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ADAPTIVE SAMPLING

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Abstract

This paper introduces a new method for signal representation. It is shown that a periodic signal is uniquely defined by its local extrema if the band limit ratio of the signal is less than an octave. A way of adaptive sampling, introduced among these lines, exhibits advantageous properties of possible interest, e.g., for the detection of the pitch frequency.

Keywords: adaptive sampling, pitch frequency detection.

Introduction

Uniform sampling is frequently used for the characterization of a time function. However, occasionally the signal is determined with its nonuniform samples. A generalization of the sampling theorem for such purpose is given in full details in (FREEMAN (1965)). Nonuniform sampling is mostly applied for analysis and synthesis of digital filters (THOMAS and LUTTE (1972); WOJTKIEWICZ et al (1985); WEINBERG and LIU (1974))

This paper is divided into two main parts. First we introduce nonuniform sampling and as a special case of this the periodic nonuniform sampling of a signal and its first derivative.

Next we show that a periodic signal is completely specified by the knowledge of its local extrema if the band limit ratio of the signal is less than an octave. Furthermore, we discuss the basic properties of the adaptive sampling.

Sampling of a Continuous-Time Function

It is often necessary to reduce a continuous-time function to a series of samples. If the interval between samples is uniformly placed then what we have is the classical form of Shannon sampling theorem. In this section, however, our attention will be focused not on this case but on nonuniform sampling (see FREEMAN (1965)).

Uniform Sampling and the Sampling Theorem

Let a continuous function $x(t)$ be bandlimited and given in the following form

$$x(t) = \int_{-B}^{B} e^{j\omega t}\, d\beta(\omega). \tag{1}$$

The function $x(t)$ can be written

$$x(t) = \sum_{k=-\infty}^{\infty} x(t_k) \frac{\sin\left(\frac{\omega_0}{2}(t-t_k)\right)}{\frac{\omega_0}{2}(t-t_k)}, \tag{2}$$

where

$$\omega_0 = \frac{2\pi}{\tau_0}, \quad t_k = k\tau_0 \quad \text{and} \quad B < \frac{\omega_0}{2}.$$

$x(t)$ is uniquely described by the knowledge of its values $x(t_k)$ for all integers $-\infty < k < \infty$.

Sampling may be regarded as a simple form of multiplication between two time-dependent quantities. The sampled signal, denoted by $y(t)$ may be approximately

$$y(t) = x(t) \cdot v(t), \tag{3}$$

where $v(t)$ is a periodic train of rectangular pulses of duration Δ. The function $v(t)$ is shown in *Fig. 1.* By Fourier expansion

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \tag{4}$$

where

$$c_k = \bar{c}_{-k} = \frac{b\Delta}{\tau_0} \frac{\sin(k\pi\Delta/\tau_0)}{k\pi\Delta/\tau_0},$$

and the overscoring indicates the complex conjugate. We have a simpler form if $b\Delta/\tau_0 = 1$. By putting Eq. (1) and Eq. (4) into Eq. (3) we obtain the following form

$$y(t) = \left[\int_{-B}^{B} e^{j\omega t}\, d\beta(\omega)\right] \cdot \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right]. \tag{5}$$

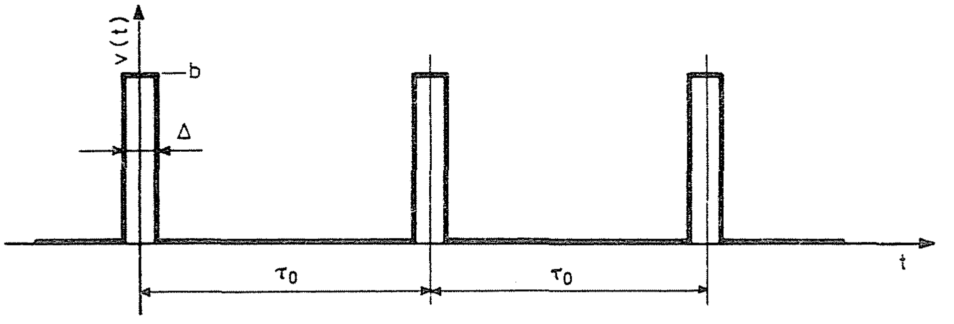

Fig. 1. Uniformly placed sampling function

Referring to Eq. (5) if $\Delta \ll \tau_0$ then we will get the sampling theorem (see FREEMAN (1965)).

Periodic Nonuniform Sampling

Consider a function $x(t)$ which is nonuniformly sampled. More distinctly assume the spacing between samples of $x(t)$ is nonuniform, but with a pattern which is periodic with period $T = n\tau_0$. This sampling pattern is shown in *Fig. 2.* We distinguish functions $v_1(t)$, $v_2(t)$,..., $v_n(t)$ and let

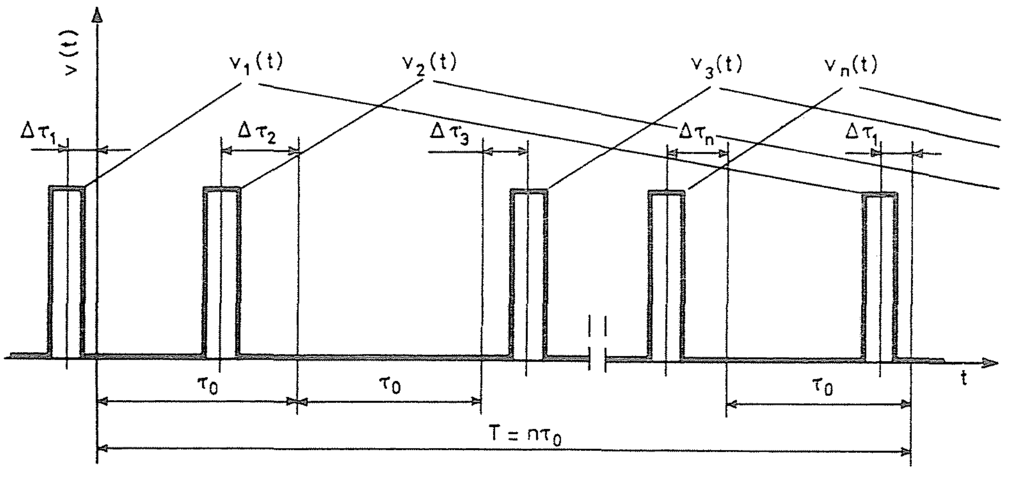

Fig. 2. Nonuniform sampling pattern periodically recurring with T

$y_i(t) = x(t) \cdot v_i(t)$, where $i = 1, 2, \ldots, n$. Using a time independent linear transformation H_i and a summation, we obtain $y_0(t)$ as it may be seen in *Fig. 3.* We would like to determine the transfer function H_i so that the

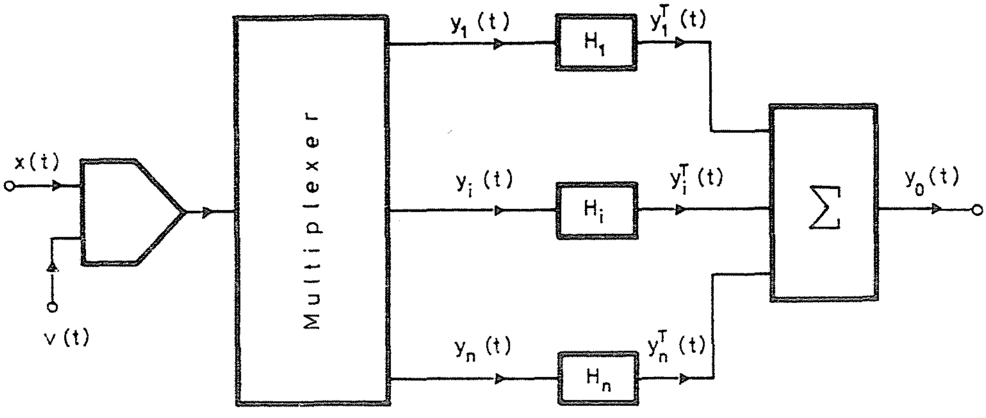

Fig. 3. Block diagram of recovering

function $x(t)$ can be recovered by filtering $y_0(t)$ with an ideal bandpass filter at interval $(-n\omega_0/2,\ n\omega_0/2$. Consider, for so doing, the Fourier series of the sampling function $v_i(t)$ which is periodic with period $T = n\tau_0$. It is known that

$$v_i(t) = \frac{1}{n} \sum_{k=-\infty}^{\infty} c_k e^{j\left[\frac{k\omega_0}{n} t + k\Delta\Phi_i\right]}, \tag{6}$$

where $\omega_0 = 2\pi/\tau_0$ and

$$\Delta\Phi_i = -\frac{2\pi}{n}\left[(i-1) + \frac{\Delta\tau_i}{\tau_0}\right]. \tag{7}$$

By letting $A_i = \exp(j\Delta\Phi_i)$ in Eq. (6)

$$v_i(t) = \frac{1}{n} \sum_{k=-\infty}^{\infty} c_k A_i^k e^{j\frac{k\omega_0}{n} t}. \tag{8}$$

Combining Eq. (1) and Eq. (8), we find the following simultaneous equations for $y_i(t)$, where $i = 1, 2, \ldots, n$ and $B = n\omega_0/2$,

$$y_1(t) \quad = \quad \left[\int_{-B}^{B} e^{j\omega t}\, d\beta(\omega)\right] \cdot \left[\frac{1}{n} \sum_{k=-\infty}^{\infty} c_k A_1^k e^{j\frac{k\omega_0}{n} t}\right],$$

$$\vdots$$

$$y_i(t) = \left[\int_{-B}^{B} e^{j\omega t}\, d\beta(\omega)\right] \cdot \left[\frac{1}{n}\sum_{k=-\infty}^{\infty} c_k A_i^k e^{j\frac{k\omega_0}{n}t}\right], \tag{9}$$

$$\vdots$$

$$y_n(t) = \left[\int_{-B}^{B} e^{j\omega t}\, d\beta(\omega)\right] \cdot \left[\frac{1}{n}\sum_{k=-\infty}^{\infty} c_k A_n^k e^{j\frac{k\omega_0}{n}t}\right].$$

By a little manipulation and introducing the transformation H_i, we can write

$$y_i^T(t) = \frac{1}{n}\sum_{k=-\infty}^{\infty}\int_{-B}^{B} c_k A_i^k H_i e^{j\left[\frac{k\omega_0}{n}+\omega\right]t}\, d\beta(\omega). \tag{10}$$

Next divide the interval $-B < \omega < B$ into subintervals, each of length B/n. Consider one of them, e.g., the subinterval $(\omega_1 = (n-1/n)B,\, \omega_2 = B)$. The spectrum in this subinterval can be expressed as a sum of a finite number of terms in Eq. (10). As just $-B < \omega < B$ is of our present interest, it is sufficient to use only the terms including $k = 0, 1, 2, \ldots, n-1$. If $k = 0$ then we should have get the spectrum of the original signal concerning this subinterval. This may be seen most clearly by referring to *Fig. 4*. For a

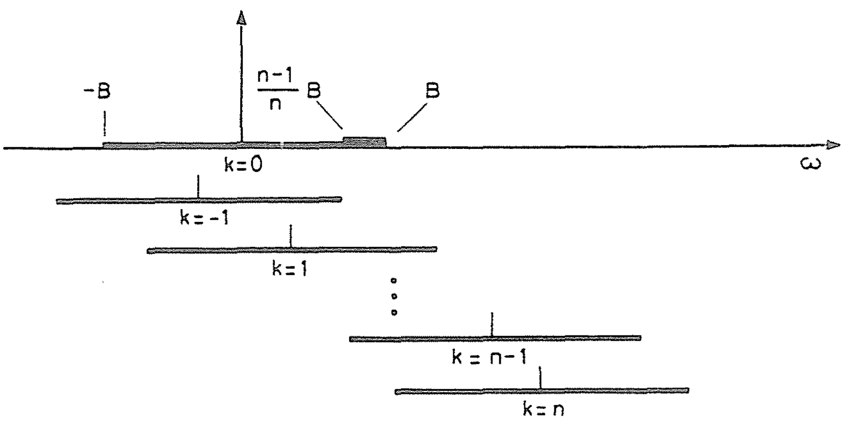

Fig. 4. The spectrum of nonuniformly sampled signal

precise recovery of the spectrum of $x(t)$ in (ω_1, ω_2) we need

$$\sum_{i=1}^{n} y_i^T(t)|_{k=0} = \sum_{i=1}^{n} \frac{1}{n} \int_{\omega_1}^{\omega_2} c_0 A_i^0 H_i e^{j\omega t}\, d\beta(\omega) = \int_{\omega_1}^{\omega_2} e^{j\omega t}\, d\beta(\omega). \qquad (11)$$

Accordingly

$$c_0 A_1^0 H_1 + c_0 A_2^0 H_2 + \cdots + c_0 A_n^0 H_n = n. \qquad (12)$$

If $k = 1, 2, \ldots, n-1$ then the sum of the spectrum must be zero. By this requirement, e.g. $k = n-1$,

$$c_{n-1} A_1^{n-1} H_1 + c_{n-1} A_2^{n-1} H_2 + \cdots + c_{n-1} A_n^{n-1} H_n = 0. \qquad (13)$$

From Eq. (12) and Eq. (13) we obtain n simultaneous equations with unknown transfer functions $H_1, H_2, \ldots, H_n$. It is instructive to write these in the following matrix form

$$\mathbf{B}\,\mathbf{H} = \mathbf{N}. \qquad (14)$$

Here

$$\mathbf{B} = \begin{bmatrix} c_0 A_1^0 & c_0 A_2^0 & \cdots & c_0 A_n^0 \\ c_1 A_1^1 & c_1 A_2^1 & \cdots & c_1 A_n^1 \\ c_2 A_1^2 & c_2 A_2^2 & \cdots & c_2 A_n^2 \\ \vdots & \vdots & & \vdots \\ c_{n-1} A_1^{n-1} & c_{n-1} A_2^{n-1} & \cdots & c_{n-1} A_n^{n-1} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Taking out the c_0 constant from Eq. (14) and cancelled with c_1/c_0, $c_2/c_0, \ldots$, c_n/c_0, it can be written as

$$c_0 \mathbf{A}\,\mathbf{H} = \mathbf{N}, \qquad (15)$$

where

$$\mathbf{A} = \begin{bmatrix} A_1^0 & A_2^0 & \cdots & A_n^0 \\ A_1^1 & A_2^1 & \cdots & A_n^1 \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & \vdots & & \vdots \\ A_1^{n-1} & A_2^{n-1} & \cdots & A_n^{n-1} \end{bmatrix}.$$

The solution of Eq. (15) may be obtained as

$$\mathbf{H} = \frac{1}{c_0}\mathbf{A}^{-1}\mathbf{N} \tag{16}$$

given that the inverse of the matrix $\mathbf{A}$ exists. The matrix $\mathbf{A}$ is recognized to be a Vandermonde matrix. If the samples are distinct at the interval T then $A_1, A_2, \ldots, A_n$ have different values, furthermore, none of them vanish, thus $\mathbf{A}^{-1}$ exists. This is valid on each subinterval of $(-B, B)$. By symmetry $H_i(-j\omega) = H_i(j\omega)$. It is to be noticed that $H_1, H_2, \ldots, H_n$ are physically nonrealizable filters.

We thus conclude that a signal $x(t)$, which is bandlimited to the interval $(-B, B)$, is completely specified by the knowledge of n arbitrarily spaced distinct samples within an interval $T = n\tau_0$ but periodically recurring with T.

Periodic Nonuniform Sampling of a Signal and Its First Derivative

Now we examine a more general form of nonuniform sampling. Consider a function $x(t)$ and its first derivative $x'(t)$ which are nonuniformly sampled. *Fig. 5* shows only the instants of sampling. We must have n distinct samples

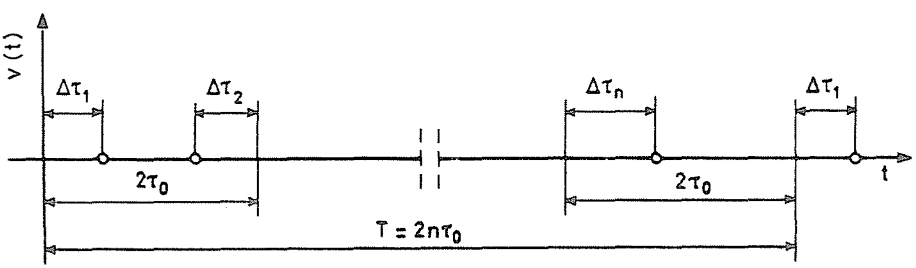

Fig. 5. Time instants for sampling of $x(t)$ and $x'(t)$

from $x(t)$ and $x'(t)$ at the same instants (all $2n$) at period $T = 2n\tau_0$. In a similar way to the one adapted in the previous section, we distinguish functions $v_1(t), v_2(t), \ldots, v_n(t)$, where $i = 1, 2, \ldots, n$ and multiply them by $x(t)$ and $x'(t)$.

Accordingly

$$
\begin{aligned}
y_1(t) &= x(t) \cdot v_1(t)\,, \\
&\vdots \\
y_i(t) &= x(t) \cdot v_i(t)\,, \\
&\vdots \\
y_n(t) &= x(t) \cdot v_n(t)\,, \\
y_{n+1}(t) &= x'(t) \cdot v_1(t)\,, \\
&\vdots \\
y_{2i}(t) &= x'(t) \cdot v_i(t)\,, \\
&\vdots \\
y_{2n}(t) &= x'(t) \cdot v_n(t)\,,
\end{aligned}
\tag{17}
$$

Therefore, the ith sampling function can be written as

$$
v_i(t) = \frac{1}{n} \sum_{k=-\infty}^{\infty} c_k A_i^k e^{j \frac{k\omega_0}{2n} t}\,. \tag{18}
$$

Observe that the period of the sampling function is $T = 2n\tau_0$ for the present case. The ith and the $2i$th sampled signals are the following

$$
\begin{aligned}
y_i(t) &= \left[\int_{-B}^{B} e^{j\omega t}\, d\beta(\omega) \right] \cdot \left[\frac{1}{2n} \sum_{k=-\infty}^{\infty} c_k A_i^k e^{j \frac{k\omega_0}{2n} t} \right], \\
y_{2i}(t) &= \left[\int_{-B}^{B} j\omega\, e^{j\omega t}\, d\beta(\omega) \right] \cdot \left[\frac{1}{2n} \sum_{k=-\infty}^{\infty} c_k A_i^k e^{j \frac{k\omega_0}{2n} t} \right].
\end{aligned}
\tag{19}
$$

Doing the multiplication in Eq. (19) and using the notation

$$
\Omega_k = \frac{k\omega_0}{2n} - \omega
$$

where $k = 0, 1, 2, \ldots, 2n-1$ furthermore after lengthy transformations we can write simultaneous equations in the following matrix form

$$
\mathbf{A}\,\mathbf{H} = \mathbf{N}\,. \tag{20}
$$

Here

$$\mathbf{A} = \begin{bmatrix} A_1^0 & \cdots & A_n^0 & \cdots & j\omega A_1^0 & \cdots & j\omega A_n^0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_1^{n-1} & \cdots & A_n^{n-1} & \cdots & j\Omega_{n-1} A_1^{n-1} & \cdots & j\Omega_{n-1} A_n^{n-1} \\ A_1^n & \cdots & A_n^n & \cdots & j\Omega_n A_1^n & \cdots & j\Omega_n A_n^n \\ \vdots & & \vdots & & \vdots & & \vdots \\ A_1^{2n-1} & \cdots & A_n^{2n-1} & \cdots & j\Omega_{2n-1} A_1^{2n-1} & \cdots & j\Omega_{2n-1} A_n^{2n-1} \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} H_1(\omega) \\ H_2(\omega) \\ \vdots \\ H_{2n}(\omega) \end{bmatrix} \quad \text{and} \quad \mathbf{N} = \begin{bmatrix} 2n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From Eq. (20) we can obtain

$$\mathbf{H} = \mathbf{A}^{-1}\mathbf{N}. \tag{21}$$

By considerations with the method presented in the previous section it can be proved that $\mathbf{A}^{-1}$ exists.

Now we can conclude that a signal $x(t)$ which is bandlimited to the interval $(-B, B)$, is completely specified by the knowledge of both n arbitrarily spaced distinct samples from $x(t)$ and in the same instants n samples from $x'(t)$ within an interval $T = 2n\tau_0$ but periodically recurring with T. (The number of all samples is $2n$.)

Periodic Nonuniform Sampling of a Periodic Signal

Let us have some heuristic expectation that a signal can be described by the knowledge of its local extrema or distances between zero crossings. This description will be called adaptive sampling in what follows as the instants of samples are not determined by a clock but are adapted to the signal. As the extrema of a signal cannot be given in a closed form, we have to look for some other way to solve the problem. For the sake of simplicity let us merely consider the periodic signals. However, the results can be generalized both in time and frequency domain.

Zero Crossings of a Periodic Signal

A bandlimited and periodic signal of practical interest always can be written in the following form

$$x(t) = \sum_{n=n_L}^{n_H} (a_n \sin n\omega t + b_n \cos n\omega t) = \sum_{n=n_L}^{n_H} c_n \sin(n\omega t + \Phi_n), \qquad (22)$$

where n is a positive integer and $n_L \le n_H$. It is sure that $x(t)$ is periodic with $T = 2\pi/\omega$ which is not always the shortest time period. If we denote the shortest time period by T_p, then $T_p \le T$. Let the instant $t = 0$ be at a zero crossing with a positive slope of signal $x(t)$ (*Fig. 6*). The signal $x(t)$

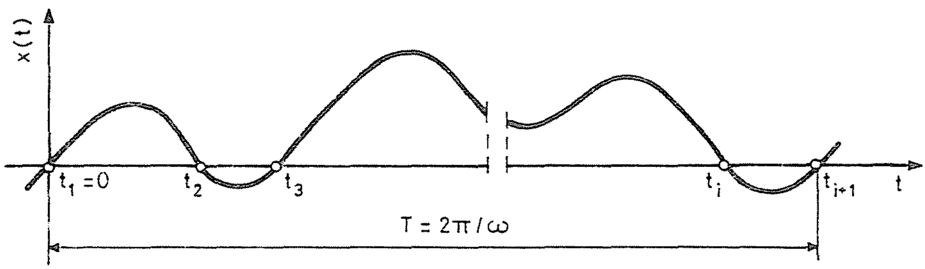

Fig. 6. Zero crossings of a signal in the interval $[0, T)$

has i zero crossings in $[0, T)$. It can be proved that

$$h(n_L\omega) \le i \le h(n_H\omega), \qquad (23)$$

where $h(n_L\omega)$ and $h(n_H\omega)$ are the numbers of zero crossings of the lowest and the highest components in Eq. (22) in the same interval. To prove this statement we write the gth primitive function of $x(t)$ as

$$G(t) = \underset{1}{\int} \cdots \underset{g}{\int} x(t)\, dt \cdots dt. \qquad (24)$$

The function $G(t)$ is periodic with period T, as well, since all of linearly transformed functions are periodic with the same period. The Fourier series of $G(t)$ have a similar form as Eq. (22). It is sufficient to examine the magnitude of components. They are the following

$$\frac{c_{n_L}}{(n_L\omega)^g}, \frac{c_{n_L} + 1}{((n_L + 1)\omega)^g}, \ldots, \frac{c_{n_H}}{(n_H\omega)^g}. \qquad (25)$$

Referring to foregoing series, it may be seen that the first term could be arbitrarily large comparing with the others if we increase the value of g. So we can obtain

$$G(t) = \frac{c_{n_L}}{(n_L\omega)^g} \sin\left[n_L\omega t + \Phi_L + g\frac{\pi}{2}\right] + \epsilon(t)\,. \tag{26}$$

The number of zero crossings $G(t)$ is given by the frequency of component $\sin(n_L\omega t + \varphi)$. Denote by $h(n_L\omega)$ the number of zero crossings of this component at interval $[0, T)$. If we make the first derivative function of $G(t)$ then we obtain the $(g-1)$th primitive function which has zero crossings at the local extrema of $G(t)$. By Rolle's theorem the derivative function must be zero at least once between two successive zero crossings. So the $(g-1)$th primitive function has at least the same number of zero crossings as the function $G(t)$. Continuing that until we get back $x(t)$, thus we may write

$$h(n_L\omega) \leq i\,. \tag{27}$$

Using now the gth derivative function of $x(t)$, we can easily prove, in a similar way, that

$$i \leq h(n_H\omega)\,. \tag{28}$$

Consequently, if a signal $x(t)$ is periodic with T and has i zero crossings at interval $[0, T)$ then the following inequality holds:

$$2n_L \leq i \leq 2n_H\,. \tag{29}$$

Adaptive Sampling of a Periodic Signal

Let $x(t)$ be a periodic signal. By Eq. (22)

$$x(t) = \sum_{n=n_L}^{n_H} c_n \sin(n\omega t + \Phi_n)\,. \tag{30}$$

Now we consider the samples of this function at the local extrema as shown in *Fig. 7*. Let

$$x_1 = x(t_1),\, x_2 = x(t_2), \ldots, x_i = x(t_i)\,,$$

and

$$\Delta t_1 = t_1 - t_0,\, \Delta t_2 = t_2 - t_1,\, \ldots, \Delta t_i = t_i - t_{i-1}\,.$$

Accordingly

$$T = \sum_{k=1}^{i} \Delta t_k\,. \tag{31}$$

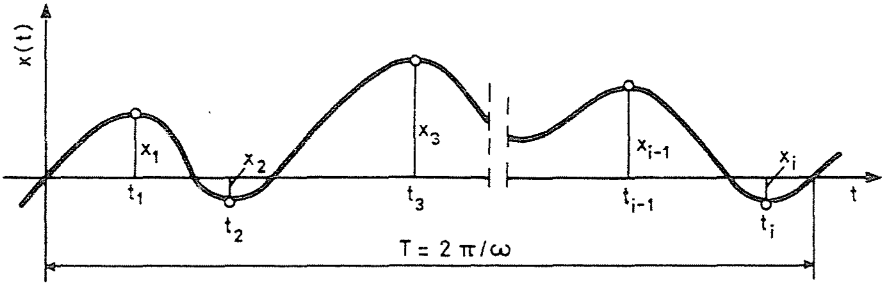

Fig. 7. Adapted samples of a signal

Statement. If $\omega_H < 2\omega_L$ then the function $x(t)$ is uniquely determined by the set $\{x_1, \Delta t_1, x_2, \Delta t_2, \ldots, x_i, \Delta t_i\}$. In other words, if the Fourier components of $x(t)$ possess a band limit ratio less than an octave then the function is completely specified by the knowledge of its adapted samples.

Proof. If we have $2i$ uniform samples at the interval $[0, T)$, as can be seen in *Fig. 8*,then $\tau_0 = T/2i$ and the frequency of sampling is

$$\omega_0 = \frac{2\pi}{\tau_0} = 2\pi\frac{2i}{T}. \tag{32}$$

From inequality Eq. (29) it results that the signal has at least $i = 2n_L$ zero

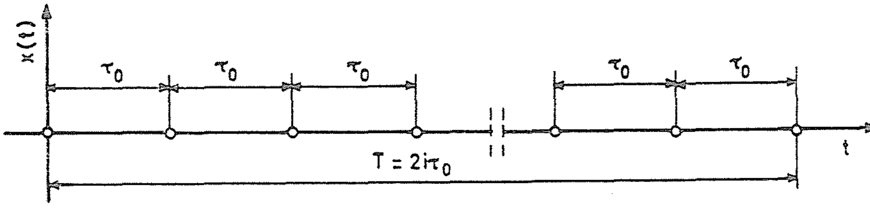

Fig. 8. Equivalent uniformly placed sampling instants

crossings and by substituting this in Eq. (32) we obtain

$$\omega_0 = \frac{2\pi}{T}(2.2n_L) = 4n_L\omega = 4\omega_L. \tag{33}$$

Since $\omega_H < 2\omega_L$, using Eq. (33) it can be written

$$2\omega_H < \omega_0. \tag{34}$$

The signal $x(t)$ is uniquely described with i uniform spaced samples, namely $2n_L \leq i$, so the inequality Eq. (34) is always satisfied. In the previous section it was proved that a bandlimited signal is completely specified by the knowledge of n arbitrary spaced distinct samples from $x(t)$ and $x'(t)$ at the interval $[0, T)$, where $T = 2n\tau_0$. The periodic signal $x(t)$ has at least i local extrema at interval $[0, T)$, hence $T = 2i\tau_0$.

Now we conclude that a periodic signal which has a band limit ratio less than an octave can be uniquely reconstructed from a set of its adapted samples.

Basic Properties of the Adaptive Sampling

Before we examine the basic properties of the adaptive sampling, let us consider the sum of periodic signals. If a signal $x_1(t)$ is periodic with T_1 and $x_2(t)$ with T_2 then the sum $x(t) = x_1(t) + x_2(t)$ can be periodic with T only if

$$T = n_1 T_1 = n_2 T_2 \,, \tag{35}$$

where n_1 and n_2 are positive integers. Let us suppose that there exists a transformation Θ which determines the time period of a periodic function, so

$$T_1 = \Theta\{x_1(t)\}\,, \quad T_2 = \Theta\{x_2(t)\}\,. \tag{36}$$

This transformation Θ is nonlinear since, as it can be seen from Eq. (35), the principle of superposition is not satisfied, that is,

$$\Theta\{x_1(t) + x_1(t)\} \neq \Theta\{x_1(t)\} + \Theta\{x_2(t)\}\,. \tag{37}$$

As we have already mentioned, the adaptive sampling is a transformation which determines a set of samples of a bandlimited periodic signal. We shall denote this with AS (Adaptive Sampling) and examine its basic properties.

Property 1. *The adaptive sampling in a geometrical sense maps similar signals into similar sets, that is*

$$\begin{aligned} AS\{x(t)\} &= \{x_1, \Delta t_1, x_2, \Delta t_2, \cdots, x_i, \Delta t_i\}\,, \\ AS\{\alpha x(\beta t)\} &= \{\alpha x_1,\, \beta\Delta t_1,\, \alpha x_2,\, \beta\Delta t_2,\, \cdots\,, \alpha x_i,\, \beta\Delta t_i\}\,. \end{aligned}$$

This property is evident by referring to *Fig. 7.* In addition, the adaptive sampling is time-invariant, since

$$AS\{x(t-\tau)\} = \{x_1,\, \Delta t_1,\, x_2,\, \Delta t_2,\, \cdots\,, x_i,\, \Delta t_i\}\,.$$

Property 2. *If a signal $x(t)$ is periodic then the adapted samples are periodic, too, that is $x_k = x_{k+i}$ and $\Delta t_k = \Delta t_{k+i}$. Thus the principle of superposition is not satisfied so*

$$AS\{x_1(t) + x_1(t)\} \neq AS\{x_1(t)\} + AS\{x_2(t)\} .$$

Every linear transformation possesses the first property but the adaptive sampling is a nonlinear transformation as we have proven before.

Property 3. *If we perturb the adapted samples of the signal $x(t)$ so that*

$$T = \sum_{k=1}^{i} \Delta t_k + \delta t_k$$

we obtain the following set of samples

$$\{x_1 + \delta x_1, \Delta t_1 + \delta t_1, \cdots, x_i + \delta x_i, \Delta t_i + \delta t_i\} .$$

Of course, this set is periodic with period T. Denote by $\tilde{x}(t)$ the function which has these samples. (This function in special case can be zero, too.) The function $\tilde{x}(t)$ may be obtained from linear transformation of $x(t)$. Thus we can write $\tilde{x}(t) = x(t) * h(t)$ where $h(t)$ denotes the weighting function of a linear filter.

Property 4. *If a signal $x(t)$ is periodic with period T then the adapted samples of each subband have to be periodic with T, as well.*

In other words in each subband there must be zero crossings with positive slopes and with the same time interval T between them.

Property 5. *If the signal $x(t)$ is periodic then all of its linearly transformed functions are periodic with the same period, so that*

$$x(t) = x(t+T), x^{(1)}(t) = x^{(1)}(t+T), \cdots, x^{(i-1)}(t) = x^{(i-1)}(t+T) .$$

Thus the adapted samples of derivatives are periodic with period T, as well. If we have a T^H hypothesis for period T then we can make a decision at an interval $T + \Delta t$. (We need not have a window with length $2T$).

Property 6. *If the function $x(t)$ has $i > 2n$ zero crossings at interval $[0, T)$ then we have more adapted samples than necessary hence the surplus can be left out of consideration.*

Property 7. *Let $x(t) = x_p(t) + \xi_t$ where $x_p(t)$ is periodic and ξ_t is a Gaussian process, moreover*

$$M(\xi_t^2) \ll \frac{1}{T} \int_{-T/2}^{T/2} x_p^2(t)\, dt.$$

The signal $x_p(t)$ is nearly linear at zero crossings. Consequently, the distance, e.g., between two zero crossings has Gaussian distribution. It can be proved that an estimated T from adapted samples is an unbiased and efficient estimate of T.

Conclusion and Outlook

The first two properties of adaptive sampling may seem contradictory, as every linear transformation possesses the first property, while every non-linear transformation possesses the second property. We have shown, that the considered adaptive sampling has both.

If we examine the instantaneous amplitude and the instantaneous frequency of a signal then it is easy to show that they have similar properties (KÜPFMÜLLER (1949); CIZEK (1987)). Furthermore, the Wigner distribution (CLAASEN and MECKLENBRÄUKER (1984)) is a member of a special class of bilinear, shift-invariant transformation and it has similar properties, too. The adaptive sampling makes it possible to modelize the human hearing. Our knowledge of the hearing (FLETCHER (1957); BÉKÉSY (1960); FELDTKELLER and ZWICKER (1967)) is in accordance with the properties of adaptive sampling.

The psychophysical investigation of hearing (GROBBEN (1971)) proved that the uncertainty relation of the spectrogram (GABOR (1947)) is not valid for short tones. (WOKUREK et al (1987)) described that the Wigner distribution allows arbitrarily high resolutions.The properties of adaptive sampling suggest that the approach may be useful for detection of pitch frequency. Combining the *AMDF* (Average Magnitude Difference Function) method (see Ross et al (1974)) with adaptive sampling may be, in this respect, of particular interest. Of course, if we have got the same time period in more subbands, a better decision is expected.

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