SINGULAR PERTURBATION PROBLEMS USING INDEFINITE LIAPUNOV FUNCTIONS

P. BÉDA

Department of Engineering Mechanics, Faculty of Electrical Engineering, Technical University, Budapest H-1521

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Abstract

With a system of differential equations consisting of many variables the singular perturbation methods are of great importance. If we know that some variables decrease faster than the others, we can reduce the number of the equations. The reduction makes their solving much more easy.

For example, if we use some numerical method, the time and storage necessary for the solutions become less than originally [1].

In this paper a singular perturbation problem, the existence and separation of the small, and large solutions of a differential equation is considered, but instead of the usual way an indefinite Liapunov function is used for the investigation.

1. Introduction

In case of a system of differential equations consisting of many variables if some practical experiences (measurements) or mathematical investigation inducate which variables will decrease faster, we can multiply their derivative with a small $\varepsilon>0$, so generally a singular perturbation problem can be described in the next form:

$$\dot{\alpha} = f(\alpha, \beta, t, \varepsilon)$$
$$\varepsilon\dot{\beta} = g(\alpha, \beta, t, \varepsilon)$$

where $f(0, 0, t, \varepsilon)=g(0, 0, t, \varepsilon)=0$.

A. Halanay investigated a singular perturbation problem in [8]. His example was as follows:

Let us see a singular perturbation problem in the form:

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{1.1}$$

where $x_1\in\mathbf{R}^n, x_2\in\mathbf{R}^m$, A_{11}, A_{21}, A_{12} and A_{22} are $n\times n$, $m\times n$, $n\times m$ and $m\times m$ matrices, respectively. Generally this equation can be reduced as $\varepsilon\approx 0$, so from equation

$$A_{21}x_1+A_{22}x_2 = 0, \quad \text{or}$$
$$x_2 = -A_{22}^{-1}A_{21}x_1$$

we have:

$$\dot{x}_1 = (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1.$$

However this reduction is not always correct. In the procedure used by Halanay we define at first a matrix $\mathbf{\Pi}(n+m \times n+m)$ as

$$\mathbf{\Pi} = \begin{bmatrix} I_1 & 0 \\ T & I_2 \end{bmatrix},$$

where I_1 and I_2 are the unit matrices of dimension $n \times n$ and $m \times m$, respectively. $\mathbf{T}$ is an unknown matrix $(n \times m)$. It can easily be proved, that

$$\mathbf{\Pi}^{-1} = \begin{bmatrix} I_1 & 0 \\ -T & I_2 \end{bmatrix}.$$

We transform the equation by the matrix $\mathbf{\Pi}$. The new variable is

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{i.e.}$$

$$\mathbf{y} = \mathbf{\Pi} x,$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The new equation for $\mathbf{y}$ is:

$$\dot{\mathbf{y}} = \mathbf{\Pi} \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon} A_{21} & \frac{1}{\varepsilon} A_{22} \end{bmatrix} \mathbf{\Pi}^{-1} \mathbf{y}$$

i.e.

$$\dot{\mathbf{y}} = \begin{bmatrix} A_{11} - A_{12} T & A_{12} \\ TA_{11} + \frac{1}{\varepsilon} A_{21} - TA_{12} T - \frac{1}{\varepsilon} A_{22} T & TA_{12} + \frac{1}{\varepsilon} A_{22} \end{bmatrix} y.$$

Assume that there exists a T which satisfies:

$$\varepsilon T(A_{11} - A_{12} T) + (A_{21} - A_{22} T) = 0. \tag{1.2}$$

Then the approximation of the differential equation is

$$\begin{aligned} \dot{y}_1 &= (A_{11} - A_{12} T) y_1 + A_{12} y_{12} \\ \varepsilon \dot{y}_2 &= (A_{22} + \varepsilon T A_{12}) y_2, \end{aligned} \tag{1.3}$$

where $T \approx A_{22}^{-1} A_{21}$, if ε is small.

From equation (1.2) we can determine matrix $\mathbf{T}$. Now let us see equation (1.3) using matrix $\mathbf{T}$. If the eigenvalues of matrix $A_{22} + \varepsilon T A_{12}$ have negative real parts, then y_2 tends to zero. It is actually so if A_{22} is a stable matrix and ε is small enough. Moreover, the smaller ε is, the faster y_2 tends to zero. However, if A_{22} is not stable the singular perturbation technique is not justified. If one does not consider the sufficient condition for applying the singular perturbation technique a wrong result can be obtained instead of a good approximation.

Note that in this case solutions y tend to the subspace of the variable y_1. (see Fig. 1.1) $y_1(t)$ can be called as large solution and $y_2(t)$ as small solution, expressing the fact that $y_2(t)$ tends to zero faster than $y_1(t)$.

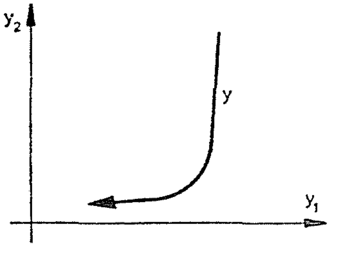

Fig. 1.1

In this paper we are going to investigate an explicite third order differential equation. We give sufficient conditions of the case in which a second order equation can be approached. It was not intended to choose parameter ε, but we studied the basic problem, the existence and the separation of the small and large solutions. For the investigation an indefinite Liapunov function has been used.

2. Some definitions used later on

2.1. Definitions about attractivity

Let us look at equation

$$\dot{x} = g(t, x), \tag{2.1}$$

where

$$g \in C[\mathbf{R}^+ \times \Omega, \mathbf{R}^n], \quad \mathbf{R}^+ = [t_0, \infty),$$

$\Omega \subset \mathbf{R}^n$, connected and open set. Denote $d(P, R)$ the distance between sets P and R.

2.1.1. Definition

$M \subset \mathbf{R}^+ \times \mathbf{R}^n$ is stable set, if $\forall \varepsilon > 0$, $\alpha > 0$ and $\forall t_1 \geqq t_0$ $\exists \delta > 0$ such that $d[y_0, M(t_1)] < \delta$ and $|y_0| < \alpha$ implies $d[y(t, t_1, y_0), M(t)] < \varepsilon$ for $\forall t \geqq t_1$. Where $y(t, t_1, y_0)$ is the solution of (2.1) with the initial value $y_0 = y(t_1, t_1, y_0)$.

2.1.2. Definition

M is uniformly stable set of (2.1) if it is stable and δ does not depend on t_1.

2.1.3. Definition

M is uniformly asymptotically stable set of (2.1) if it is uniformly stable, and for $\forall\varepsilon>0, \forall\alpha>0$ there exist $t_1>0$ and $\delta_0>0$, which does not depend on ε, such that if

$$d[y_0, M(t_1)] < \delta_0 \quad \text{and if} \quad |y_0| \leqq \alpha \quad \text{then}$$

$$d[y(t, t_1, y_0), M(t)] < \varepsilon \quad \text{for all} \quad t \geqq t_1.$$

2.1.4. Definition

M is invariant set of (2.1) if for $\forall (t_1, y_0)\in M$ the solution $y(t, t_1, y_0)$ exists if $t \geqq t_1$, and $y(t, t_1, y_0)\in M$ for all $t \geqq t_1$.

2.1.5. Definition

$\overline{\mathbf{M}}$ is uniformly attractive, if it is uniformly asymptotically stable (consequently invariant) set. The domain of the attractivity of M is the set

$$A = \left\{(\tau, s)\in\mathbf{R}^+\times\Omega : \lim_{t\to\infty} d[y(t, \tau, s), \overline{\mathbf{M}}(t)] = 0\right\}.$$

2.2. Definite and indefinite Liapunov functions

Liapunov functions are often used for stability investigations of differential equations. They generally belong to the class C^1, defined by

$$V\colon \mathbf{R}^+\times\Omega \to \mathbf{R},$$

$\Omega\subset\mathbf{R}^n$ and the origin is in Ω.

In stability investigations these functions are generally positive definite ones. Considering the behaviour of this Liapunov function, we can be informed about the trajectories. [2], [3]. In [4], [5], [6] V. Kertész is using an indefinite Liapunov function. Using this function we can separate "small" and "large" solutions. This separation is the basis of our investigation in the singular perturbation problem.

3. The main theorem and its proof

Let C be a symmetric constant $n\times n$ matrix, such that the quadratic form r^TCr, $r\in R^n$ is indefinite, $A(r, t)$ is a continuous $n\times n$ matrix function, $r\in \mathbf{R}^n$.

Consider equation

$$\dot{\mathbf{r}} = A(r, t)r. \tag{3.1}$$

Instead of $A(r, t)$, we write simply A.

We define an indefinite Liapunov function

$$\tilde{\mathbf{z}}(r) = r^T Cr$$

and another function denoted by ϱ,

$$\varrho(r, t) = \frac{r^T(CA - A^T C)r}{r^T Cr}$$

(T denotes the transpose)
and the sets

$$Q_1 = \{r\in\mathbf{R}^n\colon \tilde{\mathbf{z}}(r) > 0\}$$
$$Q_2 = \{r\in\mathbf{R}^n\colon \tilde{\mathbf{z}}(r) < 0\}.$$

Now we have two theorems:

Theorem 3.1

Supposing

$$r^T(CA + A^T C)r > 0 \quad \text{if} \quad \tilde{\mathbf{z}}(r) = 0, \quad r \neq 0 \quad \text{and} \quad t \geqq t_1,$$

then Q_1 is an invariant set of the solutions of (3.1). In this case if there exists a $\varrho_0(t)$ such that

$$\varrho_0(t) \leqq \varrho(r, t) \quad \text{in } Q_1, \text{ then}$$

$$z(t) \geqq z(t_1) \exp\Big(\int_{t_1}^{t} \varrho_0(\tau)\, d\tau\Big),$$

where $z(t) = \tilde{\mathbf{z}}(r(t))$, and $r(t)$ is a solution of (3.1) which satisfies $r(t_1)\in Q_1$.

$$\Big(\text{If } \int_{t_1}^{\infty} \varrho_0(t)\, dt > -\infty, \text{ then } \varliminf_{t\to\infty} |r(t)| > 0\Big).$$

Theorem 3.2

Supposing that

$$r^T(CA+A^TC)r<0 \quad \text{if} \quad \tilde{z}(r)=0, \quad r\neq 0 \quad \text{and} \quad t\geqq t_1,$$

then Q_2 is an invariant set of the solutions. In this case if there exists a $\varrho_0(t)$, that

$$\varrho_0(t)\leqq z(r,t) \quad \text{in } Q_2, \text{ then}$$

$$z(t)\leqq z(t_1)\exp\Big(\int_{t_1}^{t}\varrho_0(\tau)\,\mathrm{d}\tau\Big),$$

where $z(t)=\tilde{z}(r(t))$, and $r(t)$ is a solution of (3.1) which satisfies $r(t_1)\in Q_2$.

$$\Big(\text{If } \int_{t_1}^{\infty}\varrho_0(t)\,\mathrm{d}t>-\infty, \text{ then } \varliminf_{t\to\infty}|r(t)|>0\Big).$$

The following lemma [4] will be used for the proof of these theorems.

Lemma 3.1

Let us consider matrix $\mathbf{C}$ in part 3, and set Q_1. Then for every solution of (3.1) which satisfies

$$z(t)\neq 0 \quad \text{if} \quad t_1\leqq t\leqq t_2 \quad \text{for some } t_2 \quad (t_1\leqq t_2<\infty),$$

the next equation holds

$$z(t)=z(t_1)\exp\Big(\int_{t_1}^{t}\frac{r^T(\tau)\big(A^T(\tau,r(\tau))C+CA(\tau,r(\tau))\big)r(\tau)}{z(\tau)}\,\mathrm{d}\tau\Big) \tag{3.2}$$

$$t_1\leqq t\leqq t_2 \quad \text{where} \quad z(t)=r^T(t)Cr(t).$$

Proof:
As we know

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln z(t)=\frac{\frac{\mathrm{d}}{\mathrm{d}t}z(t)}{z(t)}.$$

Noticing that $\frac{\mathrm{d}}{\mathrm{d}t}z(t)=r^T(t)\big(A^T(t,r(t))C+CA(t,r(t))\big)r(t)$ we integrate both sides of the equation, and the lemma is proved.

Remark: if $z(t)\leqq 0$, then

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}z(t)}{z(t)}=\frac{\frac{\mathrm{d}}{\mathrm{d}t}|z(t)|}{|z(t)|}=\frac{\mathrm{d}}{\mathrm{d}t}\ln|z(t)|.$$

From it

$$|z(t)| = |z(t_1)| \exp \int_{t_1}^{t} \varrho \, d\tau,$$

$$z(t) = z(t_1) \exp \int_{t_1}^{t} \varrho \, d\tau,$$

(where $\varrho = \varrho(r(\tau), \tau)$).

The proof of Theorem 3.1:

Let $r(t)$ be the solution of (3.1) for which $r(t_1) \in Q_1$. So inequality

$\tilde{\mathbf{z}}(r(t_1)) > 0$ is satisfied.

As we have considered the expression

$$f(r, t) = r^T (CA + A^T C) r,$$

this is positive if $\tilde{\mathbf{z}}(r) = 0$ and $r \neq 0$, because $f(r, t)$ is continuous in its variables. (C is constant, A consists of continuous functions.)

Boundary set $\tilde{\mathbf{z}}(r) = 0$ has an open neighbourhood $K(t)$, where

$$f(r, t)|_{r \in K(t)} > 0. \quad \text{Obviously,} \quad K(t) \cap Q_1 \neq \emptyset.$$

Solution $r(t)$ is continuous and so $z(t) = \tilde{\mathbf{z}}(r(t))$ is continuous too, (See (3.2)).

Assume that a solution $r(t)$ of (3.1) satisfying the conditions of Theorem 3.1. does not remain in set Q_1. Because of the continuity of $z(t)$, there exists a $t^* > t_1$, for which $\lim_{t \to t^* - 0} z(t) = 0$, i.e. for $\forall \varepsilon > 0 \; \exists \delta$:

$$(0 <) \, t^* - t < \delta \Rightarrow (0 <) \, z(t) < \varepsilon.$$

moreover

$$z(t^*) = 0$$

and

$$\exists t_3 < t^*,$$

such that

$$\forall t \in (t_3, t^*): \quad f(r, t) > 0. \tag{3.3}$$

Let us denote

$$2\varepsilon = z(t_3) \quad (t_3 > 0).$$

We want to find a δ, for for which if

$$t \in (t^* - \delta, t^*)$$

then

$$z(t) < \varepsilon, \tag{3.4}$$

and

$$t_3 \leqq t^* - \delta.$$

We examine function $z(t) = \tilde{\mathbf{z}}(r(t))$ in domain Q_1. Here $z(t) > 0$. We may use Lemma 3.1. $(t_3 < t < t^*)$

$$z(t) = z(t_3) \exp \int_{t_3}^{t} \varrho \, d\tau,$$

From (3.4) we get

$$2\varepsilon \exp \int_{t_3}^{t} \varrho \, d\tau < \varepsilon \quad \text{if} \quad t_3 < t^* - \delta < t < t^*$$

$$\exp \int_{t_3}^{t} \varrho \, d\tau < \frac{1}{2} \tag{3.5}$$

But we know that

$$\varrho(t) > 0 \quad \text{if} \quad t \in [t_3, t^*),$$

(see (3.3) and the definition of K) i. e.

$$\int_{t_2}^{t} \varrho \, d\tau > 0$$

so inequality (3.5) can not be true, so a δ satisfying the definition of the limes does not exist. The other statement of the theorem can be derived from the lemma.

Remark:

the proof of Theorem 3.2. is similar to the proof of Theorem 3.1.

4. The application of an indefinite Liapunov function in a singular perturbation problem of a nonlinear third order differential equation

Theorem 4.1

Let $\beta > 0$ be a constant, and let

$$a_0(t, u, v, w),$$

$$a_1(t, u, v, w),$$

$$a_2(t, u, v, w)$$

be continuous functions.

Let the matrices used in Theorem 3.1. be:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \mathrm{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\beta^2 \end{bmatrix},$$

and $\mathbf{r} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \neq 0$, introducing notations

$$u = y, \quad v = y', \quad w = y'',$$

If there exist constants c, c_{ik} $(i=0, 1; k=1, 2)$ such that

$$0 < c_{01} \leqq a_0 \leqq c_{02},$$

$$0 < c_{11} \leqq a_1 \leqq c_{12},$$

$$0 < c \leqq a_2,$$

and

$$c > \frac{c_2^2}{c_1^2} + c_2,$$

where $c_2 = \max(c_{02}, c_{12})$, $c_1 = \min(c_{01}, c_{11})$, then there exists $\beta > 0$, for which the expression

$$r^T(CA + A^T C)r \tag{4.1}$$

is greater than zero if

$$r^T C r = 0. \tag{4.2}$$

Proof:

we substitute A, C and r into (4.1), and after multiplication by 1/2 we get:

$$\beta^2 a_2 w^2 + (1 + \beta^2 a_1) vw + \beta^2 a_0 uw + uv. \tag{4.1*}$$

If $r \neq 0$, we may introduce some new variables:

$$x = \frac{u}{w}, \quad y = \frac{v}{w},$$

by these (4.1) is:

$$a_2 \beta^2 + a_0 \beta^2 x + (1 + a_1 \beta^2) y + xy. \tag{4.3}$$

Denote $\chi(x, y)$ expression (4.3), and we fix the values of a_0, a_1, a_2 and vary x and y. The problem is whether

$$\chi(x, y) > 0 \tag{4.4}$$

if

$$x^2 + y^2 = 0, \tag{4.5}$$

(For $r \neq 0$ equation (4.5) is equivalent to (4.2).)

On the plane x, y expression (4.4) means a domain bounded by a hyperbola. Its equation is

$$\chi(x, y) = 0,$$

i.e.

$$y = -\beta^2 a_0 + \frac{\beta^2 a_0 (1 + a_1 \beta^2) - \beta^2 a_2}{x + (1 + a_1 \beta^2)}.$$

The asymptotes are parallel to the axis x, and axis y, and consist of points

$$\begin{aligned} x_0 &= -1 - a_1 \beta^2, \\ y_0 &= -\beta^2 a_0. \end{aligned} \tag{4.6}$$

Expression (4.5) means a circle around the origin with radius β. (4.4) and (4.5) are satisfied if the circle is

a) in the quadrant bounded by the asymptotes, in which there is no point of the hyperbola, and

b) (4.4) is satisfied between the graph of the hyperbola. (see Fig. 4.4.)

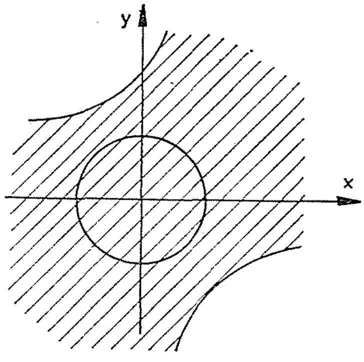

Fig. 4.4

These conditions are

a)

$$\chi(0,0) > 0, \quad \beta \leqq |x_0|, \tag{4.7}$$

i.e.

$$\beta \leqq |1+a_1\beta^2|. \tag{4.8}$$

The sign of x_0 is the same of y_0, and

$$\beta \leqq |y_0|, \quad \beta \leqq |a_0|\beta^2, \tag{4.9}$$

b)

$$\chi(x_0, y_0) > 0, \tag{4.10}$$

that is

$$a_2\beta^2 - a_0\beta^2(1+a_1\beta^2) > 0,$$

so

$$a_2 > a_0(1+a_1\beta^2).$$

Because of the conditions of the theorem $a_1, a_0 > 0$, so (4.10) is equivalent to

$$\frac{a_2-a_0}{a_1 a_0} > \beta^2.$$

(4.7) satisfies because of condition $a_2 > 0$. In (4.6) both x_0 and y_0 are negative. Inequality (4.8) is

$$\beta \leqq |1+a_1\beta^2| = 1+a_1\beta^2,$$

so

$$a_1\beta^2-\beta+1 \geqq 0.$$

If $a_1 \geqq 1/4$ then $\beta \in \mathbf{R}$, if $0<a_1<1/4$ then $\beta \in \left[\frac{1+\sqrt{1-4a_1}}{2a_1}, \infty\right)$. So if $a_1 \geqq 1/4$ then (4.9) and (4.10) hold if

$$\frac{1}{a_0} \leqq \beta < \sqrt{\frac{a_2-a_0}{a_1 a_0}}.$$

In case of $0<a_1<1/4$, (4.9) and (4.10) are true if

$$\max\left(\frac{1}{a_0}, \frac{1+\sqrt{1-4a_1}}{2a_1}\right) < \beta < \sqrt{\frac{a_2-a_0}{a_1 a_0}}.$$

These intervals are not allowed to be empty by the conditions of the theorem, because

$$\frac{1}{a_0} < \frac{1}{c_1} < \sqrt{\frac{c-c_2}{c_2^2}} < \sqrt{\frac{a_2-a_0}{a_1 a_0}}.$$

or if $a_1 \in (0, 1/4)$

$$\max\left(\frac{1}{a_0}, \frac{1+\sqrt{1-4a_1}}{2a_1}\right) < \max\left(\frac{1}{a_0}, \frac{1}{a_1}\right),$$

i.e.

$$\max\left(\frac{1}{a_0}, \frac{1+\sqrt{1-4a_1}}{2a_1}\right) < \frac{1}{c_1} < \sqrt{\frac{c-c_2}{c_2^2}} < \sqrt{\frac{a_2-a_0}{a_1 a_0}}.$$

There exists β with the required properties. Obviously, if a_0, a_1, a_2 are not constants, but satisfy the conditions stated in theorem, (4.4) remains true.

4.2. Application of indefinite Liapunov function in approximation of a third order nonlinear differential equation

Let us see the next nonlinear third order differential equation:

$$y+a_2 y''+a_1 y'+a_0 y = 0,$$

where a_0, a_1, a_2, are functions of t, y, y' and y''. Assume that the conditions of Theorem 4.1. are satisfied. In our investigations we use the notations of Theorem 4.1. Let us have a β as in the previous theorem. Because of theorem 3.1., if

$$r^T C r = 0,$$

then

$$r^T(CA+A^T C)r > 0.$$

By Theorem 3.1. Q_1 is an invariant set of the solutions. Fig. 5.1. shows the meaning of it in the space of y, y', y''. If β is large enough, set Q_1 is near the plain (u, v), so plain (u, v) is approximately an invariant set of the solutions.

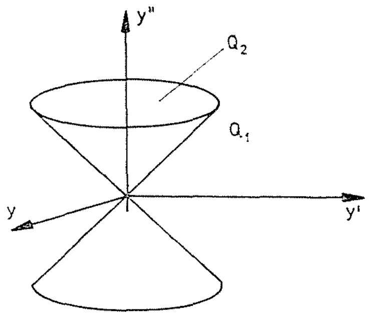

Fig. 5.1

4.3. On the behaviour of the solutions

Let Q_1 and Q_2 be satisfying the definitions used in part 3. From theorems 3.1. and 4., applying (2.5) on Fig. 5.2. in the domain between the hyperbolas

$$\chi > 0,$$

in Q_2

$$z < 0,$$

so in the expression of z the power of ϱ is negative.

$$\varrho = \frac{\chi}{z} < 0.$$

On the solutions z increases, so as in Fig. 5.2., the solutions come out of set Q_2. So set Q_1 is an attractor and its region of attractivity is Q_2.

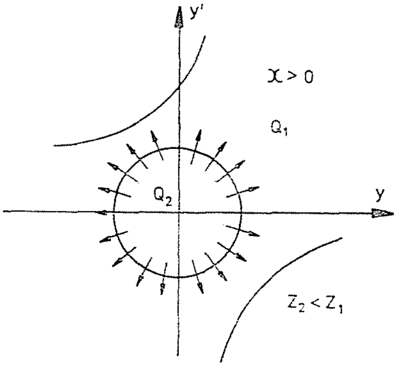

Fig. 5.2

The proof is easy because in Q_2 Lemma 3.1. holds, and on the cone Theorem 4.1 also satisfies. Thus, Q_1 is a uniformly attractive set. If β is large enough, it means that w tends faster to zero than u or v. So we encounter a typical singular perturbation problem, solved by indefinite Liapunov function and not by the singular perturbation technique.

4.4. *Numerical example*

To illustrate our results, there is a numerical example for the linear equation

$$y''' + 5y'' + y' + y = 0,$$

the coefficients of which satisfy the conditions of Theorem 4.1. if $1 \leqq \beta < 2$. The numerical approximation of the roots of its characteristical equation is

$$\lambda_1 = -4.836,$$
$$\lambda_2 = -0.082 + 0.448i,$$
$$\lambda_3 = -0.082 - 0.448i.$$

We can easily see, that the real part of λ_1 has larger absolute value, than that of the others. So the component of the solutions belonging to λ_1 decreases much faster, than that of the others.

The eigenvectors s_1, s_2, s_3 belonging to eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are approximately

$$s_1 = \begin{bmatrix} 0.043 \\ -0.207 \\ 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ -0.082 \\ -0.194 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 \\ 0.448 \\ -0.074 \end{bmatrix}.$$

Accordingly, components decreasing faster than the others are approximately in direction of basic vector

$$k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solutions tend to a plane which is close to the plane (u, v).

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Péter BÉDA H-1521 Budapest