FINITE ELEMENT METHOD FOR THE COMPUTATION OF THE ELECTROMAGNETIC FIELD OF WAVEGUIDES

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Abstract

The electromagnetic field of waveguides is customarily derived from an electric or magnetic vector potential. A functional may be written for the potentials which attains extrem value if the vector potentials satisfy the wave equation. The studied region is subdivided into triangles within which linear approximation is employed. In the present case the integral in the functional can be written in an analytical form. Having sought the extremum and taking the different boundary conditions of TM and TE modes into account, a set of linear equations is obtained. The unknowns are the nodal vector potential values. The set is solved by computer, and the method is illustrated by examples.

The electromagnetic field of waveguides is usually derived with the aid of an electric or magnetic vector potential A^{em}. With the axis of the waveguide chosen as the z axis, the z-directed vector potential can be written in Cartesian coordinates in the product form

$$A = \Theta(x, y)e^{-\gamma z} \tag{1}$$

where the term $e^{-\gamma z}$ describes the variation in the z-direction and γ is the propagation coefficient in the z-direction. Θ depends upon the transversal coordinates only. In case of harmonic time-variation Θ satisfies the wave equation

$$\Delta\Theta + k^2\Theta = 0 \tag{2}$$

where k^2 and γ^2 are related as

$$k^2 = \gamma^2 + \omega^2\mu\varepsilon. \tag{3}$$

Eq. (2) can also be obtained by equating the first variation of the functional

$$I = \int_S (\text{grad}^2\, \Theta - k^2\Theta^2)dS \tag{4}$$

with zero. Here S denotes the cross section of the waveguide in the transversal plane and the gradient should be taken with respect to the transversal coordinates. The functional (4) may attain an extremal value, if its first variation vanishes i.e.

$$\delta I = -2\int_S \eta(\Delta\Theta + k^2\Theta)dS - \oint_l \eta\frac{\partial\Theta}{\partial n}dl = 0 \tag{5}$$

where l is the curve bounding the cross section S and $\frac{\partial\Theta}{\partial n}$ is the derivative of Θ with respect to the outer normal.

The function $\eta(x, y)$ appearing here is continuous in S along with its first two derivatives and, in case of Dirichlet boundary conditions, it vanishes on l, otherwise it is arbitrary.

Eq. (5) may hold only if both the surface and line integrals are zero. The surface integral equals zero irrespective of the choice of η, if Eq. (2) is satisfied. The line integral is zero at Dirichlet boundary conditions since then $\eta = 0$ on l, whereas at homogeneous Neumann boundary conditions it vanishes because then $\frac{\partial\Theta}{\partial n} = 0$ on l. It is thus evident from Eq. (5) that the homogeneous Neumann boundary condition is a natural boundary condition of the functional (4).

The integral (4) can be approximately written with the aid of the finite element method. To this end the region S is divided into m triangles with the aid of nodes selected on the boundary and in the interior of the region and the

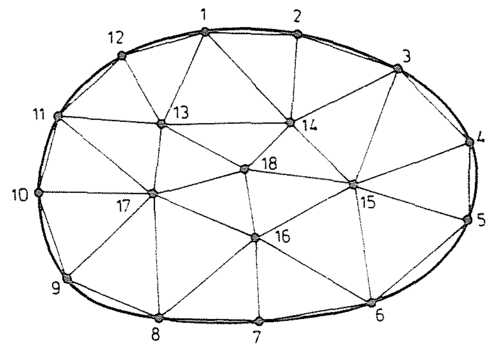

Fig. 1

curve l is approximated by a curve formed by the relevant triangle edges (Fig. 1). The triangles and their nodes are given order numbers. The global order numbers of the nodes on the curve l are $1, 2, \ldots n_1$, and those for the nodes inside the region are $n_1+1, n_1+2, \ldots, n$. The three nodes of the jth triangle are also assigned local order numbers: j_1, j_2, j_3 (Fig. 2). The function $\Theta(x, y)$ is approximated in the jth triangle by the linear function

$$\Theta(x, y) = a_{j0} + a_{j1}x + a_{j2}y, \tag{6}$$

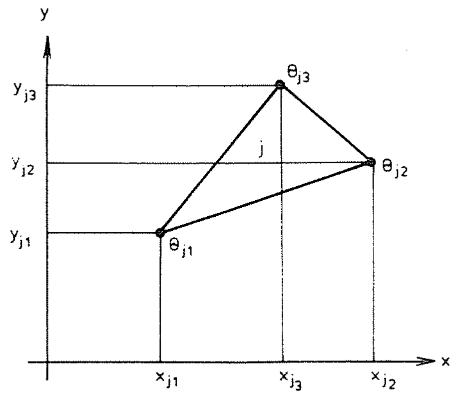

Fig. 2

the values Θ_i ($i=1, 2\ldots, n$) corresponding to the nodes of the triangular subregions are regarded as unknowns and the column matrix formed by these is denoted by $\boldsymbol{\Theta}_0$. A column matrix is also formed by the values Θ_j, corresponding to the nodes of the jth triangle and this is denoted by $\boldsymbol{\Theta}_j$. $\boldsymbol{\Theta}_j$ can be expressed with the aid of $\boldsymbol{\Theta}_0$ as

$$\boldsymbol{\Theta}_j = \boldsymbol{G}_j \boldsymbol{\Theta}_0. \tag{7}$$

The three rows of $\boldsymbol{G}_j$ correspond to the nodes of the jth triangle, and its columns to the nodes $1, 2, \ldots, n$ selected in the region S. The kth element in the ith row of $\boldsymbol{G}_j$ is 1, if the global order number of the node j, in the jth triangle is k, otherwise the element is zero.

If the coordinates of the nodes of the jth triangle are (x_{j1}, y_{j1}), (x_{j2}, y_{j2}), (x_{j3}, y_{j3}) then according to (6)

$$\boldsymbol{\Theta}_j = \begin{bmatrix} 1 & x_{j1} & y_{j1} \\ 1 & x_{j2} & y_{j2} \\ 1 & x_{j3} & y_{j3} \end{bmatrix} \begin{bmatrix} a_{j0} \\ a_{j1} \\ a_{j2} \end{bmatrix}. \tag{8}$$

Hence

$$\operatorname{grad} \Theta_j = \begin{bmatrix} a_{j1} \\ a_{j2} \end{bmatrix} = \frac{1}{D_j} \begin{bmatrix} y_{j2}-y_{j3} & y_{j3}-y_{j1} & y_{j1}-y_{j2} \\ x_{j3}-x_{j2} & x_{j1}-x_{j3} & x_{j2}-x_{j1} \end{bmatrix} \boldsymbol{\Theta}_j =$$

$$= \boldsymbol{F}_j \boldsymbol{\Theta}_j \tag{9}$$

where

$$D_j = (x_{j2}-x_{j1})(y_{j3}-y_{j2}) - (x_{j3}-x_{j1})(y_{j2}-y_{j1}). \tag{10}$$

Substituting (7) into (9)

$$\operatorname{grad} \Theta_j = \boldsymbol{F}_j \boldsymbol{G}_j \boldsymbol{\Theta}_0 = \boldsymbol{H}_j \boldsymbol{\Theta}_0. \tag{11}$$

Its square is

$$\operatorname{grad}^2 \Theta_j = \boldsymbol{\Theta}_0^T \boldsymbol{H}_j^T \boldsymbol{H}_j \boldsymbol{\Theta}_0 \tag{12}$$

where $\boldsymbol{\Theta}_0^T$ and $\boldsymbol{H}_j^T$ are obtained by transposing $\boldsymbol{\Theta}_0$ and $\boldsymbol{H}_j$. Since $\operatorname{grad}^2 \Theta_j$ is constant in the jth triangle, its integral over the triangle equals its product with the area S_j. So the first term on the right hand side of (4) can be approximately written as

$$I_1 = \int_S \operatorname{grad}^2 \Theta \, dS \approx \boldsymbol{\Theta}_0^T \sum_{j=1}^{m} S_j \boldsymbol{H}_j^T \boldsymbol{H}_j \boldsymbol{\Theta}_0. \tag{13}$$

The part over the jth triangle in the second term of the integral (4) can be expressed with the aid of the values $\Theta_{j1}, \Theta_{j2}, \Theta_{j3}$ of Θ assumed in the nodes of the jth triangle if Θ_j varies as in Eq. (6) (Fig. 2).:

$$I_{2j} = -\int_S k^2 \Theta_{j2} dS = -k^2 \frac{S_j}{6} (\Theta_{j1}^2 + \Theta_{j2}^2 + \Theta_{j3}^2 + \\ + \Theta_{j1}\Theta_{j2} + \Theta_{j2}\Theta_{j3} + \Theta_{j3}\Theta_{j1}). \tag{14}$$

Introducing the notation

$$\boldsymbol{R} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \tag{15}$$

[6] and using Eq. (7)

$$I_{2j} = -k^2 \frac{S_j}{6} \boldsymbol{\Theta}_j^T \boldsymbol{R} \boldsymbol{\Theta}_j = -k^2 \frac{S_j}{6} \boldsymbol{\Theta}_0^T \boldsymbol{G}_j^T \boldsymbol{R} \boldsymbol{G}_j \boldsymbol{\Theta}_0. \tag{16}$$

Over the cross section S:

$$I_2 = -\int_S k^2 \Theta^2 dS \approx \sum_{j=1}^{m} I_{2j} = \\ = -\frac{k^2}{6} \boldsymbol{\Theta}_0^T \sum_{j=1}^{m} S_j \boldsymbol{G}_j^T \boldsymbol{R} \boldsymbol{G}_j \boldsymbol{\Theta}_0. \tag{17}$$

According to Eqs. (13) and (17) the functional (4) can be approximated as

$$I=I_1+I_2\approx\boldsymbol{\Theta}_0^T\sum_{j=1}^{m}S_j\left[\boldsymbol{H}_j^T\boldsymbol{H}_j-\frac{k^2}{6}\boldsymbol{G}_j^T\boldsymbol{R}\boldsymbol{G}_j\right]\boldsymbol{\Theta}_0. \tag{18}$$

Introducing the notations

$$\boldsymbol{P}=\sum_{j=1}^{m}S_j\boldsymbol{H}_j^T\boldsymbol{H}_j, \tag{19}$$

$$\boldsymbol{Q}=-\frac{1}{6}\sum_{j=1}^{m}S_j\boldsymbol{G}_j^T\boldsymbol{R}\boldsymbol{G}_j \tag{20}$$

the formula

$$I\approx\boldsymbol{\Theta}_0^T(\boldsymbol{P}-k^2\boldsymbol{Q})\boldsymbol{\Theta}_0 \tag{21}$$

is obtained.

In the following the TE mode solution is first discussed and thereafter the TM one is treated.

In case of TE mode the electric vector potential A^e namely its term Θ depending upon the transversal coordinates satisfies homogeneous Neumann boundary condition on the boundary which is a natural boundary condition of the variational problem. (21) may be extremal if

$$\frac{\partial I}{\partial\boldsymbol{\Theta}_0}=2(\boldsymbol{P}-k^2\boldsymbol{Q})\boldsymbol{\Theta}_0=0. \tag{22}$$

This set of homogeneous linear equations may have nontrivial solution only if

$$\det|\boldsymbol{P}-k^2\boldsymbol{Q}|=0 \tag{23}$$

i.e. if k^2 is an eigenvalue of the matrix $\boldsymbol{Q}^{-1}\boldsymbol{P}$. Different TE mode solutions correspond to the different eigenvalues with their propagation coefficients obtainable from Eq. (3). Each of the values k^2 at a particular mode involves an eigenvector $\boldsymbol{\Theta}_0$. Its elements are the approximate values of Θ in the nodes.

In case of TM mode the term Θ of the magnetic vector potential A^m satisfies homogeneous Dirichlet boundary condition on the boundary of the region S, i.e. Θ vanishes everywhere on the boundary. Hence the first n_1 elements of the column vector $\boldsymbol{\Theta}_0$ are zero, so $\boldsymbol{\Theta}_0$ can be partitioned as follows

$$\boldsymbol{\Theta}_0=\begin{bmatrix}\boldsymbol{\Theta}_{01}\\ \boldsymbol{\Theta}_{02}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{\Theta}_{02}\end{bmatrix}. \tag{24}$$

Let the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ also be partitioned to make $\boldsymbol{P}_{11}$ and $\boldsymbol{Q}_{11}$ square submatrices of order n_1. Thus from Eq. (21)

$$I\approx[\boldsymbol{\Theta}_{01}^T\ \ \boldsymbol{\Theta}_{02}^T]\left\{\begin{bmatrix}\boldsymbol{P}_{11} & \boldsymbol{P}_{12}\\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22}\end{bmatrix}+k^2\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22}\end{bmatrix}\right\}\begin{bmatrix}\boldsymbol{\Theta}_{01}\\ \boldsymbol{\Theta}_{02}\end{bmatrix} \tag{25}$$

Hence, taking $\mathbf{\Theta}_{01}=\mathbf{0}$ into account

$$I \approx \mathbf{\Theta}_{02}^T[\boldsymbol{P}_{22}-k^2\boldsymbol{Q}_{22}]\mathbf{\Theta}_{02}.$$

This may be extremal if

$$\frac{\partial I}{\partial \mathbf{\Theta}_{02}}=2[\boldsymbol{P}_{22}-k^2\boldsymbol{Q}_{22}]\mathbf{\Theta}_{02}=\mathbf{0} \tag{27}$$

Now, the values of k^2 are the solutions of the equation

$$\det|\boldsymbol{P}_{22}-k^2\boldsymbol{Q}_{22}|=0. \tag{28}$$

The elements of the eigenvectors $\mathbf{\Theta}_{02}$ corresponding to the eigenvalues k^2 are approximations of the values of Θ in the internal nodes.

A computer program has been developed based on the method presented. With this, several cross sections have been treated to obtain the values k^2 and wavelengths λ corresponding to different modes, and to plot the electrical flux density lines of TE modes and the magnetic flux density lines of TM modes.

To test the method a rectangular cross section waveguide has been first examined (Fig. 3). The exact values and the finite element approximations of the wavelengths are summarized in Table 1. The flux plots for a quarter of the cross section have been plotted in Fig. 3.

As a second example the cross section shown in Fig. 4 has been treated. For this case the results obtained by the method of finite differences for the fundamental TM mode are given in reference [5]. The wavelengths and the flux plots are shown in Fig. 4.

Flux plots obtained for an elliptical cross section waveguide are shown in Fig. 5.

Table 1

m	n	TE mode λ_e/cm exact	TE mode λ_a/cm approx	TE mode $\frac{\lambda_a-\lambda_e}{\lambda_e}100$	TM mode λ_e/cm exact	TM mode λ_a/cm approx	TM mode $\frac{\lambda_a-\lambda_e}{\lambda_e}100$
0	1	3.60	3.59	−0.27			
1	0	4.80	4.79	−0.27			
1	1	2.88	2.86	−0.82	2.88	2.86	−0.89
2	1	2.00	1.96	−1.89	2.00	1.96	−0.49
1	2	1.69	1.66	−1.70	1.69	1.66	−1.70
2	0	2.40	2.37	−1.09			
0	2	1.80	1.78	−1.06			
2	2	1.44	1.41	−1.82	1.44	1.41	−1.82

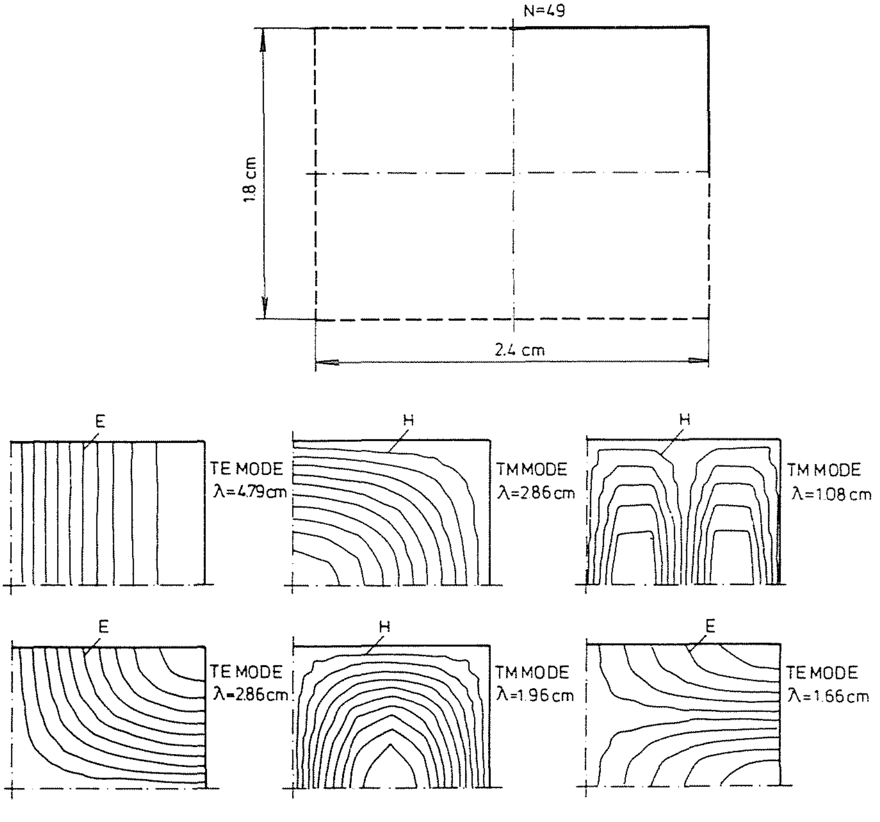

Fig. 3

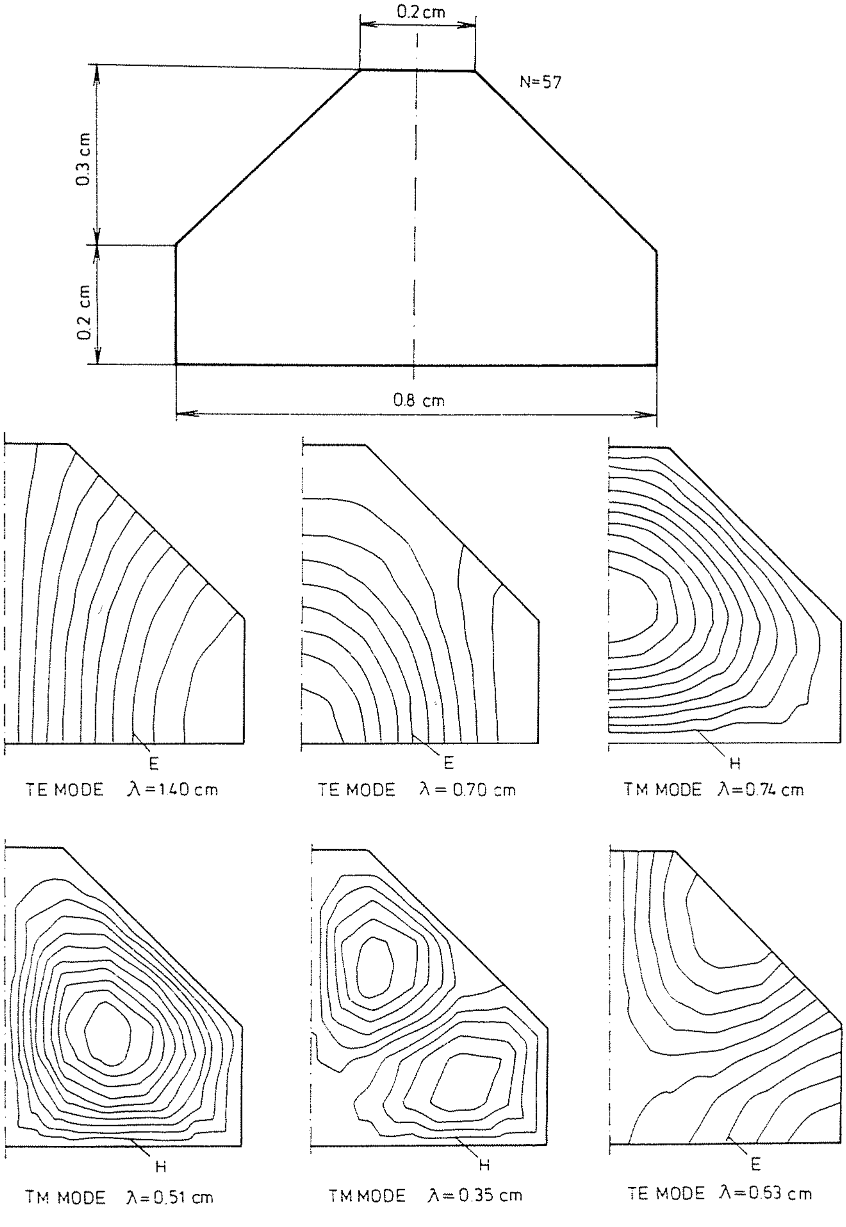
0.2 cm
N=57
0.3 cm
0.2 cm
0.8 cm
E
TE MODE λ=1.40 cm
E
TE MODE λ = 0.70 cm
H
TM MODE λ=0.74 cm
H
TM MODE λ=0.51 cm
H
TM MODE λ=0.35 cm
E
TE MODE λ=0.63 cm

Fig. 4

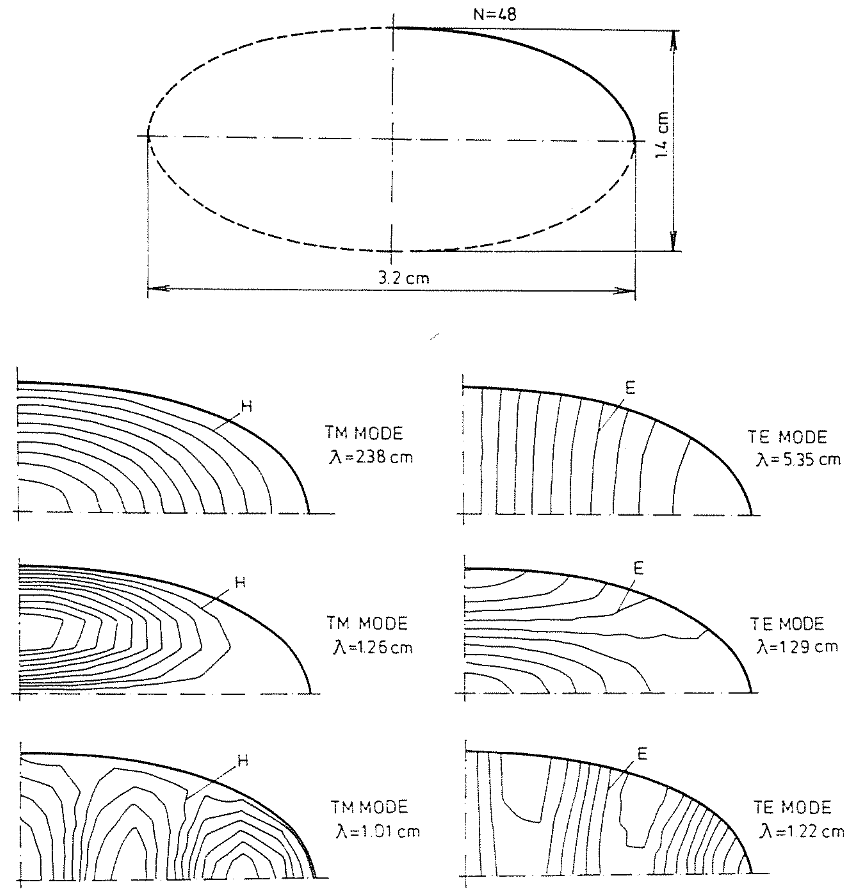

Fig. 5

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