LOW FREQUENCY FIELD OF AN EARTHING CONDUCTOR BY VARIATIONAL METHOD

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Received June 1, 1987
Presented by Prof. Dr. István Vágó

Abstract

A variational method using global approximation is extended to the analysis of the low-frequency field of exterior boundary value problems. The method is presented through the example of computing the current distribution of an earthing conductor. Using a $\mathbf{T}-\Omega$ approach, an approximate solution of Helmholtz's equation is generated with the prescribed Dirichlet and Neumann boundary conditions, and also the approximate conditions at infinity are satisfied. The boundary surfaces of the region examined are assumed to be described or approximated by piecewise analytical functions. The satisfacion of the prescribed boundary conditions is ensured by means of **R**-functions.

Introduction

Electromagnetic field problems frequently lead to the examination of unbounded regions. Integral equation methods are well suited to problems of this kind [1], [2].

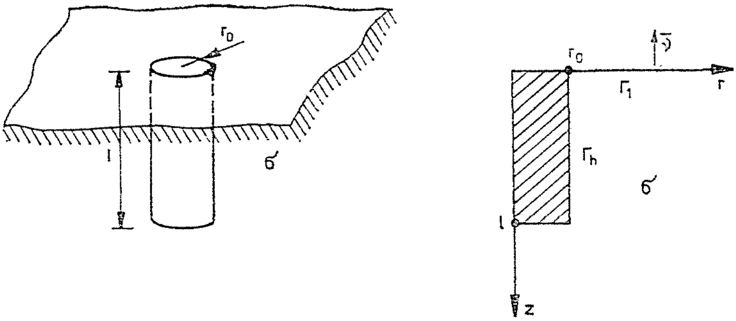
Since variational methods are primarily used to investigate closed regions, some authors employ a hybrid method combining the variational approach with the boundary element method to calculate the electromagnetic field in open regions [3], [4].

The development of the finite element and global variational methods, however, brought about a continuous effort to make the variational approach capable of treating regions with open boundaries. One method used in conjunction with the finite element method involves the introduction of so-called infinite elements [5], [6], [7]. Other authors employ ballooning techniques to solve differential equations in open regions [8], [9]. One possibility to use global approximation in conjunction with variational methods is offered by the use of R-functions. Reference [10] introduced a method for generating an approximate solution to Laplace equation in unbounded regions.

The present paper extends the global variational method to calculating low-frequency electromagnetic fields in open regions. The method is presented through the example of computing the current distribution of an earthing conductor. The problem is axi-symmetric and the potentials $\mathbf{T}-\Omega$ are introduced for the solution [11], [12], [13]. The solution of the Helmholtz equation is reduced by means of variational calculus to an extremum problem. The minimizing function is approximated by a function series in accordance with Ritz's method. The satisfaction of the prescribed Dirichlet and Neumann boundary conditions as well as of the conditions at infinity is ensured by the employment of **R**-functions [14], [15].

Differential equation and boundary conditions

Consider the following problem. An ideal cylindrical conductor of radius r_0 and length l is immersed in a medium of conductivity σ as shown in Fig. 1a. The sinusoidal current of the earthing conductor with peak value I_0 leaves the conductor over its surface. Let us examine the distribution of current density in the conducting medium in case of low-frequency excitation with the effect of the eddy current in the medium also taken into account. Let us determine the earthing impedance of the medium.



a) b)

Fig. 1. a: The arrangement of the earthing conductor *b*: Planar section of the problem

In cylindrical coordinates, a two-dimensional problem is arrived at in view of the axi-symmetry of layout. In the planar section shown in Fig. 1b, Γ_h denotes the surface of the ideal conductor. Current density enters the medium examined perpendicular to this surface. Γ_1 denotes the boundary of the conducting medium. Current density is tangential along this surface. The symmetry axis of the layout is at $r=0$. $\mathbf{v}$ is the outer normal of the region examined.

The electromagnetic field ensuing in the conducting medium is described by Maxwell equations [16]. Displacement currents are neglected, and the examination is restricted to the field in the medium of conductivity σ. Using the potentials $\mathbf{A}-\Phi$, the electromagnetic field can be described by a two-component vector potential $\mathbf{A}$ since the electric field has r and z components in the conducting medium whereas the magnetic field is φ-directed [17]. However, using the potentials $\mathbf{T}-\Omega$, the problem can be solved with the aid of a single component vector potential, too [11], [12], [13]. Since

$$\operatorname{div} \mathbf{J} = 0 \tag{1}$$

in the region examined, the current density can be written as the **curl** of a φ-directed vector potential $(\mathbf{T}=T\mathbf{e}_\varphi)$:

$$\mathbf{J} = \mathbf{curl}\,\mathbf{T}. \tag{2}$$

According to Ampère's law:

$$\mathbf{curl}\,\mathbf{H} = \mathbf{J} = \mathbf{curl}\,\mathbf{T}, \tag{3}$$

so the magnetic field intensity is

$$\mathbf{H} = \mathbf{T} - \mathbf{grad}\,\Omega, \tag{4}$$

where Ω is the magnetic scalar potential. Faraday's law yields the following for the potentials:

$$\mathbf{curl}\left(\frac{1}{\sigma}\,\mathbf{curl}\,\mathbf{T}\right) = -j\mu\omega(\mathbf{T} - \mathbf{grad}\,\Omega). \tag{5}$$

In case of homogeneous, isotropic medium, (5) is as follows:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{T} + j\mu\omega\sigma\,\mathbf{T} = j\omega\mu\sigma\,\mathbf{grad}\,\Omega. \tag{6}$$

The fact that the magnetic field is free of sources, yields the following for the scalar potential:

$$\operatorname{div}\mathbf{grad}\,\Omega = \operatorname{div}\mathbf{T}. \tag{7}$$

The divergence of the electric vector potential $\mathbf{T}$ being zero, the scalar potential Ω must obey Laplace's equation. In view of the axi-symmetry of the problem and making use of the fact that the magnetic field is φ-directed, (7) and (4) imply the solution Ω=constant in the region examined:

$$\Omega = \text{const}, \quad \mathbf{grad}\,\Omega = 0 \tag{8}$$

and the magnetic field from (4) is

$$\mathbf{H} = \mathbf{T}. \tag{9}$$

In view of (6), the vector potential $\mathbf{T}$ can be obtained as the solution of the homogeneous Helmholtz equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{T} + j\mu\omega\sigma\,\mathbf{T} = \mathbf{0}. \tag{10}$$

The vector potential obeying the differential equation (10) must satisfy the following boundary conditions.

Along the surface Γ_h of the studied region where the current density enters perpendicularly, the vector potential $\mathbf{T}$ satisfies the homogeneous Neumann boundary condition:

$$(\mathbf{curl}\,\mathbf{T}) \times |\mathbf{v}|_{\Gamma_h} = \mathbf{0}. \tag{11}$$

Along the boundary Γ_1 of the conducting medium and, in view of axi-symmetry, on the z-axis (at $r=0$), too, the current density is tangential:

$$(\mathbf{curl}\,\mathbf{T})\,\mathbf{v}|_{\Gamma_1} = 0, \quad (\mathbf{curl}\,\mathbf{T})\,\mathbf{v}|_{r=0} = 0. \tag{12}$$

Taking into account that $\mathbf{v}=-\mathbf{e}_z$ on the surface Γ_1, and that

$$\mathbf{curl\,T} = -\frac{\partial T}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial (rT)}{\partial r}\mathbf{e}_z, \tag{13}$$

the following Dirichlet boundary condition is obtained on the boundary Γ_1 of the conducting medium:

$$rT|_{\Gamma_1} = rT(z=0, r) = \text{const.} \tag{14}$$

In view of axi-symmetry, the vector potential assumes the value zero at $r=0$:

$$\mathbf{T}|_{r=0} = \mathbf{0}. \tag{15}$$

The value of the constant in eq. (14) can be determined from the condition that the value of the current entering the medium along Γ_h is I_0:

$$I_0 = \int_a (\mathbf{curl\,T})\,\mathrm{d}\mathbf{a} = \oint_l \mathbf{T}\,\mathrm{d}\mathbf{l} = T(z=0, r)\,2r\pi, \tag{16}$$

where l is the curve bounding the surface a in the plane $z=0$. In view of (16), the condition (14) is as follows:

$$T|_{\Gamma_1} = T(r, z=0) = \frac{I_0}{2r\pi}. \tag{17}$$

At infinity, both the electric and the magnetic field are zero. So, the vector potential $\mathbf{T}$ satisfying the differential equation (10) must vanish at infinity as $1/\varrho$ where ϱ is the distance of the point (r, z) from the origin in a planar section of the region:

$$\lim_{\varrho\to\infty} \mathbf{T} = \mathbf{o}(1/\varrho). \tag{18}$$

Application of the variational method, satisfaction of the boundary conditions

The approximate solution of the differential equation (10) satisfying the Neumann boundary condition (11), the Dirichlet boundary conditions (15) and (17), as well as the behaviour (18) at infinity will be generated with the aid of a variational calculus. As known form literature [18], [19], the vector potential formally extremizing the functional

$$W = \iint_{z,r} (\mathbf{curl\,T}\ \mathbf{curl\,T}^* + j\mu\omega\sigma\,\mathbf{T}\mathbf{T}^*)\,r\,\mathrm{d}r\,\mathrm{d}z - \int_{\Gamma_h} \mathbf{T}^*(\mathbf{curl\,T}\times\mathbf{v})\,\mathrm{d}\Gamma \tag{19}$$

satisfies the differential equation (10). In (19), $\mathbf{T}^*$ denotes the conjugate of $\mathbf{T}$. Since the Neumann boundary condition (11) on the boundary Γ_h of the studied region is a natural condition of the functional (19), only the satisfaction of the Dirichlet boundary conditions (15), (17) on the surface Γ_1 and at $r=0$ as well as the condition (18) on the vector potential $\mathbf{T}$ at infinity will be treated.

In order to satisfy the Dirichlet boundary conditions (15) and (17), the vector potential $\mathbf{T}$ is decomposed as the sum of two functions [20], [10]:

$$\mathbf{T} = \mathbf{T}_\delta + \mathbf{T}_\alpha, \tag{20}$$

where $\mathbf{T}_\delta$ is a known function, continuous in the region examined, which satisfies the Dirichlet boundary conditions (14), (17) on the bounding surface Γ_1 and on the symmetry axis $r=0$:

$$\mathbf{T}_\delta|_{\Gamma_1} = \frac{I_0}{2r\pi}\mathbf{e}_\varphi, \quad \mathbf{T}_\delta|_{r=0} = \mathbf{0}. \tag{21}$$

In (20), $\mathbf{T}_\alpha$ is an unknown function satisfying homogeneous Dirichlet boundary conditions on Γ_1 and at $r=0$:

$$\mathbf{T}_\alpha|_{\Gamma_1} = \mathbf{0}, \quad \mathbf{T}_\alpha|_{r=0} = \mathbf{0}. \tag{22}$$

The condition (18) on the vector potential at infinity can be satisfied by ensuring that both terms of the potential vanish at least as $1/\varrho$ at infinity. Therefore, the term $\mathbf{T}_\alpha$ of the potential is selected to make it vanish as $1/\varrho$ at infinity. The other term $\mathbf{T}_\delta$ of the potential function, which satisfies Dirichlet's boundary conditions on the boundary Γ_1 and on the symmetry axis, is constructed to make it vanish at infinity as $1/\varrho$ too:

$$\lim_{\varrho\to\infty} \mathbf{T}_\alpha = \mathbf{o}(1/\varrho), \tag{23}$$

$$\lim_{\varrho\to\infty} \mathbf{T}_\delta = \mathbf{o}(1/\varrho). \tag{24}$$

The term $\mathbf{T}_\alpha$ of the potential function (20) is approximated according to Ritz's method [20], [21] by the first n terms of a function set, complete in the studied region:

$$\mathbf{T}_\alpha \approx \mathbf{T}_n = \sum_{k=1}^{n} a_k f_k w_D e^{-\gamma\varrho}\mathbf{e}_\varphi = \mathbf{F}^T\mathbf{a}, \tag{25}$$

where $\gamma=\sqrt{j\omega\mu\sigma}$, f_k is the k-th element of the approximating function set, a_k is the k-th unknown coefficient. In (25), w_D is an R-function defined in the studied region [14], [15]. w_D ensures the satisfaction of the homogeneous Dirichlet boundary conditions on the boundary Γ_1 and at $r=0$ by the function $\mathbf{T}_n$ approximating the vector potential $\mathbf{T}_\alpha$:

$$w_D|_{\Gamma_1} = 0, \quad w_D|_{r=0} = 0. \tag{26}$$

At infinity, the function w_D satisfies the condition (23) prescribed for the function $\mathbf{T}_\alpha$:

$$\lim_{\varrho\to\infty} w_D = o(1/\varrho). \tag{27}$$

Generation of the solution, construction of the functions $\mathbf{T}_\delta$ and w_D

The functions $\mathbf{T}_\delta$ and w_D introduced for the satisfaction of the boundary conditions (15), (17), (18) are constructed with the aid of the **R**-functions established by V. L. Rvachev [14], [15].

On the boundary Γ_1 of the studied region and at $r=0$, the function $\mathbf{T}_\delta$ satisfying (21) can be constructed as follows, using the equations

$$\begin{aligned} w_1(r, z) &= z, \\ w_2(r, z) &= r \end{aligned} \tag{28}$$

of the bounding surface Γ_1 and of the symmetry axis [10]:

$$\mathbf{T}_\delta = \frac{I_0}{2\pi} \frac{r}{r^2+z^2} e^{-\gamma w_d} \mathbf{e}_\varphi, \tag{29}$$

where $\gamma = \sqrt{j\omega\mu\sigma}$, and w_d is an R-function given by (30). The function $\mathbf{T}_\delta$ given in (29) varies as $I_0/2r\pi$ along the bounding surface Γ_1 (at $z=0$) while it vanishes at $r=0$. At infinity, the function $\mathbf{T}_\delta$ given in (29) approaches zero as $1/\varrho$ in view of $\varrho = \sqrt{r^2+z^2}$.

The function w_D ensuring the satisfaction on the homogeneous Dirichlet boundary conditions prescribed for the function $\mathbf{T}_\alpha$ is constructed as follows. Since the function w_D must satisfy the conditions (26) on the boundary Γ_1 and at $r=0$, an **R**-function w_d is to be constructed by **R**-conjunction from the equations (28) of the bounding surface Γ_1 and of the symmetry axis which vanishes on the boundary Γ_1 of the studied region and at $r=0$:

$$w_d = w_1 \wedge w_2 = w_1 + w_2 - \sqrt{w_1^2 + w_2^2}. \tag{30}$$

Since the functions w_1 and w_2 are monotonously increasing at infinity,

$$\left.\begin{aligned} \mathbf{grad}\, w_1 \neq 0 \\ \mathbf{grad}\, w_2 \neq 0 \end{aligned}\right\} \quad \text{if} \quad 0 < r < \infty,\ 0 < z < \infty, \tag{31}$$

so the function w_d constructed in (30) is also monotonously increasing at infinity. Using the above properties of the function w_d, the function w_D satisfying the condition (26) and (27) can be constructed as follows:

$$w_D = \frac{w_d}{(A+\varrho)^2}, \tag{32}$$

where A is an arbitrary positive constant.

The elements of the approximating function set have been chosen to be Chebyshev polynomials. Since Chebsyhev polynomials are defined in the interval $(-1, +1)$, the points $0<r<\infty$ and $0<z<\infty$ of the infinite region have been transformed into the region $0<\xi<1$ and $0<\eta<1$, by the aid of the transformation $\xi = 2/\pi \tan^{-1}(z/a)$ and $\eta = 2/\pi \tan^{-1}(r/a)$, respectively.

The coefficients a_k in the approximate function series (25) extremizing the function (19) have been computed from a set of complex, linear equations:

$$(\mathbf{A}_r + j\mathbf{A}_i)(\mathbf{a}_r + j\mathbf{a}_i) = \mathbf{b}_r + j\mathbf{b}_i \tag{33}$$

where

$$\mathbf{A}_r = \iint_{z,r} \mathbf{curl\, F\; curl\, F}^T r\, \mathrm{d}r\, \mathrm{d}z, \qquad \mathbf{A}_i = \iint_{z,r} \mu\omega\sigma \mathbf{F}\mathbf{F}^T r\, \mathrm{d}r\, \mathrm{d}z \tag{34}$$

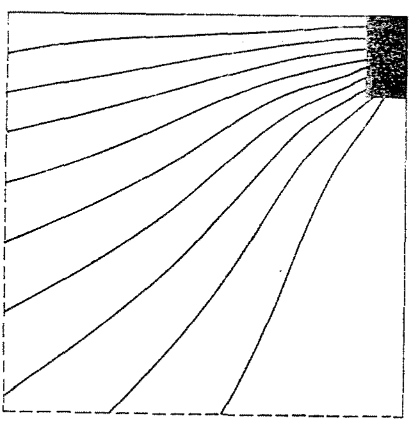
and

$$\mathbf{b}_r = \iint_{z,r} \mathbf{curl\, T}_\delta\, \mathbf{curl\, F} r\, \mathrm{d}r\, \mathrm{d}z, \qquad \mathbf{b}_i = \iint_{z,r} \mu\omega\sigma \mathbf{T}_\delta \mathbf{F} r\, \mathrm{d}r\, \mathrm{d}z. \tag{35}$$

Numerical results

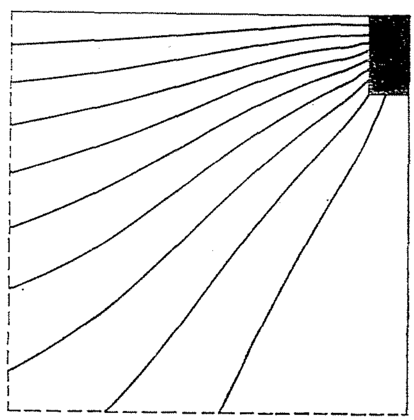
Numerical calculations have been carried out at $r_0/l = 1/2$. The number of the terms n has been chosen as $n=25$. Knowing the vector potential $\mathbf{T}$ obtained by the solution of equ. (33), the distribution of the current density has been drawn at different values of the skin depth $\delta = \sqrt{2/\mu\omega\delta}$ (Fig. 2). In Fig. 2a, the current distribution at D. C. excitation has been plotted. In the further diagrams (Figs. 2b—2f), the current density distribution at $l/\delta = 0.2$, 0.4, 0.6, 0.8 and 1.0 has been plotted. It is evident

Fig. 2. Distribution of current density in the conducting medium at different skin depths. The same current flows between any two lines



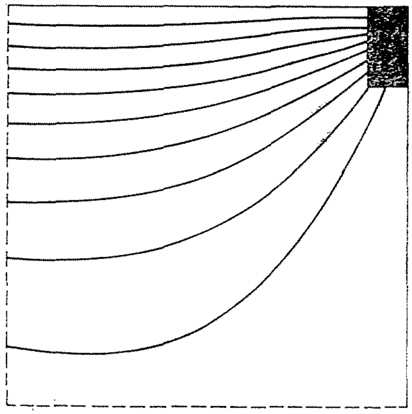
a)

a: Distribution of current density at D. C. excitation



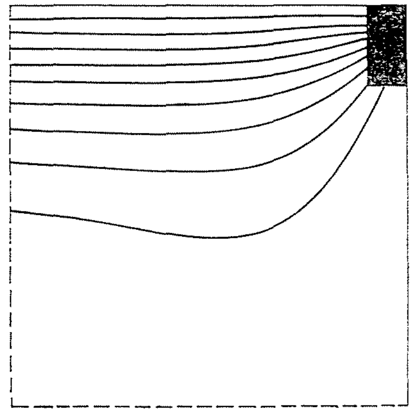
b)

b: Distribution of current density at $l/\delta = .2$



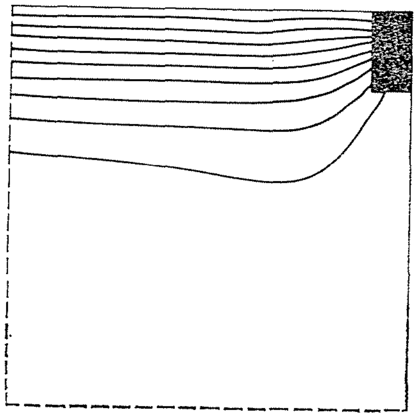
c)

c: Distribution of current density at $l/\delta = .4$



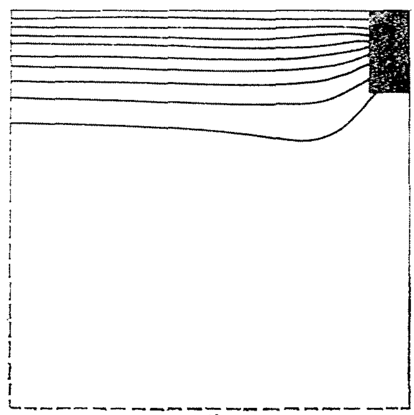
d)

d: Distribution of current density at $l/\delta = .6$



e)

e: Distribution of current density at $l/\delta = .8$



f)

f: Distribution of current density at $l/\delta = 1.$

from these diagrams that the current density tends to be confined to the vicinity of the bounding surface of the conducting medium as the frequency is increased (the skin depth is decreased).

With the aid of Poynting's vector, the impedance on the surface of the conducting medium has been computed as a function of the distance from the ideal conductor:

$$Z(z=0, r) = \frac{1}{I_0^2} \int_{r_0}^{r} \frac{1}{\sigma} (\mathbf{curl\, T}) \times \mathbf{T}^*|_z r 2\pi \, dr. \tag{36}$$

For the cases $l/\delta = 1.0$ and $l/\delta = 0.6$, the results are shown in Fig. 3.

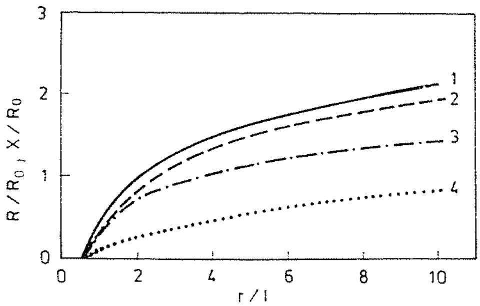


Fig. 3. Impedance on the surface of the conducting medium against the distance from the earthing conductor. Ro is D. C.resistance. On curves 1. R/R_0, 2. X/R_0 at $\delta=1$, while on curves 3. R/R_0, 4. R/R_0 at $l/\delta=0{,}6$.

R_0 is the D. C. resistance at $r=10 \cdot l$.

The results indicate that the earthing impedance approaches the diffusion value at a greater distance from the earthing conductor as the frequency increases.

In Figure 4, the earthing impedance at $r=10 \cdot l$ has been plotted against l/δ. Here, the continuous line indicates R/R_0 and the broken line X/R_0. This diagram also shows that both the resistive and reactive parts of the impedance increase with increasing frequency. It is also seen in the diagram that, although the accuracy of the approximation fluctuates, the real and the imaginary parts of the impedance approach a common value similarly to the case of an infinite half space.

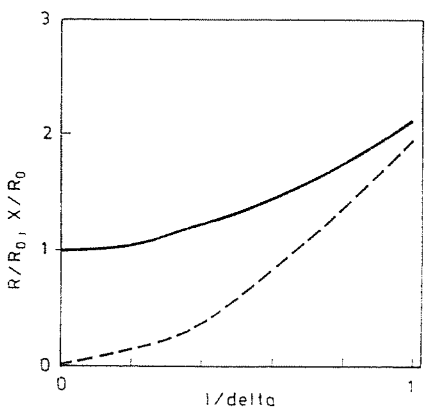


Fig. 4. Impedance at $r=10\,l$ against skin depth ——R/R_0, – – –X/R_0, R_0 is D. C. resistance

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