

HIGH ACCURACY FREQUENCY DETERMINATION FROM DISCRETE SPECTRA

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Abstract

The problem of determining the characteristics of a sine wave from its discrete spectrum is considered. The nontriviality of the problem is caused basically by a phenomenon called spectral leakage, that is, by the fact that the spectral envelope of a single sinusoid forms a bell-shaped curve, even in the ideal noiseless case. In the paper a simple and self-contained treatment of spectral leakage is presented and a computationally efficient frequency estimation method is derived, taking into consideration different types of time-domain windows.

Introduction

The development of digital signal processing involves the frequent application of Discrete Fourier Transformation (DFT) or Fast Fourier Transformation (FFT). But, at the same time, discrete spectra raise some new problems, which have been out of sight in connection with continuous spectra.

One of the important problems is the following: if the input signal to an FFT analyzer contains a sine wave, then, even in an ideal noiseless case, it is not trivial to determine its frequency with high accuracy from the discrete spectrum. This difficulty is caused by the contradiction between the continuity of input frequency and the discrete position of resulting spectral lines: if the frequency of input sinusoid is not an integer multiple of the base frequency, then this sine wave is represented by more than one spectral lines, even in an ideal noiseless case. The envelope of these lines forms a bell-shaped curve, called selectivity curve.

The phenomenon is known as spectral leakage and several attempts have been made to develop methods of higher accuracy for the estimation of the exact frequency from discrete spectra (see [1], . . . , [5]).

Here we present a simple and self-contained treatment of spectral leakage and a computationally efficient method for high accuracy frequency estimation from discrete spectra with different time domain windows.

The work outlined here has been part of a research project at the Polytechnical University of Budapest, Microwave Department, to evaluate

radiofrequency signals in course of spectrum monitoring ([6], [7], [8]). Nevertheless, the results are of wider interest, therefore, we present them in general form, disregarding to the original special motivation.

Selectivity curve for a single sinusoid with different timedomain windows

We consider the sampled signal

$$s(k\Delta t) = A \sin(2\pi f k \Delta t + \varphi) \quad (k=0, 1, \dots, N-1) \quad (1)$$

where $f = \lambda f_0$, $f_0 = 1/T$, $T = N \Delta t$.

The complex spectral lines of this signal for rectangular window are

$$S_i = \frac{1}{N} \sum_{k=0}^{N-1} s(k\Delta t) e^{-j2\pi i f_0 k \Delta t} \quad (2)$$

(by the definition of DFT). Using the particular form of $s(k\Delta t)$ and applying Euler's formula S_i can be written as

$$\begin{aligned} S_i &= \frac{A}{2jN} \sum_{k=1}^{N-1} e^{j(2\pi\lambda f_0 k \Delta t + \varphi)} \cdot e^{-j2\pi i f_0 k \Delta t} - \\ &\quad - \frac{A}{2jN} \sum_{k=0}^{N-1} e^{-j(2\pi\lambda f_0 k \Delta t + \varphi)} \cdot e^{-j2\pi i f_0 k \Delta t} \end{aligned} \quad (3)$$

After elementary calculations, using the summation formula for finite geometric series, we can get

$$\begin{aligned} S_i &= \frac{A e^{j\varphi}}{2jN} \cdot \frac{e^{j2\pi\Delta\lambda} - 1}{e^{j2\pi\Delta\lambda/N} - 1} - \\ &\quad - \frac{A e^{j\varphi}}{2jN} \cdot \frac{e^{-j2\pi\Delta\lambda} - 1}{e^{j2\pi(\Delta\lambda + 2i)/N} - 1} \end{aligned} \quad (4)$$

where $\Delta\lambda = \lambda - i$. Expression (4) gives the exact complex valued selectivity curve for a single sinusoid (with rectangular window). If λ is an integer (that is, the frequency of the sinusoid is an integer multiple of f_0) then the only nonzero value of S_i is S_λ . Otherwise (if λ is not integer), the absolute value of the S_i 's form a sampled $(\sin x)/x$ shaped curve, as we shall see below. This phenomenon is called spectral leakage. The main problem appears now as determining the frequency of the input signal from the S_i 's by means of possibly little computational effort, but at the same time as exactly as possible.

Since the exact formula for S_i is too complicated for further considerations, it is useful to derive an approximation. Let M_i and H_i be the first

and second members of expression (4), respectively. That is, $S_i = M_i + H_i$. We are going to show the legality of the approximation $S_i \cong M_i$. The relative error of this approximation is

$$e_{rel} = \left| \frac{S_i^{(exact)} - S_i^{(approx.)}}{S_i^{(exact)}} \right| = \left| \frac{(M_i + H_i) - H_i}{M_i + H_i} \right| = \left| \frac{H_i/M_i}{1 + H_i/M_i} \right| \quad (5)$$

If $|H_i/M_i|$ is small enough, then

$$e_{rel} \cong |H_i/M_i| \quad (6)$$

so it suffices to show that $|H_i/M_i| \ll 1$.

We can assume that $2\pi\Delta\lambda/N \ll 1$, which implies the approximation

$$e^{j2\pi\Delta\lambda/N} \cong 1 + j2\pi\Delta\lambda/N \quad (7)$$

can be used with negligible error. Then $|H_i/M_i|$ can be written as

$$\left| \frac{H_i}{M_i} \right| = \frac{|2\pi\Delta\lambda/N|}{|e^{-j4\pi i/N} - (1 + j(2\pi\Delta\lambda) \cdot e^{-j4\pi i/N} \cdot \frac{1}{N})|} \quad (8)$$

Because of the large value of N (e.g $N = 1024$) the last imaginary member in the denominator of expression (8) is negligible. Thus

$$\left| \frac{H_i}{M_i} \right| \cong \frac{|2\pi\Delta\lambda/N|}{|e^{-j4\pi i/N} - 1|} = \frac{\pi\sqrt{2}}{N} \cdot \frac{\Delta\lambda}{\sqrt{1 - \cos(4\pi i/N)}} \quad (9)$$

Expression (9) shows that if we consider the “important part” of the spectrum, which is the largest spectral line and its neighbourhood (where $|\Delta\lambda|$ is small) and also i is not extremely small or large in the range $(0, N/2)$, then $|H_i/M_i|$ is in fact negligible. For example, if $|\Delta\lambda| \leq 1.5$ (that is, we consider the largest spectral line and its two neighbours), $N/12 < i < 5N/12$ and $N = 1024$; then

$$|H_i/M_i| \approx 0.01$$

so the relative error is about 1%. Thus we can use the approximation $S_i \cong M_i$, that is

$$S_i \cong \frac{Ae^{j\varphi}}{2jN} \cdot \frac{e^{j2\pi\Delta\lambda} - 1}{e^{j2\pi\Delta\lambda/N} - 1} \quad (10)$$

Applying (7), the result can be made simpler:

$$S_i \cong -\frac{Ae^{j\varphi}}{4\pi} \cdot \frac{e^{j2\pi\Delta\lambda} - 1}{\Delta\lambda} \quad (11)$$

It is worth to remark that N is no longer part of the expression.

If we are interested in the shape of the selectivity curve, it is useful to take the absolute value of (11):

$$|S_i| \cong A \frac{|e^{j2\pi\Delta\lambda} - 1|}{4\pi|\Delta\lambda|} = \frac{A}{2} \left| \frac{\sin(x)}{x} \right| \quad (12)$$

where $x = \pi\Delta\lambda$. This shows clearly the main shape.

Turning back to the exact expression (4), we can realize that the second member can be regarded as a perturbation caused by an other selectivity curve of the same type, whose center is positioned symmetrically at the frequency $-f$. We refer this perturbation as *mirror-effect*. Thus, the approximative formula (10) is the result of neglecting the mirror-effect. But it still saves the periodicity with respect to N (that is, $S_{i+N} = S_i$), which is a consequence of sampling. The second approximative formula (11) does not show this periodicity any longer. For this reason, it is acceptable first of all if we are near to the center of the selectivity curve and far from the endpoints of the range $(0, N/2)$.

It is natural to ask whether the exact expression (4) can be factorized such that the different effects become separated? These effects are the amplitude and phase of the input sinusoid, the main shape of selectivity curve caused by spectral leakage, the effect of spectral periodicity caused by sampling and the mirror-effect. A somewhat lengthy but elementary calculation shows that this factorization in fact can be done and expression (4) can be reformulated as

$$S_i = \frac{1}{2} A e^{j\varphi} F_i^{(1)} F_i^{(2)} F_i^{(3)} \quad (13)$$

Using the abbreviation $x = \pi\Delta\lambda$, the factors $F_i^{(1)}$, $F_i^{(2)}$, $F_i^{(3)}$ in (13) are the following:

— Main shape of selectivity curve caused by spectral leakage:

$$F_i^{(1)} = \frac{\sin(x)}{x} e^{jx} \quad (14)$$

— Spectral periodicity caused by sampling:

$$F_i^{(2)} = \frac{2x/N}{\cos(2x/N) - 1 + j\sin(2x/N)} \quad (15)$$

— Mirror-effect:

$$F_i^{(3)} = 1 - e^{-j2\varphi} e^{-j2\arctan\left(\frac{2x}{2x-1}\right)} \cdot \frac{e^{-j2x/N} - 1}{e^{-j2x/N} \cdot e^{-j4\pi i/N} - 1} \quad (16)$$

All the considerations have been done so far for rectangular time-window. This is the simplest one, but its main disadvantage that moving away along the frequency axis from the frequency of the input sine wave, the

selectivity curve does not decrease fast enough. The rate of decreasing is roughly proportional to $1/\Delta\lambda$, as it can be seen from expression (11). This gives the approximate rate of 6 dB/octave.

The need for a rapidly decreasing selectivity curve is based on the intention that in the presence of more sinusoids we should like to handle them separately, that is, we wish to neglect the interrelation among the selectivity curves. Obviously this causes less error in the case of a more rapidly decreasing curve. A good solution of wide practical use is the well-known Hanning-window, which weights the input signal by the function

$$w(t) = \frac{1}{2}(1 - \cos(2\pi f_0 t)) \quad (17)$$

This guarantees—as we shall see below—a rate of decreasing which is roughly proportional to $1/(\Delta\lambda)^3$, that is approximately 18 dB/octave.

The price we have to pay for the less interrelation among selectivity curves belonging to different sine waves is that the top of the curve becomes flatter. In other words, there is more uncertainty in the frequency determination of a particular sinusoid. In general, selectivity curves belonging to different time-windows show the behavior which could be characterized by a certain trade-off between far-selectivity (that is, the resistance against interrelation) and the flatness of the top (the uncertainty in frequency determination).

A possible compromise between rectangular and Hanning-windows is their linear combination, called Hamming-window. It can be described by the function:

$$w(t) = \frac{1}{1+a} \left(1 - a \cos(2\pi f_0 t) \right) \quad (0 \leq t \leq T) \quad (18)$$

The parameter “ a ” varies in the range $[0,1]$. If $a=0$, we get back the rectangular window; $a=1$ gives Hanning-window and the intermediate parameter values present different Hamming-windows. We are going to give a unified treatment for the above mentioned window types, letting the results depend on parameter “ a ”.

Suppose now, the input sine wave is weighted by the window function given by (18). Denote the i^{th} complex spectral line of the corresponding DFT by $S_i^{(a)}$. Making use of trigonometrical identities we can easily express $S_i^{(a)}$ by the spectral lines of the unweighted (that is, rectangularly windowed) sinusoid:

$$S_i^{(a)} = \frac{1}{1+a} S_i - \frac{1}{2} \frac{1}{1+a} \left(S_{i-1} + S_{i+1} \right) \quad (19)$$

Substituting expression (4) for S_i we could get an exact formula for $S_i^{(a)}$, but that would be too complicated for the applications. Instead, substituting the approximative formula (11) we conclude:

$$S_i^{(a)} \cong -\frac{1}{1+a} \cdot \frac{Ae^{j\varphi}}{4\pi} \left(e^{-j2\pi\Delta\lambda} - 1 \right) \cdot \frac{(1-a)\Delta\lambda^2 - 1}{\Delta\lambda(\Delta\lambda^2 - 1)} \quad (20)$$

Naturally, for $a=0$ we get back (11), the formula for rectangular window. Substituting $a=1$, we are given the case of Hanning-window:

$$S_i^{(H)} \cong \frac{Ae^{j\varphi}}{8\pi} \cdot \frac{e^{j2\pi\Delta\lambda} - 1}{\Delta\lambda(\Delta\lambda^2 - 1)} \quad (21)$$

The improved far-selectivity of Hanning-window (compared to the rectangular one) can easily be read from (21): the rate of decreasing is roughly proportional to $1/\Delta\lambda^3$, which corresponds to about 18 dB/octave.

Determination of the characteristics of input sinusoid from selectivity curve

Assuming the input signal is a single sine wave, we wish to determine its frequency, amplitude and phase from its DFT. The nontriviality of this problem is caused basically by spectral leakage.

Obviously, we need to make use of the information hidden in the spectral lines of great magnitude, since the smaller ones are already not reliably measured because of the unavoidable presence of noise. Denote the index of spectral line of maximum magnitude by i_0 and let

$$\alpha = \frac{|S_{i_0}^{(a)}|}{|S_{i_0-1}^{(a)}|} \quad (22)$$

Substituting formula (20), after cancellations we get

$$\alpha = \frac{2 + \Delta\lambda}{1 - \Delta\lambda} \cdot \frac{1 - (1-a)\Delta\lambda^2}{|(1-a)\Delta\lambda^2 + 2(1-a)\Delta\lambda - a|} \quad (23)$$

The relation between α and $\Delta\lambda$ can be easily tabulated and since α and i_0 are known from the DFT, thus the frequency of the input sine wave can be reached as $f = (i_0 + \Delta\lambda)f_0$, after determining the value of $\Delta\lambda$ from (23).

In the most important special cases expression (23) becomes much simpler. If $a=0$ (rectangular window), then

$$\alpha = \frac{\Delta\lambda + 1}{|\Delta\lambda|} \quad (24)$$

which yields

$$\Delta\lambda = \begin{cases} \frac{1}{\alpha-1} & \text{if } |S_{i_0+1}| > |S_{i_0-1}| \\ -\frac{1}{\alpha-1} & \text{if } |S_{i_0+1}| < |S_{i_0-1}| \\ 0 & \text{if } |S_{i_0+1}| = |S_{i_0-1}| \end{cases} \quad (25)$$

In the case of Hanning-window ($a = 1$) expression (23) becomes

$$\alpha = \frac{2 + \Delta\lambda}{1 - \Delta\lambda} \quad (26)$$

which yields

$$\Delta\lambda = \frac{\alpha - 2}{\alpha + 1} \quad (27)$$

Having determined the frequency f , then the amplitude A and phase φ can be calculated in a straightforward way by resubstituting $\Delta\lambda$ into the expression of S_i , plausibly taking i_0 for i .

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