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# INSTRUMENTATION PHASE LOCKED LOOP

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## Abstract

Near to optimal design of the high-precision phase-locked loop (PLL), which is intended for use in phase-sensitive measuring instruments is considered in this paper. Minimisation of the steady-state phase error and the transient response time is established as a guide post for design to meet accuracy requirements for measurement applications. Optimization of the parameters of the second order and the third order linearized PLLs is carried out. The conclusion is drawn that the third order PLL exceeds the second order one not only by the far better steady-state accuracy, but also by the faster transient response. The computer aided simulation is used for studying the nonlinear effects in the real nonlinear PLL. It is concluded that the suboptimal third order PLL is able to satisfy the requirements for the phase error within the limits of $\pm 0.1°$.

*Keywords:* phase-locked loop, phase accuracy, steady-state error, transient response time, phase-sensitive measurements.

## Introduction

All the phase-sensitive measuring instruments, such as lock-in amplifiers, gain/phase meters, vector or phase angle voltmeters, and complex spectrum analyzers, require a well-defined reference signal for their normal operation (MEADE, 1983). To extract a normalized reference signal from the noisy and unstable input signal the selective frequency and phase synchronization systems based on the phase-locked loops (PLL) are proposed for use in measuring instruments (MIN et al., 1988).

In spite of more than 30 years of experience in using PLLs in communication and control (LINDSAY, 1972) the conventional PLLs are designed not well enough to meet the accuracy requirements for measurement applications. As it became evident, there are no sufficiently simple and effective design methods for the precision or instrumentation PLL workout. Because of this fact the need for elaboration of design methods has arisen. Solutions of some design problems, e.g. the methods for minimisation of the steady-state phase error and the phase settling time in the third order

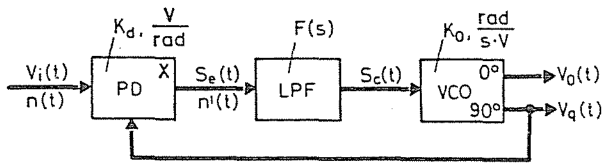


*Fig. 1.* Block diagram of the PLL

PLL are proposed in this paper on the basis of further development of the author's earlier works (MIN, 1985, 1987).

## Operating Principle and Signal Model

The PLL in *Fig. 1* contains a multiplying type phase detector PD with the gain $K_d$, V/rad, a low pass filter LPF with the transfer function $F(s)$, and a voltage controlled oscillator VCO with the gain $K_0$, rad/sV (GARDNER, 1979). The VCO has the in-phase (0°) and quadrature (90°) outputs. The latter one is used to establish the negative feed-back path from the VCO output to the PD. The input signal

$$V_i(t) = A_i \sin\Big(\omega_i t + \int_0^t \Omega_i(t)\mathrm{d}t + \Phi_i(t)\Big) \tag{1}$$

has the stable frequency component $\omega_i$ and the variable component $\Omega_i(t)$. The variable phase of the input signal (1) is denoted by $\Phi_i(t)$. In general, the input signal (1) is accompanied by the additive noise $n(t)$. The quadrature feedback signal

$$V_q(t) = \cos\Big(\omega_i t + K_0 \int_0^t S_c(t)\mathrm{d}t + \psi_0\Big) \tag{2}$$

has the normalized amplitude, and due to the negative feedback its frequency difference from the input frequency will be fully compensated by the controllable component

$$\Omega_0(t) = K_0 \cdot S_c(t)\,. \tag{3}$$

Only the minimal phase error $\Theta_e(t)$ may be required to form the sufficient error signal for generating the control voltage $S_c(t)$ at the input of the VCO to keep the PLL in lock, see *Eqs.* (2) and (3), and *Fig. 1.*

The error signal $S_e(t)$ and the converted noise $n'(t)$ at the PD output are the products of multiplication of signals at the PLL input by the feedback signal (2):

$$S_e(t) = (A_i/2)K_d \sin\Theta_e(t) + (A_i/2)K_d \sin\phi(t)\,, \tag{4}$$

where the phase error $\Theta_e(t)$ is equal to

$$\Theta_e(t) = \int_0^t \Big(\Omega_i(t) - \Omega_0(t)\Big)\mathrm{d}t + \Phi_i(t) - \psi_0 \tag{5}$$

and the current phase $\phi(t)$ of the high frequency disturbing component may be described as follows:

$$\phi(t) = 2\omega_i t + \int_0^t \Big(\Omega_0(t) + \Omega_i(t)\Big)\mathrm{d}t + \Phi_i(t) + \psi_0\,. \tag{6}$$

The control signal $S_c(t)$ is formed from the error signal (4) by suppressing the disturbing component and converted noise $n'(t)$ in the LPF. The control signal causes the frequency shift $\Omega_0$, which will compensate the phase error, (see *Eqs.* (3) and (5)). The corresponding signal model of the PLL is represented in *Fig. 2*.

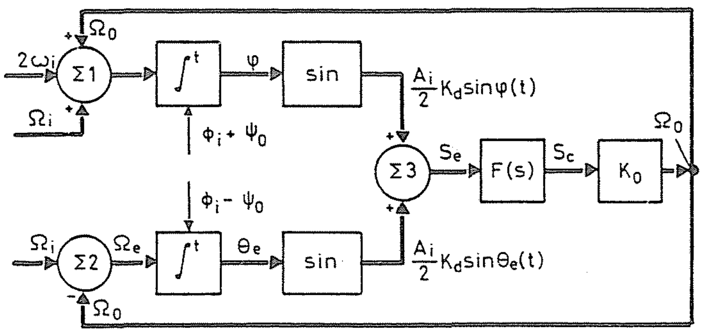


*Fig. 2.* Signal model of the PLL

## Linear model

Since the very small phase error $\Theta_e$ is able to keep the system in lock the sine wave nonlinearity of the PD due to *Eq.* (4) may be removed

($\sin\Theta_e \approx \Theta_e$) in the signal model in *Fig. 2*. Removing also the high frequency disturbing component of the error signal with the current phase $\phi(t)$, assuming the complete suppression of it in the LPF, (see *Eqs.* (4) and (6)), the simple linear model of the PLL is obtained. The performance of the PLL may be described now by the aid of transfer functions and corresponding Bode diagrams, (see *Fig. 3*), using the Laplace transform.

The linearized PLL without LPF (see *Fig. 3*) essentially contains an ideal integrator and can be described by the transfer function

$$L(s) = \frac{(A_i/2)K_d K_0}{s} = \frac{1}{T_1 s}\,. \tag{7}$$

The LPF is given by the transfer function $F(s)$. The corresponding Bode diagrams are given in *Fig. 3*.

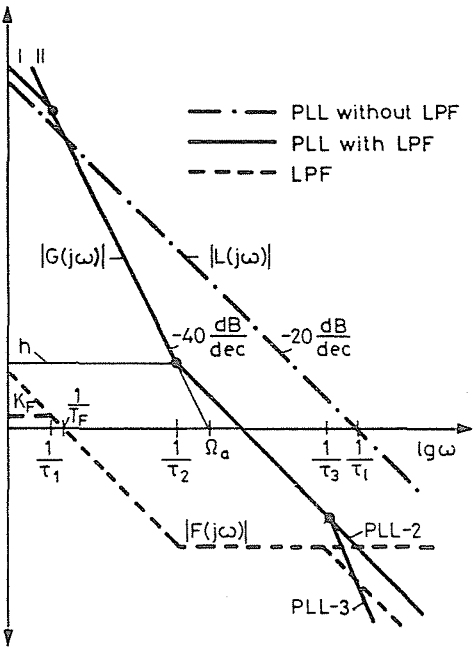


*Fig. 3.* Open loop Bode diagrams of the linearized PLL and LPF

The 1st and 2nd order type I, and, in particular, type II (containing an ideal integrator) LPF are used in precision PLLs (MIN, 1987) to get smaller steady-state phase error. Therefore the following 1st and 2nd order type II LPFs are proposed:

$$F_1(s) = \frac{\tau_2 s + 1}{T_F s}\,, \tag{8}$$

$$F_2(s) = \frac{\tau_2 s + 1}{T_F s(\tau_3 s + 1)}. \tag{9}$$

As a result, the following 2nd and 3rd order (PLL–2 and PLL–3 in *Fig. 3*) open loop transfer functions $G_2(s)$ and $G_3(s)$ are obtained for the linearized PLL with LPF:

$$G_2(s) = L(s)F_1(s) = \frac{\tau_2 s + 1}{T_a^2 s^2}, \tag{10}$$

$$G_3(s) = L(s)F_2(s) = \frac{\tau_2 s + 1}{T_a^2 s^2(\tau_3 s + 1)}, \tag{11}$$

where

$$T_a = \sqrt{T_1 \cdot T_F} \qquad \text{and} \qquad \Omega_a = 1/T_a. \tag{12}$$

The parameters

$$n = \frac{\tau_2}{\tau_3} \qquad \text{and} \qquad h = G(j\omega_2) = \left(\frac{\tau_2}{T_a}\right)^2, \tag{13}$$

where $\omega_2 = 1/\tau_2$, are introduced for further considerations. The closed loop transfer function $H(s)$ for the phase error $\Theta_e(s)$ can be found on the basis of open loop ones, see *Eqs.* (10) and (11):

$$H(s) = \frac{\Theta_e(s)}{\Theta_i(s)} = \frac{1}{1 + G(s)}, \tag{14}$$

where the Laplace transform of the variable component of the current input phase, (see *Eq.* (1)), is the following:

$$\Theta_i(s) = \frac{\Omega_i(s)}{s} + \Phi_i(s). \tag{15}$$

## Suppression of Disturbances

Let us take into account here only the higher frequency disturbances, which arise inherently in the PLL due to converting the input signal according to *Eq.* (1) in the phase detector PD, (see *Eqs.* (4) and (6)). As a result, the steady-state phase error $\Theta_e(t)$ appears, the value of which depends on the filtering properties of the low pass filter LPF, see *Fig. 2*.

Assuming that only the small phase errors occur, we can use the linearized model, having $\sin\Theta_e \approx \Theta_e$, and the time domain expression for the disturbing signal is the following:

$$d(t) = (A_i/2)K_d \sin\phi(t) \tag{16}$$

on the basis of *Eq.* (4) and *Fig. 2*.

Without losing generality, let us presume now that there is no frequency and phase difference between the input and output signals of the PLL at the starting moment. It means that $\Omega_i(t) = 0$ and $\Phi_i(t) = 0$, and therefore the variable component $\Omega_0(t)$ of the output frequency, (see *Eqs.* (2) and (3)), is induced only by the steady-state phase error $\Theta_e(t)$, which is caused by the above mentioned disturbances. As it follows, (see *Eqs.* (5) and (6)), the steady-state error is

$$\Theta_e(t) = -\int_0^t \Omega_0(t)\mathrm{d}t\,, \tag{17}$$

and the disturbing signal $d(t)$ can be obtained in the following form:

$$d(t) = (A_i/2)K_d\sin\Big(2\omega_i t + \Theta_e(t) + \psi_0\Big)\,. \tag{18}$$

For a more robust approximation, let us assume that the small phase error $\Theta_e(t)$ is negligible in *Eq.* (18). It means that the feedback from the output to the adder $\Sigma 1$ in the signal model in *Fig. 2* is too weak to have a significant role, and the following transfer function is valid for describing the action of disturbances:

$$D(s) = \frac{\Theta_e(s)}{d(s)} = (2/A_iK_d)\cdot\frac{G(s)}{1+G(s)}\,. \tag{19}$$

To get the expression for the phase due to the harmonic disturbance signal removing $\Theta_e(t)$ as negligible in *Eq.* (18), we can insert $s \rightarrow j\omega$ into *Eq.* (19) and get:

$$\Theta_e(j\omega) = D(j\omega)d(j\omega)\,. \tag{20}$$

Using the gain and phase frequency response functions $K(\omega)$ and the $\Phi(\omega)$ of the closed loop PLL, which are obtained from the complex transfer function $D(j\omega)$, we can define the first-order harmonic of the phase error:

$$\Theta_{e1}(\omega t) \approx K(2\omega_i)\sin\Big(2\omega_i t + \psi_0 + \Phi(2\omega_i)\Big)\,. \tag{21}$$

Inserting now expression (21) into *Eq.* (18) and repeating the procedure, we can see that the steady-state phase error contains solely a negligible amount of higher harmonics, but what is more important – it contains the significant nonzero stationary value:

$$\overline{\Theta}_e \approx (1/2)K(2\omega_i)\sin\Phi(2\omega_i)\,. \tag{22}$$

At higher frequencies where $(2\omega_i)T_a \gg 1$ (see *Fig. 3*), asymptotic approximations for frequency responses are valid. For the 3rd and 2nd order PLLs the corresponding gain frequency responses have the forms:

$$K_3(\omega) \approx \frac{n}{(\omega T_a)^2} \qquad \text{and} \qquad K_2(\omega) \approx \frac{\sqrt{h}}{\omega T_a}\,. \tag{23}$$

Inserting the asymptotic expressions (23) into the *Eqs.* (21) and (22), the simple rules will be obtained for calculating the components of the steady-state phase error, for which $\omega = 2\omega_i$.

For calculating the stationary value of the phase error (see *Eq.* (22)) it is important to emphasize that the asymptotic values for the phase responses in the 2nd and 3rd order PLLs are

$$\Phi_2(\infty) = -\pi/2 \qquad \text{and} \qquad \Phi_3(\infty) = -\pi\,, \tag{24}$$

respectively.

Using *Eq.* (24) it may be concluded that the stationary value of the steady-state phase error can be removed almost perfectly in the 3rd order PLL because of $\sin(-\pi) = 0$, but in the 2nd order PLL the stationary value has the maximum level because of $\sin(-\pi/2) = -1$, see *Eq.* (22).

## Minimisation of Settling Time

First let us consider the transient response of the linearized system (MIN, 1987) on the basis of the characteristic equation

$$\left(\frac{\sqrt{h}}{n}\right)T_a^3 s^3 + T_a^2 s^2 + \left(\sqrt{h}\right)T_a s + 1 = 0 \tag{25}$$

of the closed-loop transfer function $H(s)$ of the 3rd order PLL, see *Eqs.* (13) and (14). The phase step $\Phi_i(\tau) = \text{const.}$, and the frequency step $\Omega_i(\tau) = \text{const.}$ are considered as the input actions, whereas $\tau = t/T_a$. Our task is to find the optimal values of the parameters $h$ and $n$, see *Eq.* (13), to get the fastest damping of transient responses, and to minimize in this way the settling time $\tau_\varepsilon = t_\varepsilon/T_a$, where $\varepsilon$ is the value of the allowable transient response error.

Under the above-mentioned conditions (MIN, 1985, 1987), the following results are obtained for the maximum values of the real roots, or for the real parts of the complex conjugate pairs of roots of the characteristic *Eq.* (25), see *Figs. 4* and *5*:

1. For $1<n\leq 5$ (interval $AB$ in *Figs. 4* and *5*),

$$\text{opt } h_2 = \frac{4n^2}{(n+1)^2}, \tag{26}$$

and the coefficients of roots

$$s_1 = -a/T_a\,, \qquad s_{2,3} = (-b \pm j\beta)/T_a$$

have the following values:

$$a = 1\,, \qquad b = \frac{(n-1)}{4} \qquad \text{and} \qquad \beta = \frac{\sqrt{-n^2+10n+7}}{4}. \tag{27}$$

The natural angular frequency and the damping factor of roots $s_{2,3}$ are

$$\omega_n = \frac{1}{T_a}\cdot\sqrt{\frac{n+1}{2}}, \qquad \zeta = \frac{n-1}{2\sqrt{2(n+1)}}. \tag{28}$$

2. For $5 < n < 9$ (interval $BC$ in *Figs. 4* and *5*),

$$\text{opt } h_1 = \frac{2n^2}{9n-27}, \tag{29}$$

and the coefficients of roots ($a = b$) have the values

$$a = b = \sqrt{\frac{n-3}{2}}, \qquad \beta = \sqrt{\frac{9-n}{2}} \tag{30}$$

and

$$\omega_n = \frac{\sqrt{3}}{T_a}, \qquad \zeta = \sqrt{\frac{n-3}{6}}. \tag{31}$$

3. For $n = 9$ (point $C$ in *Figs. 4* and *5*),

$$\text{opt } h_3 = 3\,, \qquad \beta = 0\,, \qquad \zeta = 1\,, \tag{32}$$

and all the three roots are real and equal

$$s_1 = s_2 = s_3 = -\frac{\sqrt{3}}{T_a}. \tag{33}$$

It is remarkable that *Eq.* (33) gives the greatest possible value for the smallest root.

4. For $9<n<\infty$ (intervals $CD$ and $CEF$ in *Fig. 4*, and interval $CD$ in *Fig. 5*) two of the three real roots will coincide, and therefore, also two

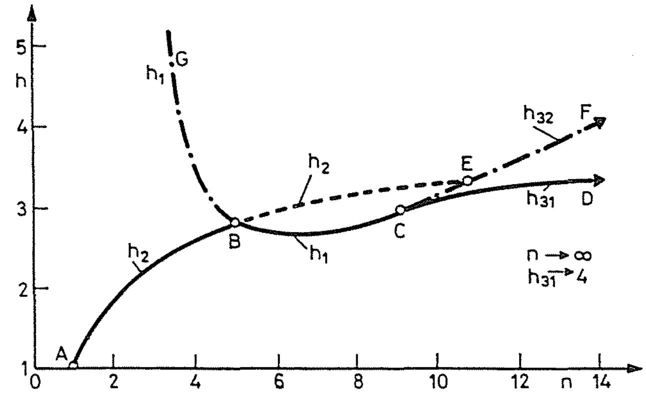


*Fig. 4.* Optimal values of the parameter $h$ versus parameter $n$ in the linearized PLL

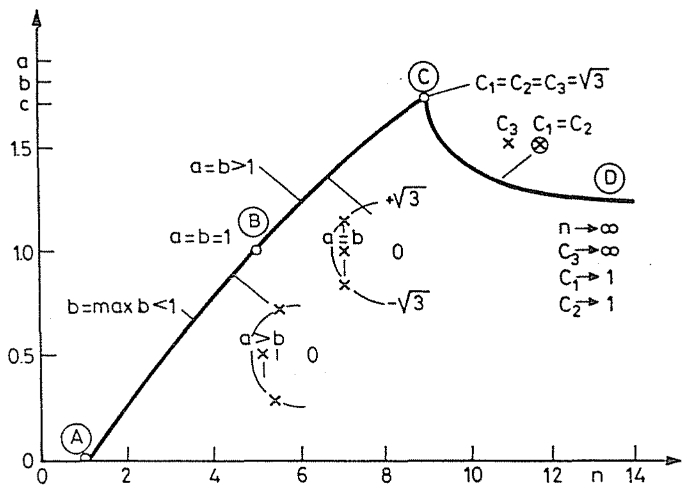


*Fig. 5.* Optimal values of roots of the linearized PLL transfer function ($h$ = opt $h$) versus parameter $n$

of the three coefficients, namely $c_1 = c_2$ (interval $CD$ ) or $c_2 = c_3$ (interval $CEF$), are equal. Only the case of interval $CD$ (parameter $h_{31}$) satisfies the optimality conditions, and equations for the parameters and coefficients are very complicated.

5. For $n \to \infty$ (point $D$ in *Figs. 4* and *5*),

$$\text{opt } h_{31} = 4\,, \tag{34}$$

and we get the conventional 2nd order PLL with real and equal roots

$$s_1 = s_2 = -\frac{1}{T_a}\,. \tag{35}$$

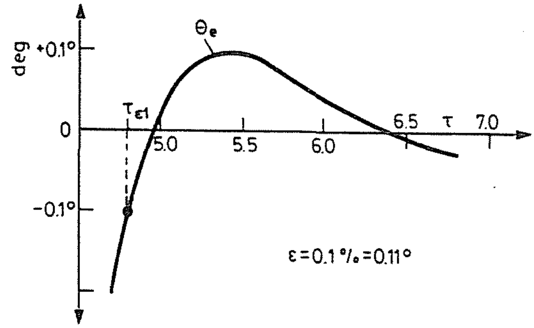


*Fig. 6.* Final part of the phase error transient response to the input phase step $\Phi_i = 2$ rad in the linearized optimal 3rd order PLL ($n=6.3$, $h=2.67$), $\tau = t/T_a$

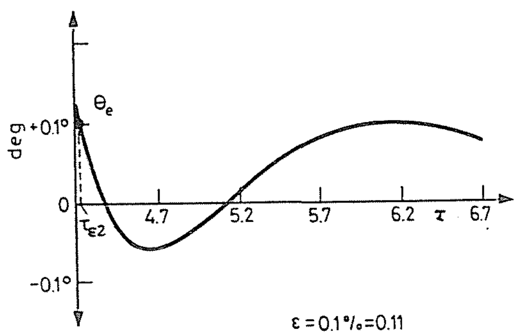


*Fig. 7.* Final part of the phase error transient response to the input frequency step $\Omega_i = 2/T_a$ in the linearized optimal 3rd order PLL ($n=6.1$, $h=2.67$)

To find the optimal values for the parameter $n$, let us observe the final parts of the optimal transient responses in *Fig. 6* (response to the input phase step) and in *Fig. 7* (response to the input frequency step).

We have to determine the minimal values for the parameter $n$, which guarantees the value of the amplitude, closely equal to the permissible error $\varepsilon$, of the first oscillation after the settling time $\tau_\varepsilon = t/T_a$. Mathematically it can be found by taking the derivatives of the transient response function with respect to the time $\tau = t/T_a$, and setting this derivative equal to zero. The results are given in *Fig. 8.* Minimal settling times $\tau_\varepsilon$ versus permissible error $\varepsilon$ in the optimal linearized 3rd order PLL are given in *Fig. 9.* Computed settling times $\tau_\varepsilon$ versus input frequency step in the third order nonlinear PLL are given in *Fig. 10.* The curves in *Fig. 10* show that the optimized 3rd order PLL exceeds the 2nd order one up to two times, if the relative frequency step $\Omega_i \cdot T_a < 2$.

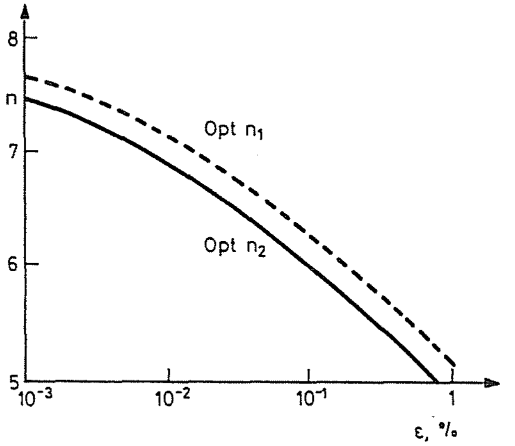


*Fig. 8.* Optimal values of the parameter $n$ (opt $n_1$ for the input phase step, and opt $n_2$ for the input frequency step) versus allowable settling error $\varepsilon$

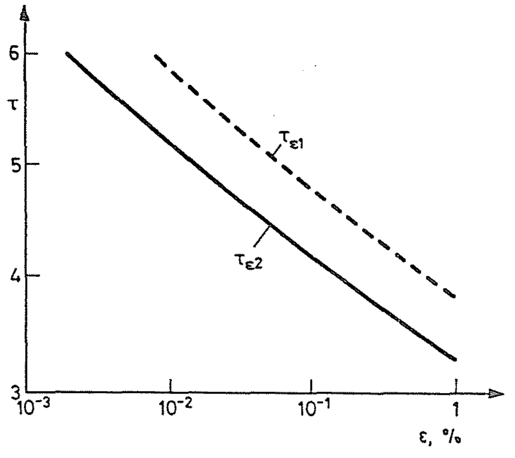


*Fig. 9.* Minimal settling times ($\tau_{\varepsilon 1}$ for the input phase step, and $\tau_{\varepsilon 2}$ for the input frequency step) versus allowable settling error $\varepsilon$

## Conclusions

The development of methods for minimisation of the steady-state error and the settling time enables us to design the suboptimal third order PLL, which is able to meet the requirements for measurement applications. Because of the far better suppression of disturbances, (see *Eqs.* (23) and (24)) and thanks to the 1.5 to 2.0 times faster transient response (see *Fig. 10*),

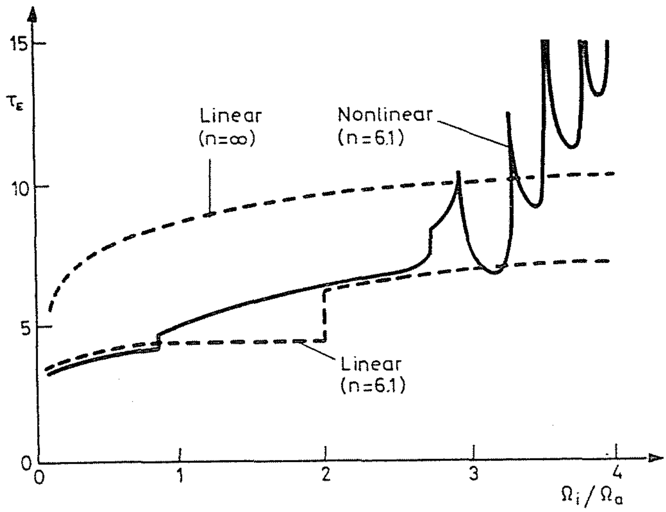


*Fig. 10.* Settling times $\tau_\varepsilon = t_\varepsilon / T_a$ of the phase error transient response versus input frequency step $\Omega_i \cdot T_a$ in the optimal nonlinear PLL (allowable settling error $\varepsilon = 0.1°$)

the third order PLL greatly surpasses the performance of the conventional second order one. It can be stated that the high-precision instrumentation PLL may be built only on the basis of the optimal third order loop.

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