos \nu T + e^{-2\alpha T}}\,. \tag{30}$$

Now let us compare the expression in square brackets of the denominator of *Eq.* (28) with the denominator of *Eqs.* (29–30). It is shown in App. 2 that

$$c_i c_u + \frac{s_1^2}{LC\omega_d^2} = 1 \tag{31}$$

and

$$d_{12}d_{13} = e^{-2\xi\omega_n T}\,. \tag{32}$$

This gives the remarkable result:

$$\xi\omega_n = \alpha\,, \tag{33}$$

i. e. the damping coefficient is not depending on the duty cycle D!

Taking into consideration the results obtained for the undamped circuit, i. e. $\omega^* = \omega D$, the *apparent damping factor* ζ can be introduced:

$$\zeta = \frac{1}{D} \cdot \xi\,. \tag{34}$$

So the damped frequency can be expressed as follows:

$$\nu = \omega_n D \cdot \sqrt{1-\zeta^2} = \omega_n \cdot \sqrt{D^2 - \xi^2} . \tag{35}$$

The value of ν can be computed from the equivalence of the second terms in denominators:

$$\cos \nu T = \frac{d_{12}c_i + d_{13}c_u}{2e^{-\xi\omega_n T}} . \tag{36}$$

The values of ν computed from *Eq.* (36) as a function of D can be seen in *Fig. 4*. The curve justifies fully the *Eq.* (35).

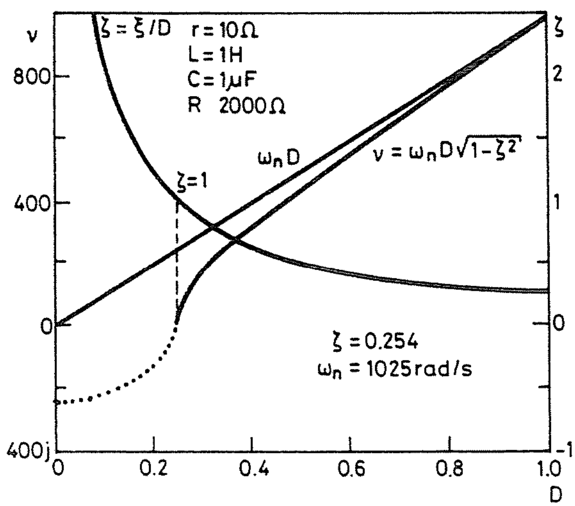

Fig. 4. Computed values of ν as a function of D. The dotted line represents imaginary values

As a consequence of invariability of the damping coefficient $\alpha = \xi\omega_\text{n}$, the value of $\frac{L}{r}$ and CR must also be invariable, and so the modified values are:

$$r^* = \frac{r}{D^2} \qquad \text{and} \qquad R^* = R . \tag{37}$$

This relationship satisfies the idea of reduction of the input side values to the output side. The reduced input voltage will be

$$U^* = \frac{U}{D} . \tag{38}$$

Conclusion

From the viewpoint of the discrete values at the end of each switching cycle, the boost chopper can be replaced by an equivalent circuit without switch whose parameters and input voltage depend on the duty cycle D, see *Fig. 5.*

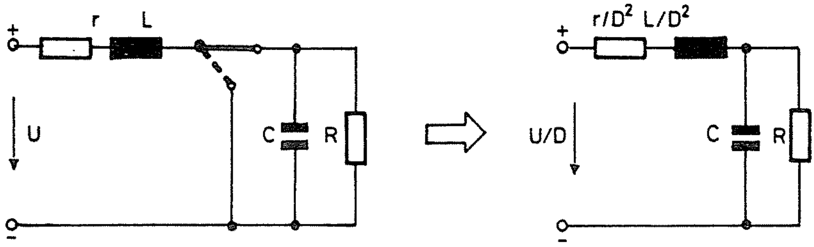

Fig. 5. The equivalent circuit of the boost chopper

This equivalency is based on the validity of the relationships:

$$U_{oav} = \frac{U}{D} \qquad \text{and} \qquad I_{av} = \frac{I_{oav}}{D}, \tag{39}$$

where the index 'av' means the average value for the cycle time T. The relationships (39) assume a linear variation of u and i (see *Fig. 1*) and require so a cycle time T small enough with regard to the smallest time constant.

Appendix 1.

Partial fraction expansions for *Eq.* (9)

$$\frac{z}{(z-1)\cdot(z^2-2cz+1)} = \frac{1}{2(1-c)}\cdot\frac{z}{z-1} - \frac{1}{2(1-c)}\cdot\frac{z(z-c)}{(z^2-2cz+1)}$$
$$-\frac{zs}{2s(z^2-2cz+1)}, \tag{A.1.1}$$

$$\frac{z(z-c)}{(z-1)\cdot(z^2-2cz+1)} = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z(z-c)}{(z^2-2cz+1)}\right.$$
$$\left.+\frac{s}{1-c}\cdot\frac{zs}{(z^2-2cz+1)}\right]. \tag{A.1.2}$$

Appendix 2.

$$c_i = c_1 + a_i s_1 ,$$

$$c_u = c_1 + a_u s_1 ,$$

$$a_i + a_u = -\frac{2\xi}{\sqrt{1-\xi^2}} + \frac{1}{\omega_d} \cdot \left(\frac{1}{CR} + \frac{r}{L}\right) = 0 \quad \text{because} \quad \frac{1}{CR} + \frac{r}{L} = 2\xi\omega_n ,$$

$$a_i \cdot a_u = -\frac{\xi^2}{1-\xi^2} + \frac{r}{R+r} \cdot \frac{1}{1-\xi^2} ,$$

$$c_i \cdot c_u + \frac{s_1^2}{LC\omega_d^2} = c_1^2 + s_1^2 \cdot \left[-\frac{\xi^2}{1-\xi^2} + \frac{r}{R+r} \cdot \frac{1}{1-\xi^2} + \frac{R}{R+r} \cdot \frac{1}{1-\xi^2}\right] = 1 , \tag{A.2.1.}$$

$$d_{12} d_{13} = e^{-2\xi\omega_n t_1 - (T-t_1)\cdot\left(\frac{r}{L} + \frac{1}{CR}\right)} = e^{-2\xi\omega_n T} . \tag{A.2.2.}$$

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