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GENERALIZED MODES IN WAVEGUIDES WITH INHOMOGENEOUS DIELECTRIC

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Abstract

It is shown that at certain singular values of the complex frequency the field in a waveguide with inhomogeneous dielectric cannot be described as a superposition of modes, because the eigenfunctions do not form a complete set. At these singular complex frequencies the fields of two different modes become identical and this common field is a special form of a generalized mode which exists only at singular complex frequencies. The generalized mode has a more complicated mathematical form than the usual mode. As for certain waveguides singular complex frequencies may assume pure imaginary values, these generalized modes may exist also at pure harmonic time dependence. In another paper it will be shown that the modal expansion is possible at singular complex frequencies, too, if generalized modes are used in addition to the usual modes.

Keywords: waveguide, inhomogeneous dielectric, modes, modal expansion.

Introduction

According to an important statement of the theory of electromagnetism the field in a waveguide with homogeneous dielectric can be described as a superposition of modes if the variation with time is harmonic. As the modal expansion based on this fact is a very useful tool for solving problems connected with waveguides, the question may arise whether a similar statement is true in the case of an inhomogeneous dielectric, a question which is not yet answered in the literature.

This problem can be treated more conveniently if the modes are defined supposing not only the harmonic, but the more general time dependence of the form $\exp(st)$, where the complex frequency s may assume any complex value. Simple physical interpretation is possible only if $\mathrm{Re}s \geq 0$, and only the case $\mathrm{Re}s=0$ has practical importance, but a boundary value problem (abbreviated further on as b.v.p.) which is related to a waveguide has mathematical solutions for any value of s. It will be shown in this paper that the field in a waveguide with inhomogeneous dielectric cannot be described as a superposition of modes at certain singular values of s, and for certain waveguides some of these singular values are pure imaginary. The modal expansion may not be used at these singular frequencies unless the concept of mode is generalized, which will be done in this paper.

Generalized Modes

Consider a lossless waveguide the axis of which is parallel to the z-axis of a Cartesian coordinate system, and in which the permittivity ε and the permeability μ depend only on the coordinates x and y. The electric and magnetic field are written as:

$$\mathcal{E} = (\mathbf{E}_{\mathrm{T}} + E_z\mathbf{k})\exp(st), \tag{1}$$

$$\mathcal{H} = (\mathbf{H}_{\mathrm{T}} + H_z\mathbf{k})\exp(st). \tag{2}$$

Here $\mathbf{k}$ denotes the unit vector parallel to the z-axis, $\mathbf{E}_{\mathrm{T}}$ and $\mathbf{H}_{\mathrm{T}}$ are perpendicular to it and together with E_z and H_z , do not depend on the time. In consequence of Maxwell's equations, $\mathbf{E}_{\mathrm{T}}$ satisfies the equation

$$\mu\mathrm{curl}_{\mathrm{T}}(\frac{1}{\mu}\mathrm{curl}_{\mathrm{T}}\mathbf{E}_{\mathrm{T}}) - \mathrm{grad}_{\mathrm{T}}(\frac{1}{\varepsilon}\mathrm{div}_{\mathrm{T}}\varepsilon\mathbf{E}_{\mathrm{T}}) - \frac{\partial^2\mathbf{E}_{\mathrm{T}}}{\partial z^2} + s^2\varepsilon\mu\mathbf{E}_{\mathrm{T}} = 0, \tag{3}$$

where the subscript T of the vector operations denotes that they must be performed with respect to the transverse coordinates x and y, only.

The usual modes, which will be called simple modes in this paper, are solutions of equation (3) in the form

$$\mathbf{E}_{\mathrm{T}}(x,y,z) = \mathbf{e}(x,y)\exp(-\gamma z). \tag{4}$$

With the notation

$$\lambda = \gamma^2 \tag{5}$$

the vector function $\mathbf{e}(x,y)$ satisfies the equation

$$\mu\mathrm{curl}(\frac{1}{\mu}\mathrm{curl}\,\mathbf{e}) - \mathrm{grad}(\frac{1}{\varepsilon}\mathrm{div}\,\varepsilon\mathbf{e}) + s^2\varepsilon\mu\mathbf{e} = \lambda\mathbf{e}. \tag{6}$$

As the wall of the waveguide is assumed to be an ideal conductor, the tangential component of the electric field vanishes on it, so $\mathbf{e}$ fulfils the following boundary conditions at the wall:

$$\mathbf{n}\times\mathbf{e} = 0, \tag{7}$$

$$\operatorname{div} \varepsilon \mathbf{e} = 0. \tag{8}$$

Here $\mathbf{n}$ denotes the normal vector to the wall, and (8) is a consequence of the fact that $E_z = 0$ at the wall. In the b.v.p. described by equations (6)-(8) λ acts as eigenvalue. The differential equation (6) has a sense only in those points of the cross-section A of the guide in which the functions $\varepsilon(x, y)$ and $\mu(x, y)$ and their derivatives are continuous. If the cross-section is composed of several regions inside of which this condition is fulfilled, boundary conditions must be considered along the curves separating these regions (MAGOS, 1979). More precisely, a generalized solution of the b.v.p. can be defined in a Soboliev space, see e. g. (LADYZHENSKAIA and URAL'TSEVA, 1973).

We shall now examine solutions of equation (3) in the form

$$\mathbf{E}_\mathrm{T} = \sum_{k=0}^{K} \mathbf{d}_k(x, y)\ (\gamma z)^k\ \exp(-\gamma z). \tag{9}$$

Such a solution can be regarded as the generalization of the simple mode given by (4), and so it will be called generalized mode of order K. A generalized mode of first order can be written in the form

$$\mathbf{E}_\mathrm{T} = \Big(\mathbf{d}(x, y) + \gamma z \mathbf{e}(x, y)\Big)\ \exp(-\gamma z), \tag{10}$$

where the notations $\mathbf{d} = \mathbf{d}_0$ and $\mathbf{e} = \mathbf{d}_1$ have been introduced. It can be easily proved that $\mathbf{e}$ must satisfy equation (6) and the boundary conditions (7)-(8), which explains the notation, and $\mathbf{d}$ must satisfy the equation

$$\mu \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{d}\right) - \operatorname{grad}\left(\frac{1}{\varepsilon}\operatorname{div}\varepsilon\mathbf{d}\right) + s^2\varepsilon\mu\mathbf{d} = \lambda(\mathbf{d} - 2\mathbf{e}) \tag{11}$$

and the boundary conditions

$$\mathbf{n} \times \mathbf{d} = 0, \tag{12}$$

$$\operatorname{div} \varepsilon \mathbf{d} = 0. \tag{13}$$

For a fixed eigenvalue λ_n and eigenfunction $\mathbf{e}_n$ of the b.v.p. (6)-(8) equations (11)-(13) describe an inhomogeneous b.v.p., the homogeneous version of which has $\mathbf{e}_n$ as solution. According to Fredholm's alternative the solubility of such an inhomogeneous problem can be examined with the solutions of the adjoint homogeneous problem, see e. g. (LADYZHENSKAIA and URAL'TSEVA, 1973). It can be shown that the adjoint problem of the eigenvalue problem (6)-(8) is defined by the equation

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mu g\right) - \varepsilon \operatorname{grad}\left(\frac{1}{\varepsilon}\operatorname{div}\mathbf{g}\right) + s^{*2}\varepsilon\mu\mathbf{g} = \nu\mathbf{g} \tag{14}$$

and the boundary conditions

$$\mathbf{n} \times \mathbf{g} = 0, \tag{15}$$

$$\operatorname{div}\varepsilon\mathbf{g} = 0, \tag{16}$$

where s^* is the complex conjugate of s. The eigenvalues of the two b.v.p. are in mutual correspondence through the equation

$$\nu_n = \lambda_n^*, \tag{17}$$

and the multiplicities of the corresponding eigenvalues are equal. The solutions of both b.v.p. can be regarded as elements of the $L^2(A)$ space formed by vector functions that are given by two complex components in the cross-section A. This Hilbert space has the following scalar product:

$$(\mathbf{u}, \mathbf{v}) = \iint_A (u_x v_x^* + u_y v_y^*)\, dx dy. \tag{18}$$

Let $\mathbf{e}_m$ and $\mathbf{g}_n$ denote eigenfunctions corresponding to the eigenvalues λ_m and ν_n , respectively. In this case

$$(\mathbf{e}_m, \mathbf{g}_n) = 0 \text{ if } \nu_n \neq \lambda_m^*. \tag{19}$$

The eigenfunctions $\mathbf{g}_n$ are related to the magnetic field of the simple modes, because the vector $\mathbf{H}_\mathrm{T}$ in (2) can always be given in the form

$$\mathbf{H}_\mathrm{T} = (\mathbf{k} \times \mathbf{g}_n^*) \exp(-\gamma_n z). \tag{20}$$

According to Fredholm's alternative the inhomogeneous b.v.p. (11)-(13) with a fixed eigenvalue λ_n and eigenfunction $\mathbf{e}_n$ has solutions if and only if in the sense of the scalar product (18) $\mathbf{e}_n$ is orthogonal to those eigenfunctions of the adjoint b.v.p. which correspond to the eigenvalue $\nu_n = \lambda_n^*$. Then because of (19), $\mathbf{e}_n$ is orthogonal to all the eigenfunctions of the adjoint b.v.p. In this paper an eigenfunction $\mathbf{e}_n$ will be called singular if it is orthogonal to all the eigenfunctions of the adjoint b.v.p., otherwise it will be called regular. If no singular eigenfunction belongs to an eigenvalue, the latter will be called regular, otherwise singular. A value of the complex frequency s will be called singular if a singular eigenvalue belongs to it, otherwise it will be called regular. It can be shown that both the complex frequency $s = 0$ and the eigenvalue $\lambda_n = 0$ are regular. It can also be shown that with the exception of these two cases a complex frequency s and a corresponding eigenfunction $\mathbf{e}_n$ are singular if and only if

$$\iint\limits_A \left(s^2 \varepsilon \left(e_{nx}^2 + e_{ny}^2 \right) + \frac{1}{\mu} \left(\frac{\partial e_{ny}}{\partial x} - \frac{\partial e_{nx}}{\partial y} \right)^2 \right) \mathrm{d}x \mathrm{d}y = 0. \tag{21}$$

In the case of a regular eigenvalue λ_n only the trivial solution $\mathbf{e} = 0$ of the b.v.p. (6)-(8) may figure in (10). If $\mathbf{e} = 0$, the b.v.p. (11)-(13) has the eigenfunction $\mathbf{e}_n$ as solution, and so the generalized mode of first order given by (10) is reduced to the simple mode (4). Thus the simple mode can be regarded as a trivial form of the generalized mode. The nontrivial form of the generalized mode exists only for singular eigenvalues. Similarly, a generalized mode of first order can be regarded as a trivial form of the generalized mode of second order, in which $\mathbf{d}_2 = 0$. It can be shown that the nontrivial form of (9) with $K = 2$ exists if and only if the solutions $\mathbf{d}$ of the b.v.p. (11)-(13) are also orthogonal to all the eigenfunctions of the adjoint b.v.p. This is a strongly degenerated case, which will not be treated in detail.

The transversal electric field of the generalized mode of first order is given by (10). The following formulae give the other components figuring in (1)-(2):

$$E_z = \frac{1}{\gamma\varepsilon} \left(\mathrm{div}\varepsilon\mathbf{d} + (1 + \gamma z)\mathrm{div}\varepsilon\mathbf{e} \right) \exp(-\gamma z), \tag{22}$$

$$H_z = -\frac{1}{s\mu} (\mathrm{curl}\, \mathbf{d} + \gamma z \mathrm{curl}\, \mathbf{e}) \mathbf{k} \exp(-\gamma z), \tag{23}$$

$$\mathbf{H}_{\mathrm{T}} = \frac{\mathbf{k}}{\gamma s} \times \left(\mathrm{curl} \frac{1}{\mu} \mathrm{curl}\, \mathbf{d} + s^2 \varepsilon \mathbf{d} + (1 + \gamma z)(\mathrm{curl} \frac{1}{\mu} \mathrm{curl}\, \mathbf{e} + s^2 \varepsilon \mathbf{e}) \right) \exp(-\gamma z). \tag{24}$$

In these formulae s is a singular complex frequency, $\mathbf{e}$ a singular eigenfunction and $\mathbf{d}$ a solution of the b.v.p. (11)-(13).

Characterization and Types of Singular Complex Frequencies and Eigenvalues

In the b.v.p. (6)-(8) the complex frequency s may be regarded as a complex variable. Then the eigenvalue problem defines a function $\lambda(s)$ having an infinity of branches. The branch points of this function represent the most frequent type of singular complex frequency and eigenvalue.

In order to see the details let us consider a branch $\lambda_n(s)$ of the function $\lambda(s)$ in the neighbourhood of a point s_0. If $\lambda_{n0} = \lambda_n(s_0)$ is not a singular eigenvalue, s_0 is a regular point of the function $\lambda_n(s)$, which is shown by the fact that the Taylor's series

$$\lambda_n(s) = \sum_{k=0}^{\infty} \lambda_{nk}(s^2 - s_0^2)^k \tag{25}$$

exists. Details of this series expansion are given by MAGOS (1979). If λ_{n0} is a simple eigenvalue, the coefficient λ_{n1} is given by

$$\lambda_{n1} = \frac{(\varepsilon\mu \mathbf{e}_{n0}, \mathbf{g}_{n0})}{(\mathbf{e}_{n0}, \mathbf{g}_{n0})}, \tag{26}$$

where $\mathbf{e}_{n0}$ and $\mathbf{g}_{n0}$ denote eigenfunctions corresponding to the eigenvalues λ_{n0} and $\nu_{n0} = \lambda_{n0}^*$, resp. If λ_{n0} is a singular simple eigenvalue, i. e. if

$$(\mathbf{e}_{n0}, \mathbf{g}_{n0}) = 0, \tag{27}$$

and if in addition $(\varepsilon\mu \mathbf{e}_{n0}, \mathbf{g}_{n0}) \neq 0$, the coefficient λ_{n1} and with it the series (25) do not exist. This means that s_0 is a singular point of the function $\lambda_n(s)$.

In order to show that if s_0 is a singular point, it is a branch point, let us try the series

$$\lambda_n(s) = \lambda_{n0} + \sum_{k=1}^{\infty} \Lambda_k \left(\sqrt{s^2 - s_0^2}\right)^k \tag{28}$$

as a generalization of the series (25). It can be shown that the series (28) exists even if (27) is true, and

$$\Lambda_1^2 = -2\lambda_{n0} \frac{(\varepsilon\mu \mathbf{e}_{n0}, \mathbf{g}_{n0})}{(\mathbf{d}_0, \mathbf{g}_{n0})}, \tag{29}$$

where $\mathbf{d}_0$ is a solution of the b.v.p. (11)-(13) with $\lambda = \lambda_{n0}$ and $\mathbf{e} = \mathbf{e}_{n0}$. As $\sqrt{s^2 - s_0^2}$ has two values, the series (28) gives two branches of the function $\lambda(s)$ in the neighbourhood of the point s_0, which is a common branch point of the two branches.

If the product $\varepsilon\mu$ is constant, especially if the dielectric is homogeneous, the function $\lambda(s)$ has no branch points and splits into independent branches of the form

$$\lambda_n(s) = c_n + \varepsilon\mu s^2, \tag{30}$$

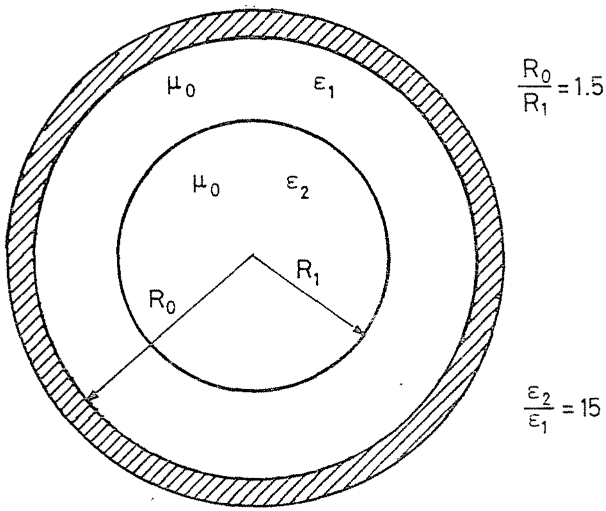

Fig. 1. Circular waveguide with two dielectrics.

where c_n is a positive real constant. If in this case $\mathbf{e}_n$ is an eigenfunction of the b.v.p. (6)-(8), then $\mathbf{g}_n = \varepsilon \mathbf{e}_n$ is an eigenfunction of the adjoint b.v.p. and so $(\mathbf{e}_n, \mathbf{g}_n) \neq 0$, which means that in this case no singular complex frequencies exist.

In other cases the singular complex frequencies can be determined with the aid of (21). An example is shown in (MAGOS, 1986) where the branch points of $\lambda(s)$, which are always singular complex frequencies, are determined for a rectangular waveguide with two dielectrics. At this arrangement no branch point can be found on the imaginary axis of the s-plane. But this is not true e. g. for the waveguide in *Fig. 1*, which was examined by CARLIN (1974) in another connection. In *Fig. 2* the eigenvalues are given along the imaginary axis of the s-plane as a function of the normed frequency $\overline{\omega} = \sqrt{\mu_0 \varepsilon_1} R_0 \omega$ for the two lowest modes depending on the cylindrical coordinate ϕ in the form $\cos \phi$. The real and imaginary parts of the normed eigenvalues $\overline{\lambda}_n = R_0^2 \lambda_n$ are represented by continuous and dotted lines, respectively. (In order to get these curves the eigenvalue problem (6)-(8) was reduced with the aid of the finite difference method to an algebraic eigenvalue problem and the latter was solved numerically with the aid of Wilkinson's method.) Obviously, the complex frequencies

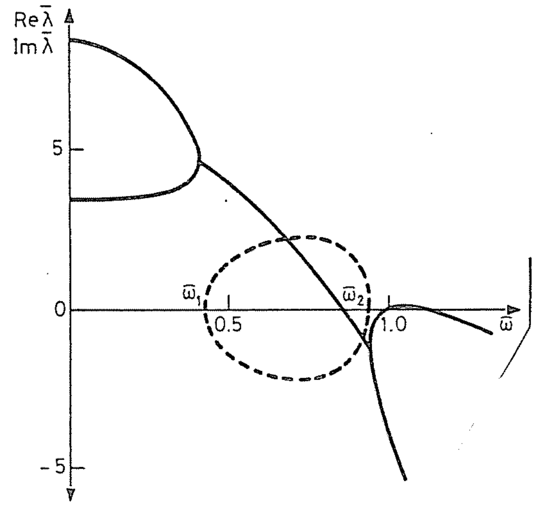

Fig. 2. Eigenvalues along the imaginary axis of the s-plane for the two lowest modes of the waveguide in *Fig. 1*.

$s_1 = j\omega_1$ and $s_2 = j\omega_2$ are common branch points of the represented branches of the function.

The previously described phenomena appear also in the case of two coupled lines if they are a model of coupled TM and TE modes. If the coupling is between the two shunt capacitors, the series impedance matrix and the shunt admittance matrix of the coupled lines can be expressed as

$$\mathbf{Z}(s) = \begin{pmatrix} sL_s + \frac{1}{sC_s} & 0 \\ 0 & sL_s \end{pmatrix}, \tag{31}$$

$$\mathbf{Y}(s) = \begin{pmatrix} sC_p & sC_{12} \\ sC_{12} & sC_p + \frac{1}{sL_p} \end{pmatrix}. \tag{32}$$

If $\lambda_i(s)$ is an eigenvalue of the matrix $\mathbf{Z}(s)\mathbf{Y}(s)$, similarly to the case of the waveguides, the frequency dependent propagation constant $\gamma_i(s)$ can be determined from the equation $\lambda_i = \gamma_i^2$. The eigenvalues λ_i can be expressed as

$$\lambda_{1,2} = As^2 + B \pm \sqrt{Cs^4 + Ds^2 + E}, \tag{33}$$

where

$$A = L_s C_p, \quad B = \frac{1}{2}\left(\frac{C_p}{C_s} - \frac{L_s}{C_p}\right), \quad C = L_s^2 C_{12}^2,$$

$$D = \frac{L_s C_{12}^2}{C_s}, \quad E = \frac{1}{4}\left(\frac{C_p}{C_s} - \frac{L_s}{C_p}\right)^2 .$$

The functions $\lambda_1(s)$ and $\lambda_2(s)$ can be regarded as two branches of a function $\lambda(s)$, which has four branch points in the s-plane. If $D^2 > 4CE$, i. e. if

$$| C_{12} | > | C_p - \frac{L_s}{L_p} C_s |, \tag{34}$$

these branch points are on the imaginary axis. At the real frequencies corresponding to the pure imaginary branch points the two propagation constants are equal and a generalized mode exists, i. e. terms in the form $z \exp(\pm\gamma z)$ figure in the formulae that describe the waves propagating along the two coupled lines.

If in the waveguide of *Fig. 1* the ratio $\varepsilon_2/\varepsilon_1$ is diminished starting from the value 15, the two branch points come closer to each other on the imaginary axis, and at the value $\varepsilon_2/\varepsilon_1 = 7.45$ they coincide in the point $s_0 = j1.1/(\sqrt{\mu_0\varepsilon_1}R_0)$, afterwards they quit the imaginary axis as a pair of branch points, symmetric to it. For the transitional value $\varepsilon_2/\varepsilon_1 = 7.45$ there is instead of two, only one singular complex frequency on the imaginary axis, the value s_0, and s_0 is not a branch point. If $s = s_0$, not only the denominator of the right side of (26) is zero, but the numerator also. In the neighbourhood of s_0 both branches of $\lambda(s)$ can be given by series in the form (25) with equal values of λ_{n0}. The two different values of the coefficient λ_{n1} cannot be calculated with the aid of (26), but in a more complicated way not detailed here.

It is theoretically possible that not only two, but three branches of $\lambda(s)$ have a common branch point at a value s_0. For these branches the series (28) does not exist, because on the right side of (29) the denominator is zero and the numerator is not, but the three branches have a common series consisting of the powers of $\sqrt[3]{s^2 - s_0^2}$. It can also be shown that a nontrivial form of the generalized mode of second order exists for the singular complex frequency s_0.

These results concerning a singular eigenvalue λ_{n0} which is simple and belongs to the singular complex frequency s_0 can be summarized as follows. Most frequently but not necessarily s_0 is a branch point of the function $\lambda(s)$, and most frequently but not necessarily it is a common branch point of only two branches. Whether the singular complex frequency s_0 is a branch point or not, two or rarely more branches of $\lambda(s)$ have the same value λ_{n0} in the point s_0, and there the same eigenfunction belongs to these branches.

Two branches of the function $\lambda(s)$ may have the same value λ_{n0} at a regular complex frequency s_0, too, but in this case the regular eigenfunctions belonging to the two branches are linearly independent, i. e. λ_{n0} is a double regular eigenvalue. It is also possible that three branches have the same value λ_{n0} in a point s_0, and one branch is regular in s_0, and the two others have a common branch point there. In this case λ_{n0} is similarly a double eigenvalue, and one of the eigenfunctions corresponding to it is singular, but the other linearly independent eigenfunction is regular.

In a more general way the following statements can be formulated. If only one branch of the function $\lambda(s)$ has a value λ_{n0} in a point s_0, the eigenvalue λ_{n0} is surely regular. If several branches have the same value λ_{n0} in the point s_0, let the multiplicities of the eigenvalues belonging to these branches be summed in the points of a small neighbourhood of s_0. If the neighbourhood is small enough, this sum has the same value N in all points, may be except s_0. If N equals the multiplicity M of the eigenvalue λ_{n0}, the eigenvalue is regular. If $M < N$, the eigenvalue is singular, and for s_0 and λ_{n0} a nontrivial form of the generalized mode (9) with $K = N - M$ exists, or several generalized modes exist for which the sum of the parameters K equals $N - M$.

Conclusions and Summary

If the product $\varepsilon\mu$ is not constant in the cross-section of a waveguide, for an infinity of discrete values of s the adjoint eigenvalue problems described by (6)-(8) and (14)-(16), resp. have an eigenfunction which is orthogonal to all the eigenfunctions of the other problem. These complex frequencies, eigenfunctions and the eigenvalues corresponding to them can be called singular. If λ_n is a singular eigenvalue of the b.v.p. (6)-(8), the eigenvalue $\nu = \lambda_n^*$ of the adjoint problem is also singular and inversely. If an eigenvalue is singular, a generalized mode in the form (9) belongs to it.

If ε and μ are constant in the cross-section, the eigenvalue problems (6)-(8) and (14)-(16) are of the same form, i. e. they are self-adjoint. Consequently the eigenfunctions form a complete set in $L^2(A)$, and in such a waveguide a modal expansion is possible, i. e. the field can be described as a superposition of modes, see e.g. (JONES, 1964). It will be shown in another paper that in waveguides with inhomogeneous dielectric the modal expansion is also possible at complex frequencies which are regular, i. e. not singular, because at regular frequencies the eigenfunctions of the b.v.p. (6)-(8) and (14)-(16), respectively, form complete biorthonormal sets in $L^2(A)$.

Obviously, at a singular complex frequency the eigenfunctions of the b.v.p. (6)-(8) cannot form a complete set of functions in $L^2(A)$, because a function, the singular eigenfunction of the adjoint b.v.p. is orthogonal to all the eigenfunctions in the set. Consequently at a singular frequency the modal expansion is not possible in the usual sense. On the other hand at singular complex frequencies and only at these complex frequencies nontrivial forms of generalized modes exist, and approaching a singular complex frequency two or more modes, which have the same field there, fuse into a generalized mode. This fact suggests the supposition that the modal expansion is also practicable at singular complex frequencies if generalized modes are used in addition to the simple modes. This supposition will be proved in another paper. As for certain waveguides singular complex frequencies may assume pure imaginary values, these theoretical results have importance for the harmonic time dependence, too.

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