APPROXIMATE SOLUTION OF LAPLACE EQUATION IN UNBOUNDED REGIONS BY VARIATIONAL METHOD

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Abstract

The paper extends the global element variational method to the determination of the static and stationary electric and magnetic field in unbounded regions. An approximating solution of Laplace equation in the studied region is obtained with the aid of **R**-functions defined on the boundaries approximated by analytical functions with the solution satisfying the Dirichlet and Neumann boundary conditions on the boundaries and behaving appropriately in infinity. The application of the method is illustrated by an example.

Introduction

At the determination of static and stationary electric or magnetic fields the potential function satisfying Laplace equation is to be derived with the prescribed Dirichlet or Neumann boundary conditions fulfilled. In layouts with complex geometries the solution is usually determined by numerical methods with the aid of computers. Among the numerical procedures, the method of integral equations and the variational methods are the most widespread. Variational methods may be realized by finite element or global element techniques.

Variational methods are mainly employed for the analysis of bounded regions. There are, however, cases when the studied region extends to infinity. In case of such unbounded regions, only a few of the numerical methods are capable of deriving the function satisfying Laplace equation and behaving appropriately in infinity. In such cases the method of integral equations [13], Trefftz's method [8] or mixed methods obtained by combining variational and integral techniques [12] or variational and Trefftz's method [7] are usually employed to determine the potential function. In some cases it is permissible to close the unbounded region by a surface with Dirichlet or Neumann boundary conditions at a large distance from the boundaries. The variational method is applicable to the analysis of such simplified, bounded regions.

The paper deals with the investigation of unbounded regions. It is shown how the global element variational method can be extended to the determination of the static or stationary electric and magnetic field of the region without closing the geometrical space.

It is assumed in the paper that the boundary surfaces or, in case of planar problems, the bounding curves can be described or approximated by piecewise analytical functions. The method to be presented produces an approximate solution of Laplace equation which, besides satisfying the prescribed Dirichlet or Neumann boundary conditions, behaves appropriately in infinity. The problem set is solved in the paper with the aid of **R**-functions [5], [6].

The application of the method is illustrated by an example.

Laplace equation and the boundary conditions

At the investigation of electrodes or magnetic poles, the problem is to determine the field vectors of the electric or magnetic field in the region between the electrodes or the magnetic poles.

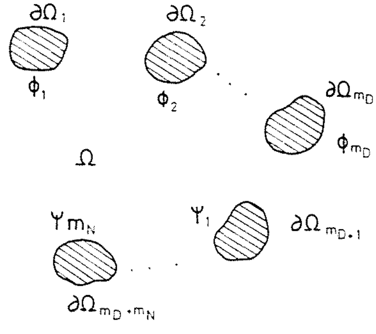

Fig. 1

In the unbounded layout (Fig. 1) let Ω denote the region investigated,

$$\partial\Omega = \bigcup_{i=1}^{m} \partial\Omega_i$$

the boundary of this possibly multi-connected region and $\mathbf{v}_i$ the outer normal on $\partial\Omega_i$. Let us assume that the boundary section $\partial\Omega_i$ of the studied region can be described or approximated by piecewise analytical functions.

On the boundary sections $\partial\Omega_p(p=1, 2, \ldots, m_D)$ the value $\Phi_p(p=1, 2, \ldots, m_D)$ of the potential function is given and on the boundary sections $\partial\Omega_q(q=1, 2, \ldots, m_N)$ the value $\Psi_q(q=1, 2, \ldots m_N)$ of the normal derivative of the potential function is prescribed $(m_D+m_N=m)$.

In the problems investigated, the electric or magnetic field is assumed to have no sources outside the boundaries. This means that the static electric field is brought about by the charge of the electrodes, the stationary magnetic field by the flux of the poles and the stationary electric field by the current of the electrodes. So, since there is no charge outside the electrode surfaces and no current in the regions between the magnetic poles:

$$\rho=0, \qquad \text{or} \quad \mathbf{J}=\mathbf{0}. \tag{1}$$

The electric or magnetic scalar potential defined by

$$\mathbf{E}=-\mathbf{grad}\,\Phi^e, \qquad \text{or} \quad \mathbf{H}=-\mathbf{grad}\,\Phi^m \tag{2}$$

has to satisfy the Laplace equation [1]

$$\Delta\Phi=0, \tag{3}$$

where Φ denotes the electric scalar potential Φ^e or the magnetic one Φ^m.

In unbounded regions, the unique solution of the Laplace equation (3) can only be determined if, beside the given boundary conditions, the behaviour of the potential function is fixed in infinity. In a static electric field the total charge of the arrangement, in case of magnetic poles the total flux and in a stationary electric field the total current of the arrangement has to be specified. This means that for a closed surface $\partial\Omega$ in infinity, the following condition has to be fulfilled:

$$\lim_{r\to\infty} \oint_{\partial\Omega} \mathbf{grad}\,\Phi \mathbf{dS}=-K. \tag{4}$$

In a static electric field $K=Q_0/\varepsilon$, in the field of magnetic poles $K=\psi_0/\mu$ and in a stationary electric field $K=I_0/\sigma$ where Q_0, ψ_0, I_0 are the total charge, flux, current of the boundaries in the arrangement, and ε, μ, σ are medium characteristics.

In case of Dirichlet boundary condition on the boundary

$$\partial\Omega_D=\bigcup_{p=1}^{m_D} \partial\Omega_p$$

of the studied region $\Omega(m_N=0,\ m=m_D$, Fig. 1), the value of the potential function is defined on $\partial\Omega_D$ beside the condition (4):

$$\Phi_{|\partial\Omega_p}=\Phi_p, \qquad p=1, 2, \ldots, m_D. \tag{5}$$

In case of Neumann boundary condition on the boundary

$$\partial\Omega_N=\bigcup_{q=1}^{m_N} \partial\Omega_q$$

of the studied region ($m_D=0,\ m=m_N$, Fig. 1), the normal derivatives of the potential function are given on $\partial\Omega_N$ beside the condition (4):

$$\mathbf{v}_q \mathbf{grad}\, \Phi_{|\partial\Omega_q} = \Psi_q\,, \qquad q=1,2,\ldots,m_N\,. \tag{6}$$

$\Psi_q(q=1,2,\ldots,m_N)$ is proportional to the surface charge density in a static field and to the normal component of the magnetic flux density in the field of magnetic poles and of current density in stationary electric field. In unbounded regions, in case of Neumann boundary condition, the value of K in Eq. (4) is obtained from the prescribed values Ψ_q on $\partial\Omega_q(q=1,2,\ldots,m_N)$ as

$$K=\sum_{q=1}^{m_N}\int_{\partial\Omega_q}\Psi_q\,\mathrm{d}S\,. \tag{7}$$

In case of Neumann boundary condition, since the problem is defined upto one constant only, the potential of some point P is to be specified:

$$\Phi(P)=\Phi_P\,, \qquad P\in\Omega\,, \tag{8}$$

with Φ_P being finite in addition to the conditions (4), (6), (7) in order to produce a unique solution.

In case of mixed boundary conditions (Fig. 1), Dirichlet boundary condition (5) is prescribed on the boundary $\partial\Omega_D$ and Neumann boundary condition (6) on the boundary $\partial\Omega_N$ beside the condition (4) prescribing the behaviour of the potential function in infinity $(m=m_D+m_N)$.

As it is known from the literature [1], [3], in case of homogeneous medium, the determination of the function satisfyin Laplace equation (3) can be reduced by variational calculus to finding the function extremizing the functional:

$$I=\frac{1}{2}\int_{\Omega}\mathbf{grad}^2\,\Phi\,\mathrm{d}V-\sum_{q=1}^{m_N}\int_{\partial\Omega_q}\Phi\Psi_q\,\mathrm{d}S \tag{9}$$

The function extremizing the functional (9) can be approximated according to Ritz's method by the linear combination of the first n elements of a function set entire in the studied region Ω [2], [3]. So, with the aid of the global element variational method, the function approximating the solution of the differential equation (3) can be obtained. The satisfaction of the Dirichlet boundary conditions is, however, to be enforced separately.

As regards the satisfaction of the boundary conditions, several ways of approximation are feasible. A possibility is to derive a solution satisfying both the differential equation and the boundary conditions approximately [7]. If the surfaces bounding the studied region can be approximated by piecewise analytical functions, the employment of R-functions permits the derivation of a solution satisfying the differential equation approximately but exactly fulfilling the boundary conditions on the surfaces approximated by analytical functions [10], [11]. This last approach is used in this paper.

Satisfaction of the boundary conditions

On the boundary $\partial\Omega_N$ of the studied region Ω, the Neumann boundary condition (6) is a natural boundary condition of the functional (9) and so its satisfaction follows from the extremization of the functional. Therefore, it suffices to treat the Dirichlet boundary condition (5) prescribed on the boundary $\partial\Omega_D$ of the studied region as well as the satisfaction of the condition (4) prescribing the approximate behaviour of the potential function in infinity.

The function extremizing the functional (9) will be sought as a sum of functions differentiable in the studied region Ω in order to satisfy the boundary conditions (4), (5):

$$\Phi = \Phi_\alpha + \Phi_\delta + \Phi_\beta . \tag{10}$$

In the potential function (10), Φ_δ and Φ_β are assumed to be known and Φ_α is unknown. The term Φ_δ of the potential function Φ should be selected so that it satisfies the Dirichlet boundary condition (5) prescribed on the boundary $\partial\Omega_D$ of the studied region Ω. The other two terms of the potential function, the functions Φ_α and Φ_β are to satisfy homogeneous Dirichlet boundary conditon on this surface:

$$\Phi_{\delta|\partial\Omega_p} = \Phi_p , \qquad p = 1, 2, \ldots, m_D , \tag{11}$$

$$\Phi_{\alpha|\partial\Omega_D} = 0 , \qquad \varphi_{\alpha|\partial\Omega_D} \tag{12}$$

$$\Phi_{\beta|\partial\Omega_D} = 0 . \qquad \varphi_{\beta|\partial\Omega_D} \tag{13}$$

The behaviour of the potential function Φ in infinity is described by the function Φ_β. The functions Φ_α and Φ_δ should yield zero if the condition (4) is applied to them:

$$\lim_{r\to\infty} \oint_{\partial\Omega} \mathbf{grad}\, \Phi_\beta \, \mathbf{dS} = -K , \tag{14}$$

$$\lim_{r\to\infty} \oint_{\partial\Omega} \mathbf{grad}\, \Phi_\alpha \, \mathbf{dS} = 0 , \tag{15}$$

$$\lim_{r\to\infty} \oint_{\partial\Omega} \mathbf{grad}\, \Phi_\delta \, \mathbf{dS} = 0 . \tag{16}$$

In case of $K \neq 0$, a function Φ_β varying as $1/r$ in the three-dimensional case or as $\ln(1/r)$ in the planar case satisfies the condition (14). Therefore, the function Φ_β is selected in the three-dimension and planar cases as

$$\begin{aligned} \Phi_\beta &= \Phi_0 w_D \frac{1}{r_0 + r} , \\ \Phi_\beta &= \Phi_0 w_D \ln\left(\frac{1}{r_0 + r}\right) , \end{aligned} \tag{17}$$

respectively, where r is the distance from the origin of the coordinate system and r_0 is a constant. w_D is a known function positive in the studied region Ω vanishing on the boundaries $\partial\Omega_D$. Hence w_D ensures that Φ_β satisfies the condition (13). The function w_D is selected so that it tends to one in infinity and its gradient decays at least as $1/r^3$ in the three-dimensional case and as $1/r^2$ in the planar case:

$$w_{D|\partial\Omega_D} w_{D|\partial\Omega_D} = 0 , \tag{18}$$

$$\lim_{r\to\infty} w_D = 1 ,$$

$$\lim_{r\to\infty} \frac{\partial w_D}{\partial r} = o\left(\frac{1}{r^3}\right) \text{ in the } 3D \text{ case} \tag{19}$$

$$\lim_{r\to\infty} \frac{\partial w_D}{\partial r} = o\left(\frac{1}{r^2}\right) \text{ in the planar case}$$

The method of constructing the function w_D satisfying the conditions (18), (19) is presented in the next section.

In Eq. (17) Φ_0 is an unknown constant. The value of Φ_0 is obtained from the condition (14). The surface in infinity is approximated by a sphere in the threedimensional case and by a cylinder in the planar case. From Eq. (14), using (17), (19), the relationship

$$\lim_{r\to\infty} \Phi_0\left[\frac{1}{r_0+r}\frac{\partial w_D}{\partial r} + w_D\frac{\partial}{\partial r}\left(\frac{1}{r_0+r}\right)\right]4\pi r^2 = -K$$

yields $\Phi_0 = K/4\pi$ in the three-dimensional case, and the relationship

$$\lim_{r\to\infty} \Phi_0\left\{\ln\left(\frac{1}{r_0+r}\right)\frac{\partial w_D}{\partial r} + w_D\frac{\partial}{\partial r}\left[\ln\left(\frac{1}{r_0+r}\right)\right]\right\}2\pi r = -K$$

yields $\Phi_0 = K/2\pi$ in the planar case.

In case $K = 0$ i.e. if the total charge or current of the electrodes or the total flux of the magnetic poles is zero, the value $\Phi_0 = 0$ sets the term Φ_β of the potential function zero, so it can be disregarded.

The unknown term Φ_α in the potential function is approximated according to Ritz's method [2], [3] by the linear combination of the first n elements of a function set entire in the region Ω:

$$\Phi_\alpha \approx \Phi_n = \sum_{k=1}^{n} a_k f_k w_D , \tag{20}$$

where f_k is the k-th element of the approximating function set ($k = 1, 2, \ldots, n$), $a_k (k = 1, 2, \ldots, n)$ are the unknown coefficients. In Eq. (20) w_D is the function satisfying the conditions (18), (19). Since, according to (18), w_D vanishes on the boundary $\partial\Omega_D$ of the studied region Ω, the expansion (20) approximating the function Φ_α satisfies the homogeneous Dirichlet boundary condition prescribed for the function Φ_α. The expansion (20) satisfies the condition (15) prescribed for the function Φ_α in infinity if, beside the function w_D satisfying the conditions (18), (19), the approximating functions $f_k (k=1, 2, \ldots, n)$ are selected from a function set entire in the unbounded region [4]. The element f_k, a function of the coordinate variables x_1, x_2, x_3 is chosen as the product of the functions $f_i(x_1), f_j(x_2), f_l(x_3)$ depending upon a single variable:

$$f_k(x_1, x_2, x_3) = f_i(x_1) f_j(x_2) f_l(x_3) .$$

In accordance with the work of Mikhlin [4], the functions f_i, f_j, f_l are selected in Cartesian coordinates, for example for the variable x_1 as

$$f_i(x_1) = \cos [2i \tan^{-1}(x_1)]$$

or

$$f_i(x_1) = \sin [(2i-1) \tan^{-1}(x_1)]$$

according to symmetry conditions. In cylindrical coordinates Mikhlin [4] recommends, beside the above functions, the use of Bessel functions for the radial variation and of harmonic functions for the azymuthal variation.

The application of *R*-functions

On the basis of the work of Rvachev [5], [6], the employment of ***R***-functions permits the condition of the function w_D satisfying the conditions (18), (19) and of the function Φ_δ satisfying the conditions (11), (16).

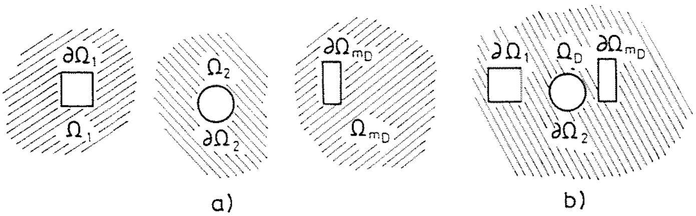

Fig. 2

For the construction of the functions Φ_δ and w_D, the studied region is constructed from subregions $\Omega_p (p=1, 2, \ldots, m_D)$ with Dirichlet boundary condition prescribed on their boundaries $\partial\Omega_p (p=1, 2, \ldots, m_D)$ (Fig. 2a). With the intersection of the regions $\Omega_p (p=1, 2, \ldots, m_D)$ a region Ω_D including $\Omega (\Omega \subset \Omega_D)$ can be constructed with Dirichlet boundary conditions only prescribed on its boundaries $\partial\Omega_D$ (Fig. 2):

$$\Omega_D = \bigcap_{p=1}^{m_D} \Omega_p \,. \tag{21}$$

The bounding surfaces of the regions Ω_p are approximated by at least twice differentiable functions w_p satisfying the following condition

$$w_p(P) \begin{cases} =0, & \text{if} \quad P \in \partial\Omega_p\,, \\ >0, & \text{if} \quad P \in \Omega_p\,, \quad \text{but} \quad P \notin \partial\Omega_p\,, \\ <0, & \text{if} \quad P \notin \Omega_p\,, \end{cases}$$

$$\mathbf{grad}\, w_p(P) = \mathbf{0}\,, \quad \text{if} \quad P \in \Omega_p\,, \quad p=1,2,\ldots,m_D\,. \tag{22}$$

The $\mathbb{R}$-function satisfying the condition (22) vanishes on the boundary $\partial\Omega_p$ of the region Ω_p, is positive and monotonously increasing with the distance from the surface.

In the knowledge of the functions $w_p (p=1,2,\ldots,m_D)$ defined in the regions Ω_p, a function w can be constructed with the aid of $\mathbb{R}$-conjunction [5], [10] on the region Ω_D obtained from the regions Ω_p according to Eq. (22) as

$$w = \bigwedge_{p=1}^{m_D} w_p\,, \tag{23}$$

which is zero everywhere on the boundary $\partial\Omega_D$ and in infinity it increases at least according to r^2 in the three-dimensional case and at least linearly in the planar case.

The function w_D appearing in the terms Φ_α and Φ_β of the potential function (10) satisfies the conditions (18), (19) if it is constructed from the function w defined by (23) in Ω_D as

$$w_D = \frac{w}{A+w}\,, \tag{24}$$

where A is an arbitrary, finite constant. The function w_D given by (24) vanishes on the boundary $\partial\Omega_D$ of the studied region Ω, tends to one in infinity in accordance with the properties of the function w defined in Eq. (23). Since the function w increases in infinity as r^2 in three-dimensional case and as r in the panar case, it is easy to see that the gradient of w_D decays as $1/r^3$ in the three-dimensional case and as $1/r^2$ in the planar case. It hence follows that the function w_D given by Eq. (24) satisfies the condition (19).

In term Φ_δ of the potential function (10) satisfies the Dirichlet boundary condition given by Eq. (11) on the boundary $\partial\Omega_D$ of the studied region and it behaves in infinity as specified in Eq. (16). The function Φ_δ is constructed with the aid of the $\mathbb{R}$-functions $w_p (p=1,2,\ldots,m_D)$ defined on the boundaries $\partial\Omega_p$ of the region Ω_D and satisfying the condition (22) as

$$\Phi_\delta = \frac{\sum_{p=1}^{m_D} \Phi_p \bigwedge_{i=1, i\neq p}^{m_D} w_i}{\sum_{p=1}^{m_D} \bigwedge_{i=1, i\neq p}^{m_D} w_i} . \tag{25}$$

The appropriate behaviour in infinity of the function Φ_δ given in (25) is ensured by the behaviour of the **R**-functions $w_p (p=1, 2, \ldots, m_D)$ associated with the surfaces $\partial\Omega_p (p=1, 2, \ldots, m_D)$. It can be shown that the function given by Eq. (25) satisfies both the conditions (11) and (16).

Application of the method

To illustrate the above method, the static electric field of two rectangular, infinitely long conductors is investigated (Fig. 3). The voltage U between the electrodes is given, the total charge of the arrangement is zero. The planar region is unbounded, its planar section is shown in Fig. 3b. Choosing the potential of the left electrode as $\Phi_2 = -U/2$, $\Phi_1 = U/2$ is obtained. The **R**-function describing the bounding curves of the electrodes are:

$$w_1 = [(x-d)^2 - a^2] \vee (y^2 - b^2) ,$$
$$w_2 = [(x+d)^2 - a^2] \vee (y^2 - b^2) ,$$

and

$$w = w_1 \wedge w_2$$

where $\vee$ denotes **R**-disjunction and $\wedge$ denotes **R**-conjunction [4], [5].

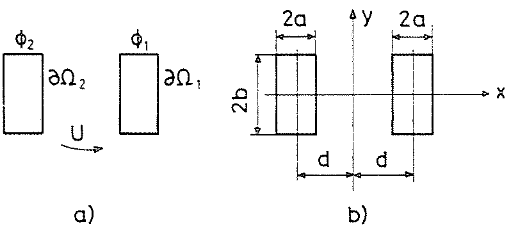

Fig. 3

In the approximate potential function, since the total charge is zero, $\Phi_\beta = 0$ as well as

$$\Phi_\delta = \frac{\Phi_1 w_2 + \Phi_2 w_1}{w_1 + w_2}$$

and

$$w_D = \frac{w}{1+w}.$$

Making use of symmetry, the elements of the approximating expansion are selected as follows [4]:

$$f_k = T_i(x)\,T_j(y)\,, \quad i=1,2,\ldots,n_1\,,$$
$$j=1,2,\ldots,n_2\,,$$
$$k=1,2,\ldots,n\,,$$

where

$$T_i(x) = \sin\left[(2i-1)\tan^{-1}(x)\right],$$
$$T_j(y) = \cos\left[2j\tan^{-1}(y)\right].$$

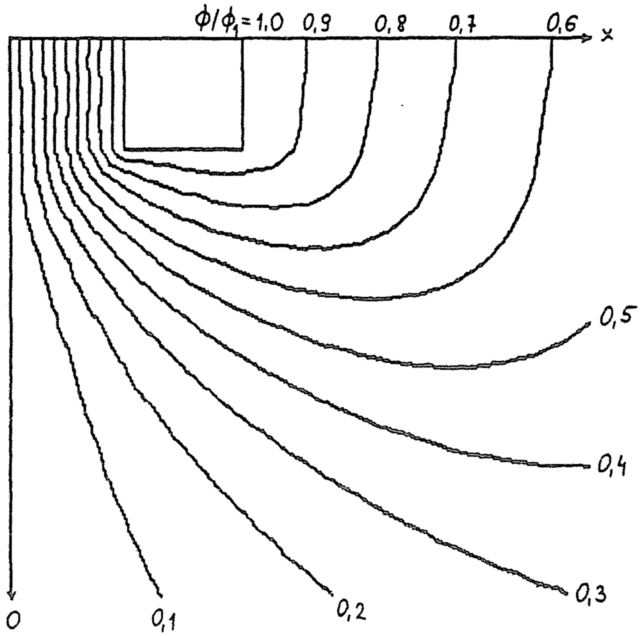

Fig. 4

For the unkown coefficients $a_k(k=1,2,\ldots,n)$ of the approximating function (20), a set of linear equations is obtained:

$$\boldsymbol{Aa} = \boldsymbol{b} \tag{26}$$

where $\boldsymbol{A}$ is an n-th order square matrix with the i-th element of its k-th row being

$$\{A\}_{k,1} = \int_\Omega \mathbf{grad}\,(w_D f_k)\,\mathbf{grad}\,(w_D f_i)\,\mathrm{d}V, \tag{27}$$

$\boldsymbol{b}$ is an n-element column vector, its k-th element is

$$\{\boldsymbol{b}\}_k = \int_\Omega \mathbf{grad}\,(\Phi_\delta + \Phi_\beta)\,\mathbf{grad}\,(w_D f_k)\,\mathrm{d}V - \sum_{q=1}^{m_N} \int_{\partial\Omega_q} (f_k w_D)\,\Psi_q\,\mathrm{d}S\,, \tag{28}$$

$\boldsymbol{a}$ is the n-element column vector of the unknown coefficients.

Since the studied region Ω extends to infinity, the integrals (27) and (28) are impropriate. Their numerical evaluation is simplified by the fact that the integrand vanishes in infinity at least as $1/r^3$ in the three-dimensional case and as $1/r^2$ in the planar case.

With the aid of the above method, the approximate potential function of the arrangement has been determined at the values $U=20$ V, $b/a=3$, $d/a=2$, $n=12$, $n_1=n_2=3$. The equipotential lines have been drawn for a quadrant of the layout (Fig. 4). The potential difference between any two lines is constant ($\mathrm{d}\Phi=1$ V). The diagram clearly shows that the electric field in the region between the electrodes is almost homogeneous.

The capacity per unit length of the arrangement has been computed from the energy of the field and has been obtained as C = 24.63 pF/m. This value is in good correspondence with values obtained by other approximating methods [1], [12].

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