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SEMI-MARKOV APPROACH FOR DETERMINING THE RELIABILITY PARAMETERS OF COMPLEX SYSTEMS

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Abstract

In the reliability investigation of complex systems it sometimes becomes necessary to deal with distributions of a general type. That mainly occurs in the case of system repair works. This problem may be solved by modelling the system with a semi-Markov model. In this paper, we derive closed-form expressions for the determination of reliability parameters, like availability and mean sojourn times. These results are the generalization of those given in the literature for the Markov case. This fact is shown in the paper, as well.

Introduction

In the last few years the author has participated in several research projects dealing with the reliability of complex electronic systems. As a result of these research efforts several program packages have been developed for modelling, analysis, and simulation of complex systems.

By these program packages a static and dynamic analysis can be done on systems which can be modelled with continuous-time Markov chains. It means that it is only possible to investigate such systems in which both the failure and their repair times are exponentially distributed.

The homogeneous Markov chains usually give a good model on the failure processes of electronic systems. However, in the case of non electronic (e.g. mechanical) systems, or of a more precise investigation of repaired electronic systems, it becomes necessary to deal with some different (general) time distributions.

The description of these systems requires a more general model than the Markov one. TOMKÓ (1984; 1986) has dealt with this problem for the case of machine interference using the inhomogeneous semi-Markov model. In this paper, the homogeneous semi-Markov model presented in (HOWARD, 1971) will be used to derive expressions on the mean sojourn times in a given class of states and thus on the reliability parameters of the system such as the availability, MTFF, MTTF, etc. In addition, the validity of these results in Markov models is shown.

Section 1 of the paper gives a concise review of semi-Markov processes and their basic notations. In Section 2 the steady state solution is determined for interval transition probabilities. The problem of sojourn times will be discussed in Part 3. In the last two Parts the reliability parameters will be determined and then it will be shown that the results of (BUZACOTT, 1970) are a special case in which all the distributions are of the exponential type.

The Semi-Markov Processes

The semi-Markov process considered in this paper is a stochastic process whose successive state occupancies are governed by transition probabilities of a Markov process, but whose stay in any state is described by an arbitrarily distributed random variable that depends on the state presently occupied and on the state to which the next transition will be made.

It is important to state that the semi-Markov process considered is homogeneous and has a finite state space.

Let p_{ij} denote the probability that a semi-Markov process that entered state i on its last transition will enter state j on its next transition. The transition probabilities p_{ij} must satisfy the same equations of the Markov case:

$$p_{ij} \geq 0, \qquad \sum_{i=1}^{N} p_{ij} = 1, \qquad i, j = 1, 2, \dots, N, \tag{1}$$

where N is the total number of states in the system. The state space will be denoted by $\mathcal{X} = \{X_1, X_2, \dots, X_N\}$.

The holding times τ_{ij} have the probability density functions $h_{ij}(t)$. We assume that the $\overline{\tau}_{ij}$ of all the holding time distributions are known and finite. Let the core function of the semi-Markov process be defined as

$$c_{ij}(t) = p_{ij} \cdot h_{ij}(t)\,. \tag{2}$$

For the sake of matrix representation, we introduce the box operator for denoting the congruent matrix multiplication (HOWARD, 1971). Thus, the core matrix is

$$\mathbf{C}(t) = \mathbf{P} \,\Box\, \mathbf{H}(t)\,, \tag{3}$$

where $\mathbf{C}(t)$, $\mathbf{P}$ and $\mathbf{H}(t)$ are defined by $c_{ij}(t)$, p_{ij} and $h_{ij}(t)$ as entries, respectively.

The waiting time τ_i of state i is the holding time of this state without a condition on the destination state, whose probability density function is

$$w_i(t) = \sum_{j=1}^{N} p_{ij} \cdot h_{ij}(t) \tag{4}$$

and whose mean is

$$\overline{\tau}_i = \sum_{j=1}^{N} p_{ij} \cdot \overline{\tau}_{ij} \,. \tag{5}$$

The Interval Transition Probabilities

The interval transition probability $\phi_{ij}(t)$ is the probability that the semi-Markov process will be in state j at time t, given that it entered state i at time zero. The fundamental integral equation of $\phi_{ij}(t)$ can be given as (HOWARD, 1971. *Eq. 11.3.1*):

$$\phi_{ij}(t) = \delta_{ij} W_i^c(t) + \sum_{k=1}^{N} p_{ik} \int_0^t h_{ik}(\tau) \phi_{kj}(t-\tau)\, d\tau \,, \tag{6}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

$$i, j = 1, 2, \ldots, N, \qquad t \geq 0 \,,$$

where $W_i^c(t)$ denotes the complementary probability distribution of τ_i, that is $P[\tau_i > t]$.

The same equation in a matrix form is:

$$\mathbf{\Phi}(t) = \mathbf{W}\ ^c(t) + \int_0^t \mathbf{C}\ (\tau) \mathbf{\Phi}(t-\tau) d\tau \,. \tag{7}$$

This matrix integral equation can be solved on any value of t by traditional numerical methods. However, the steady-state probabilities can be determined by using the Laplace transform analysis. The transformed interval transition probabilities can be given as (HOWARD, 1971, *Eq. 11.5.6)*:

$$\mathbf{\Phi}^*(s) = [\mathbf{I} - \mathbf{C}^*(s)]^{-1} \mathbf{W}^{c*}(s) \,. \tag{8}$$

The existence of this solution is provided since the embedded Markov chain associated with the semi-Markov process considered in this paper is always irreducible and has a finite state space, and is thus ergodic. Therefore, the semi-Markov process is ergodic, as well.

The limiting interval transition probabilities are

$$\mathbf{f} = \frac{1}{\overline{\tau}} \Pi \mathbf{M} \,, \tag{9}$$

where $\mathbf{f}=\{\phi_i\}$ is the limiting int. tr. pr. vector of the semi-Markov process, $\Pi=\{\pi_i\}$ is the limiting state pr. vector of the associated embedded Markov chain, $\mathbf{M}=\{\overline{\tau}_i\}$ is the diagonal matrix of the mean waiting times, and finally, $\overline{\tau}$ is the average time between transitions

$$\overline{\tau}=\sum_{j=1}^{N}\pi_j\overline{\tau}_j\,. \tag{10}$$

Sojourn Times in State Class

Let ν_{ij} be the total amount of time the system will spend in state j if it started in state i. The indicator function $\kappa_{ij}(t)$ is equal to 1 if the system is in state j at time t, 0 otherwise. Then

$$\nu_{ij}=\int_0^{\infty}\kappa_{ij}(t)dt \tag{11}$$

and the expected conditional sojourn time of state j, $\overline{\nu}_{ij}$, is

$$\overline{\nu}_{ij}=\int_0^{\infty}\overline{\kappa_{ij}(t)}dt=\int_0^{\infty}\phi_{ij}(t)dt=\int_0^{\infty}\phi_{ij}(t)e^{-st}dt|_{s=0}=\phi_{ij}^{*}(0)\,, \tag{12}$$

where $\phi_{ij}^{*}(0)$ is the transformed int.tr.pr. at $s=0$.

For further studies, let us partition the state space so that $\mathcal{R}=\{X_1,\dots,X_k\}\in\mathcal{X}$ contains the states belonging to a given state class, while $\mathcal{S}=\mathcal{X}\backslash\mathcal{R}=\{X_{k+1},\dots,X_N\}$, all the others. Thus, we can introduce the following partition on all the vectors and matrices concerning the state space:

$$a=\{a_R,a_S\}$$

and

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{RR} & \mathbf{A}_{RS}\\ \mathbf{A}_{SR} & \mathbf{A}_{SS}\end{bmatrix}. \tag{13}$$

Our task now is to determine the mean sojourn time $\overline{\nu}$ in state class $\mathcal{R}$ given that the process is started in some state $i\in\mathcal{R}$ at time $t=0$ with probability 1, that is

$$\sum_{i=1}^{k}\phi_i(0)=1\,.$$

Next we shall disregard the state transitions inside $\mathcal{S}$, and those from $\mathcal{S}$ to $\mathcal{R}$, thus $\mathbf{P}_{SR}=\mathbf{O}$ and $\mathbf{P}_{SS}=\mathbf{O}$, since we shall investigate the process until the first transition from $\mathcal{R}$ to $\mathcal{S}$.

According to equation (8)

$$\mathbf{\Phi}^*(s) = [\mathbf{I} - \mathbf{C}^*(s)]^{-1}\mathbf{W}^{c*}(s) = [\mathbf{I} - (\mathbf{P} \,\Box\, \mathbf{H}^*(s))]^{-1}\mathbf{W}^{c*}(s).$$

If we let $\overline{\mathbf{N}} = \{\overline{\nu}_{ij}\}$ and by taking into account (8) and the fact that $\mathbf{H}^*(0)=\mathbf{U}$, and

$$\lim_{s\to 0} \mathbf{W}^{c*}(s) = \mathbf{M} = \{\overline{\tau}_i\},$$

then

$$\overline{\mathbf{N}}_{RR} = [\mathbf{I}_{kk} - \mathbf{P}_{RR}]^{-1}\mathbf{M}_{RR}, \tag{14}$$

where $\overline{\mathbf{N}}_{RR}=\{\overline{\nu}_{ij}\}$, $i,j=1, \dots, k$, is the $k\times k$ matrix of the mean sojourn times of the states of $\mathcal{R}$ and $\mathbf{I}_{RR}$ is the $k\times k$ size identity matrix.

In the case of a given initial distribution vector on the state class $\mathcal{R}$, $\mathbf{f}_R(0) = \{\phi_1(0), \dots, \phi_k(0)\}$, the mean sojourn time of state class $\mathcal{R}$ can be given as:

$$\overline{\nu} = \mathbf{f}_R(0)\,\overline{\mathbf{N}}_{RR} l_k, \tag{15}$$

where l_k is the column vector of k ones.

Reliability Parameters

The problem of determining the reliability parameters, is thus reduced to the problem of choosing the appropriate initial distribution vector $\mathbf{f}_R(0)$ in (15).

From the definition of the *Mean Time to First Failure*, MTFF, it is obvious that if X_1 denotes the best state of the system then

$$\text{MTFF} = \{1, 0, \dots, 0\}\overline{\mathbf{N}}_{RR} l_k. \tag{16}$$

In the case of the *Mean Time To Failure*, MTTF, the initial distribution must be the normalized limiting one upon state class $\mathcal{R}$, thus

$$\text{MTTF} = \frac{\mathbf{f}_R}{\mathbf{f}_R \cdot l_R}\overline{\mathbf{N}}_{RR} l_k. \tag{17}$$

The determination of the *Mean Up Time*, MUT, needs a different treatment, since it is necessary to determine the probability distribution of the process on the state class $\mathcal{R}$ given that it is started in $\mathcal{S}$, and it has just entered any $j \in \mathcal{R}$ state. Thus, let ${}_e\phi_{ij}(t)$ be the probability that the process which is started in any $i \in \mathcal{S}$ at time 0, will leave the state class $\mathcal{S}$ at time t and will enter any $j \in \mathcal{R}$. The above event can occur if the process, starting in state $i \in \mathcal{S}$, enters some state $r \in \mathcal{S}$ in time τ, and then it transits to state $j \in \mathcal{R}$ in one step after a holding time $t-\tau$. These considerations can be summed in the following integral equation:

$$ {}_e\phi_{ij}(t) = \int_0^t \sum_{r=k+1}^{N} \phi_{ir}(\tau) p_{rj} h_{rj}(t-\tau) d\tau \,. \tag{18} $$

$$ j = 1, 2, \ldots, k, $$
$$ i = k+1, \ldots, N, \quad t \geq 0. $$

By transforming (18) and putting it into matrix form we have

$$ {}_e\phi^*_{SR}(s) = \phi_{SS}(s) [\mathbf{P}_{SR} \,\square\, \mathbf{H}_{SR}(s)] \,. \tag{19} $$

The limiting solution of (19) is

$$ {}_e\mathbf{\Phi}_{SR} = \lim_{s \to 0} s\, {}_e\mathbf{\Phi}^*_{SR}(s) = \lim_{s \to 0} s \mathbf{\Phi}^*_{SS}(s) \lim_{s \to 0} (\mathbf{P}_{SR} \square \mathbf{H}^*_{SR}(s)) \,, $$

$$ {}_e\mathbf{\Phi}_{SR} = \mathbf{\Phi}_{SS} \cdot \mathbf{P}_{SR} \,. \tag{20} $$

Since the process is assumed to be started in state class $\mathcal{S}$ with probability one, the unconditional probability vector can be given as

$$ {}_e\mathbf{f}_R = \frac{\mathbf{f}_S}{\mathbf{f}_S l_S} \mathbf{\Phi}_{SS} \cdot \mathbf{P}_{SR} = \mathbf{f}_S \cdot \mathbf{P}_{SR} \,, \tag{21} $$

where $\mathbf{f}_S = \{\phi_{k+1}, \ldots, \phi_N\}$.

By inserting the normalized form of (21) into (15)

$$ \mathrm{MUT} = \frac{\mathbf{f}_S \mathbf{P}_{SR}}{\mathbf{f}_S \mathbf{P}_{SR} l_k} \overline{\mathbf{N}}_{RR} l_k \,. \tag{22} $$

The problem of the *Mean Down Time*, MDT, is the same as that of MUT with the inverse interpretation of $\mathcal{R}$ and $\mathcal{S}$, thus

$$ \mathrm{MDT} = \frac{\mathbf{f}_{RS} \mathbf{P}_{RS}}{\mathbf{f}_R \mathbf{P}_{RS} l_{N-k}} \overline{\mathbf{N}}_{SS} l_{N-k} \tag{23} $$

and finally

$$ \mathrm{MCT} = \mathrm{MUT} + \mathrm{MDT} \,. \tag{24} $$

Back to the Markov model

In this Section, we shall prove that the results obtained are in correspondence with those obtained previously for Markov models. The fact that the Markov processes are a special class of the semi-Markov processes has been proved by HOWARD (1971) [Ch.12]. This paper will carry on the case to give the special form of *Eq. (15)* for the sojourn times of Markov processes. Hence, the continuous-time Markov process is an independent continuous-time semi-Markov process in which the waiting time in each state is exponentially distributed, possibly with a different exponential waiting time density function for each state. That is

$$w_i(t) = \lambda_i e^{-\lambda_i t}, \quad i = 1, \dots, N. \tag{26}$$

For the sake of convenience, we continue introducing the matrix formulation, so let $\mathbf{\Lambda} = \langle \lambda_i \rangle$ be an N by N diagonal matrix whose ith diagonal element is λ_i. Since the mean of a variable that is exponentially distributed with rate parameter λ is $1/\lambda$, the mean waiting time matrix is just the inverse of $\mathbf{\Lambda}$,

$$\mathbf{M} = \langle \overline{\tau}_i \rangle = \mathbf{\Lambda}^{-1} = \langle 1/\lambda_i \rangle. \tag{27}$$

The $\mathbf{\Lambda}$ and $\mathbf{M}$ matrices and the transition probability matrix thus provide a complete description of the continuous-time Markov process.

The interval transition probabilities $\phi_{ij}(t)$ of the continuous-time Markov process are the probability that the process occupies state j at time t given that it occupied state i at time zero. In finding $\phi_{ij}(t)$, we use result (8) and *Eq. (26)* to have

$$\begin{aligned} \mathbf{\Phi}^*(s) &= [\mathbf{I} - \mathbf{P} \,\square\, \mathbf{H}^*(s)]^{-1}\mathbf{W}^{c*}(s) \\ &= [\mathbf{I} - \mathbf{w}^*(s)\mathbf{P}]^{-1}\mathbf{W}^{c*}(s) \\ &= [(\mathbf{W}^{c*}(s))^{-1} - (\mathbf{W}^{c*}(s))^{-1}\mathbf{w}^*(s)\mathbf{P}]^{-1} \end{aligned} \tag{28}$$

Because of the exponential nature of the waiting times given in *Eq. (26)* and the Laplace transform of them, we have

$$\mathbf{w}^*(s) = \langle w_i^*(s) \rangle, \quad w_i^*(s) = \frac{\lambda_i}{s + \lambda_i}$$

and

$$\mathbf{W}^{c*}(s) = \langle W_i^{c*}(s) \rangle, \quad W_i^{c*}(s) = \frac{1}{s}[1 - w_i^*(s)] = \frac{1}{s + \lambda_i}.$$

Therefore *Eq. (28)* assumes the form

$$\mathbf{\Phi}^*(s) = [s\mathbf{I} + \mathbf{\Lambda}(\mathbf{I} - \mathbf{P})]^{-1}. \tag{29}$$

Now we define the *Transition Rate Matrix* **A** of the continuous-time Markov process by

$$\mathbf{A} = -\mathbf{\Lambda}(\mathbf{I} - \mathbf{P}) = \mathbf{\Lambda}(\mathbf{P} - \mathbf{I})\,. \tag{30}$$

If we insert the transition rate matrix **A** into *Eq. (29)*, we obtain a simple expression on the ITPs of the process

$$\mathbf{\Phi}^*(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\,. \tag{31}$$

The solution of *Eq. (31)* is

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)\,. \tag{32}$$

Finally, we define $\phi_i(t)$ as the probability that the Markov process occupies state i at time t and let $\mathbf{f}(t)$ be the row vector of state probabilities for all states, then

$$\mathbf{f}(t) = \mathbf{f}(0)\mathbf{\Phi}(t) = \mathbf{f}(0)\exp(\mathbf{A}t)\,, \tag{33}$$

where $\mathbf{f}(0)$ is the initial distribution of the process.

The above result is the well-known solution for the continuous-time Markov process (BUZACOTT, 1970), where the matrix **A** is equal to the infinitesimal operator of the Markov process.

We turn now to deal with the question of sojourn times. Let us suppose again that the state space has two disjunct sets of states, namely $\mathcal{R}$ and $\mathcal{S}$. Thus the transition rate matrix can be partitioned in the following way,

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}, \tag{34}$$

where the matrices **B**, **C**, **D** and **E** are of the size $k\times k$, $k\times(N-k)$, $(N-k)\times k$ and $(N-k)\times(N-k)$, respectively.

Turning back to *Eq. (15)*, we found that the mean sojourn time of state class $\mathcal{R}$ can be given as

$$\overline{\nu}_R = \mathbf{f}_R(0)\overline{\mathbf{N}}_{RR} l_k\,, \tag{35}$$

where $\mathbf{f}(0)$ is the initial probability distribution for state class $\mathcal{R}$, and l_k is the column vector of k ones. Furthermore the element $\overline{\nu}_{ij}$ of the matrix $\overline{\mathbf{N}}_{RR}$ gives the mean sojourn time in state j given that the process started in state $i \in \mathcal{R}$ until the process leaves this state class for the first time.

According to *Eq. (14)*, we have

$$\overline{\mathbf{N}}_{RR} = [\mathbf{I}_{kk} - \mathbf{P}_{RR}]^{-1}\,\mathbf{M}_{RR}\,. \tag{36}$$

In the Markov process case, it is obvious that

$$\mathbf{M}_{RR} = \langle \overline{\tau}_1, \dots, \overline{\tau}_k \rangle = \langle 1/\lambda_1, \dots, 1/\lambda_k \rangle = \mathbf{\Lambda}_{RR}^{-1} \tag{37}$$

and since this matrix is a diagonal one then

$$\begin{aligned} \overline{\mathbf{N}}_{RR} &= [\mathbf{I}_{kk} - \mathbf{P}_{RR}]^{-1} \mathbf{\Lambda}_{RR}^{-1} \\ &= [\mathbf{\Lambda}_{RR}(\mathbf{I}_{kk} - \mathbf{P}_{RR})]^{-1} = (-\mathbf{B})^{-1} \end{aligned} . \tag{38}$$

Therefore the mean sojourn time for the continuous-time Markov process in the set of states $\mathcal{R}$ can be simply given as

$$\overline{\nu}_R = \mathbf{f}_R(0)(-\mathbf{B})^{-1} l_k \tag{39}$$

which is the same result as that derived by BUZACOTT (1970) for these processes.

Using *Eq. (39)*, it becomes easy to show that the determination of the reliability parameters in the Markov model can be executed by the same formulas of the semi-Markov case since this is just a special case of it.

This fact is of particular interest as one can construct in this way a complete and general algorithm on the investigations of general systems including the Markov as well as the semi-Markov models.

Conclusion

We derive in this paper a semi-Markov approach for determining the steady state reliability parameters of systems in which the operation (failure) and repair times may have different (not necessarily exponential) distributions. One can by this approach give explicit expressions for the reliability parameters in terms of only matrix and vector quantities. Thus we obtained a useful way for developing a software package. Furthermore, it is shown that the Markov model is, of course, just a special case of the general semi-Markov model and thus both models can be investigated with the same apparatus.

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