CHARACTERISTIC IMPEDANCE OF WAVEGUIDES USED IN QUASI-TEM MODE

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Summary

The voltage and current of a quasi-TEM mode have been defined as linear functionals of the electric and magnetic field in the two-conductor waveguide. The functional has been chosen so that the error that arises by neglecting the higher modes generated at the junction of two different waveguides is as small as possible. A new definition of the characteristic impedance has been given by means of the so defined voltage and current.

Introduction

If the electromagnetic field of a lossless two-conductor waveguide is a pure TEM mode, the wave propagation can be described by means of the transmission-line theory. In this case the waveguide of a given length can be characterized as a two-port. The inductance and capacitance per unit length, denoted by L and C, resp. determine the parameters of this two-port through the phase constant and the characteristic impedance.

TEM mode occurs only in waveguides over the cross section of which the product of the permittivity ε and the permeability μ is constant. If this condition is not fulfilled, only quasi-TEM (qTEM) mode can propagate in the waveguide. The theory of transmission lines is usually applied in this case, too, because the qTEM mode turns into TEM mode at zero frequency, but this procedure is acceptable at low frequencies only. A better approximation is achieved, if the phase constant in the two-port parameters is not substituted by the value $\beta = \omega\sqrt{LC}$ resulting from the theory of transmission lines, but by its exact value resulting from the solution of the eigenvalue problem related to the waveguide. Similarly, the characteristic impedance is not substituted by the value $Z_0 = \sqrt{L/C}$ given by the theory of transmission lines. The characteristic impedance considered as more exact is defined in many different ways. The two definitions mostly used in the literature, one of which is based on the effective permittivity [1] and the other on the transferred power [2], sharply contradict each other. The different definitions conduced to polemic [3]—[6] with ambiguous conclusions, because neither the theoretical basis nor the practical purposes were clearly formulated.

The application of the transmission-line theory in the case of qTEM modes has several practical advantages, but the results are not exact, because the higher modes generated at the junction of two different waveguides cannot be taken into account, though they have an important role in the fulfilment of the boundary conditions. In this paper the theory of transmission lines is generalized in such a way that the results have the smallest error possible in the case of qTEM modes. This generalization includes, of course, a new definition of the characteristic impedance, which is more deeply justified by the theory than the old ones and gives the same result as the definition based on the effective permittivity, if the scalar valued permeability is constant over the cross section.

Quasi-TEM modes and the theory of transmission lines

The theory of transmission lines defines two scalar quantities, a voltage and a current in every cross section of a waveguide used in TEM mode. This cannot be done so simply at qTEM modes, because the integrals of the electric and magnetic field strength along curves connecting the two conductors and enclosing one conductor, resp. depend upon the choice of the curve in the given cross section.

With the aim of generalizing the definitions of the previously mentioned two scalar quantities, all the waveguides having a given doubly connected region A as cross section are considered simultaneously. These waveguides differ in the functions describing the permittivity and permeability over the cross section. It suffices to consider the transversal component $\mathbf{E}_T$ and $\mathbf{H}_T$ of the electric and magnetic field strength, resp. because the longitudinal components are also unambiguously determined by them. The functions $\mathbf{E}_T$ and $\mathbf{H}_T$ are regarded as elements of the Hilbert space $\mathscr{H}$ of the vectorial functions square integrable over the region A with the usual inner product

$$(\mathbf{u}, \mathbf{v}) = \int_A \mathbf{u}^* \mathbf{v} \, \mathrm{d}A. \tag{1}$$

It is a natural generalization that the voltage U and current I belonging to $\mathbf{E}_T$ and $\mathbf{H}_T$, resp. are defined as linear continuous functionals on the Hilbert space $\mathscr{H}$. It is known that such functionals can be written in the form of inner products:

$$U = (\mathbf{u}, \mathbf{E}_T) \quad I = (\mathbf{i}, \mathbf{H}_T). \tag{2}$$

Now the adequate vectors $\mathbf{u}$ and $\mathbf{i}$ must be found. The first postulate is that the functionals defined by (2) should give the usual voltage and current at zero frequency. As it is proved in Appendix I, it follows from this that the vector

i has no normal component along the boundary curves of the region A and

$$\operatorname{div} \mathbf{u} = 0 \qquad \operatorname{div} \mathbf{i} = 0 \tag{3}$$

$$\oint_{L_1} (\mathbf{n} \times \mathbf{u}) \mathrm{d}\mathbf{L} = 1 \qquad \int_{L_2} (\mathbf{i} \times \mathbf{n}) \mathrm{d}\mathbf{L} = 1, \tag{4}$$

where **n** denotes the unit vector normal to the cross section A, L_1 is an arbitrary closed curve enclosing one conductor and L_2 is an arbitrary curve connecting the two conductors.

The functionals defined by vectors **u** and **i** satisfying the relationships (3) and (4) can be given direct interpretation. As it was mentioned, the integrals of the electric field strength of a qTEM mode along curves connecting the two conductors give different values for different curves, and integrals of the magnetic field strength behave similarly. The voltage and current defined by relationships (2) signify the mean of these different values in the following sense. Let us fix the lines of force of the vector field **u** in the usual manner, i.e. the flux between any two lines of force should have the same value. Let us integrate the vector $\mathbf{E}_T$ along these lines of force. If the density of the lines of force is increased beyond any limit, the mean value of these integrals converges to the voltage U defined by (2). The current I defined by (2) can be interpreted similarly, if the lines of forces of the vector field **i** are used at the averaging. It has to be noted that the power transferred by the qTEM mode can be calculated only approximately on the basis of U and I except at zero frequency, where they give the exact power.

In the following, the curl of the vectors **u** and **i** will be determined considering the principles mentioned in the Introduction. Let us investigate the electromagnetic field in two joined waveguides. They are assumed to have the same cross section A, but the functions describing the permittivity and permeability over their cross sections are different. At the junction higher modes appear beside the qTEM mode excited from outside. These modes are rapidly damped at frequencies used in qTEM lines. That is why it does not give rise to grave error that the theory of transmission lines cannot take into account the higher modes. The subscripts 1 and 2 will refer to the two waveguides, and so let us denote at the junction of the two waveguides the transversal components of the field strengths of the qTEM mode by $\mathbf{E}_{T1}, \mathbf{H}_{T1}$ and $\mathbf{E}_{T2}, \mathbf{H}_{T2}$, resp. and the transversal components of all the higher modes together by $\mathbf{e}_{T1}, \mathbf{h}_{T1}$ and $\mathbf{e}_{T2}, \mathbf{h}_{T2}$, resp. In consequence of the boundary conditions

$$\begin{aligned} \mathbf{E}_{T1} + \mathbf{e}_{T1} &= \mathbf{E}_{T2} + \mathbf{e}_{T2} \\ \mathbf{H}_{T1} + \mathbf{h}_{T1} &= \mathbf{H}_{T2} + \mathbf{h}_{T2}. \end{aligned} \tag{5}$$

The theory of transmission lines does not describe the boundary conditions by the accurate equations

$$\begin{aligned}(\mathbf{u}, \mathbf{E}_{T1})+(\mathbf{u}, \mathbf{e}_{T1})&=(\mathbf{u}, \mathbf{E}_{T2})+(\mathbf{u}, \mathbf{e}_{T2})\\(\mathbf{i}, \mathbf{H}_{T1})+(\mathbf{i}, \mathbf{h}_{T1})&=(\mathbf{i}, \mathbf{H}_{T2})+(\mathbf{i}, \mathbf{h}_{T2})\end{aligned} \tag{6}$$

derived from (5), but by the approximating equations

$$\begin{aligned}U_1&=U_2 \quad \text{i.e.} \quad (\mathbf{u}, \mathbf{E}_{T1})=(\mathbf{u}, \mathbf{E}_{T2})\\I_1&=I_2 \quad \text{i.e.} \quad (\mathbf{i}, \mathbf{H}_{T1})=(\mathbf{i}, \mathbf{H}_{T2}),\end{aligned} \tag{7}$$

which arise, if the higher modes are neglected. These equations would be exact only if the inner products $(\mathbf{u}, \mathbf{e}_{T1})$, $(\mathbf{u}, \mathbf{e}_{T2})$, $(\mathbf{i}, \mathbf{h}_{T1})$ and $(\mathbf{i}, \mathbf{h}_{T2})$ equalled zero. Unfortunately, such vectors **u** and **i** do not exist. Accordingly, the vectors **u** and **i** have to be chosen suitably so that the absolute values of these inner products should be as small as possible.

It is proved in Appendix II that if **u** and **i** satisfy the conditions already formulated, the previous four inner products always vanish at zero frequency. So Eqs (7) are acceptable approximations of Eqs (6) at properly low frequencies. The error of these approximations is small, if the quantities

$$m_j=\left|\frac{(\mathbf{u}, \mathbf{e}_{Tj})}{(\mathbf{u}, \mathbf{E}_{Tj})}\right| \quad n_j=\left|\frac{(\mathbf{i}, \mathbf{h}_{Tj})}{(\mathbf{i}, \mathbf{H}_{Tj})}\right|, \qquad j=1,2 \tag{8}$$

are small. These quantities have the following bounds:

$$m_j \leqq \frac{\|\mathbf{u}\|\,\|\mathbf{e}_{Tj}\|}{|(\mathbf{u}, \mathbf{E}_{Tj})|} \quad n_j \leqq \frac{\|\mathbf{i}\|\,\|\mathbf{h}_{Tj}\|}{|(\mathbf{i}, \mathbf{H}_{Tj})|}, \qquad j=1,2. \tag{9}$$

These upper bounds and with them the error of the approximation are reduced for all the possible vectorial functions $\mathbf{e}_{Tj}$ and $\mathbf{h}_{Tj}$, if **u** and **i** are chosen so that the quantities

$$s_j=\frac{|(\mathbf{u}, \mathbf{E}_{Tj})|}{\|\mathbf{u}\|} \quad t_j=\frac{|(\mathbf{i}, \mathbf{H}_{Tj})|}{\|\mathbf{i}\|}, \qquad j=1,2 \tag{10}$$

are as large as possible.

According to Appendix III t_1 is maximum, if curl $\mathbf{i}=c_1$ curl $\mathbf{H}_{T1}$ with an arbitrary constant c_1. Similarly, t_2 is maximum, if curl $\mathbf{i}=c_2$ curl $\mathbf{H}_{T2}$, which contradicts the previous condition except at zero frequency, where curl $\mathbf{H}_{T1}=$ $=$curl $\mathbf{H}_{T2}=0$. So the choice curl $\mathbf{i}=0$ is the most advisable, because it guarantees that the values t_1 and t_2 are near the possible maxima, at least if the frequency is not too large. It follows similarly from Appendix III that **u** is advisably chosen so that it has no tangential components along the boundary curves of the cross section, and curl $\mathbf{u}=0$.

In consequence of the previous statements vectors **u** and **i** must fulfil the relationships

$$\begin{aligned} \operatorname{div} \mathbf{u} = 0 \quad \operatorname{curl} \mathbf{u} = 0 \\ \operatorname{div} \mathbf{i} = 0 \quad \operatorname{curl} \mathbf{i} = 0 \end{aligned} \tag{11}$$

and the relationships (4), furthermore the tangential component of **u** and the normal component of **i** must vanish along the boundary curves of the cross section. It is obvious that vectors **u** and **i** describe the electric and magnetic field of the waveguide in the case of constant permittivity ε and permeability μ. In consequence of relationships (4) the charge per unit length equals ε for the electric field **u**, and the flux per unit length equals μ for the magnetic field **i**. The two vectors can be expressed in terms of each other as follows:

$$\mathbf{i} = \frac{\mathbf{n} \times \mathbf{u}}{\int\limits_{L_2} \mathbf{u} \, d\mathbf{L}} \quad \mathbf{u} = \frac{\mathbf{i} \times \mathbf{n}}{\oint\limits_{L_1} \mathbf{i} \, d\mathbf{L}}. \tag{12}$$

So, it suffices to determine one of the vectors **u** and **i**.

Definition of the characteristic impedance

If the permittivity and permeability are scalar quantities, and if the z axis is parallel to the direction of propagation of the qTEM mode, the transversal components of the electric and magnetic field strength depend on the co-ordinate z in the following way:

$$\begin{aligned} \mathbf{E}_T(z) = \mathbf{E}^+ \exp(-j\beta z) + \mathbf{E}^- \exp(j\beta z) \\ \mathbf{H}_T(z) = \mathbf{H}^+ \exp(-j\beta z) + \mathbf{H}^- \exp(j\beta z), \end{aligned} \tag{13}$$

where

$$\mathbf{E}^- = r\mathbf{E}^+ \quad \mathbf{H}^- = -r\mathbf{H}^+, \tag{14}$$

and r is a scalar constant. In this case the following equations arise from the definition (2) of the voltage and current:

$$\begin{aligned} U(z) &= U^+ \exp(-j\beta z) + U^- \exp(j\beta z) \\ I(z) &= \frac{U^+}{Z_c} \exp(-j\beta z) - \frac{U^-}{Z_c} \exp(j\beta z), \end{aligned} \tag{15}$$

where

$$Z_c = \frac{(\mathbf{u}, \mathbf{E}^+)}{(\mathbf{i}, \mathbf{H}^+)} = -\frac{(\mathbf{u}, \mathbf{E}^-)}{(\mathbf{i}, \mathbf{H}^-)}, \tag{16}$$

and Eq. (16) gives obviously the definition of the characteristic impedance. Relationships (15) are the most important equations of the transmission-line theory, and they, together with all their consequences can be applied to the qTEM modes in the approximation previously detailed, if the characteristic impedance is defined by Eq. (16).

The following formulae can be derived from definition (16) on the basis of Eqs. (12) and the relationships between $\mathbf{E}^+$ and $\mathbf{H}^+$:

$$Z_c = \frac{\beta}{\omega} \int_{L_2} \mathbf{u}\, d\mathbf{L} \frac{(\mathbf{u}, \mathbf{E}^+)}{(\mathbf{u}, \varepsilon\mathbf{E}^+)} \tag{17}$$

$$Z_c = \frac{\omega}{\beta \oint_{L_1} \mathbf{i}\, d\mathbf{L}} \frac{(\mathbf{i}, \mu\mathbf{H}^+)}{(\mathbf{i}, \mathbf{H}^+)} . \tag{18}$$

Here the curves L_1 and L_2 have the same meaning as before. This formulae give a very simple relationship for the frequency dependence of the characteristic impedance, if either the permittivity or the permeability is constant over the cross section. If the permeability is constant, the inductivity per unit length is given by the following formula:

$$L = \frac{\mu}{\oint_{L_1} \mathbf{i}\, d\mathbf{L}} . \tag{19}$$

It follows from Eqs. (18) and (19) that the relationship

$$Z_c(\omega) = Z_c(0) \frac{v(\omega)}{v(0)} = Z_c(0) \sqrt{\frac{\varepsilon_{eff}(0)}{\varepsilon_{eff}(\omega)}} , \; \mu = \text{const.} \tag{20}$$

describes the frequency dependence of the characteristic impedance, where v denotes the phase velocity, and ε_{eff} denotes the effective permittivity defined in the usual way. If the permittivity is constant over the cross section, the formula

$$C = \frac{\varepsilon}{\int_{L_2} \mathbf{u}\, d\mathbf{L}} \tag{21}$$

gives the capacitance per unit length. It follows from Eq. (17) and (21) that the frequency dependence of the characteristic impedance is described by the relationship

$$Z_c(\omega) = Z_c(0) \frac{v(0)}{v(\omega)} , \qquad \varepsilon = \text{const.} \tag{22}$$

In these cases Denlinger [1] defines the characteristic impedance through the formulae (20) and (22) on an intuitive basis. If both ε and μ vary over the cross section, the definition of Denlinger differs from that one suggested in this paper, because his definition was formulated only in an intuitive way. Eq. (20) gives one of the two generally used definitions of the characteristic impedance of microstrips. The ideas presented in this papeı are new arguments for this definition and against the other definition based on transferred power.

The reflection coefficient is an important characteristic of the wave propagation. Its value is determined in the theory of transmission lines by means of the characteristic impedance. The error of the so calculated reflection coefficient is now investigated in the case already treated, i.e. for two waveguides of the same cross section that are connected one after the other. Let us fix the point $z=0$ at the junction of the two waveguides and define the quantities r_1 and r_2 by Eq. (14), i.e. $\mathbf{E}_1^- = r_1 \mathbf{E}_1^+$ and $\mathbf{E}_2^- = r_2 \mathbf{E}_2^+$, which means that at the junction $\mathbf{E}_{\mathrm{T}j} = (1+r_j)\mathbf{E}_j^+$. If the first waveguide is excited, the value of r_2 is determined by the termination of the second one. Though generally it is not easy to calculate r_2, its value is assumed to be known. Let us introduce the following new notations:

$$p_j = \frac{(\mathbf{u}, \mathbf{e}_{\mathrm{T}j})}{(\mathbf{u}, \mathbf{E}_j^+)} \quad q_j = \frac{(\mathbf{i}, \mathbf{h}_{\mathrm{T}j})}{(\mathbf{i}, \mathbf{H}_j^+)}, \, j=1,2 \tag{23}$$

$$Z = \frac{(\mathbf{u}, \mathbf{E}_{\mathrm{T}2} + \mathbf{e}_{\mathrm{T}2})}{(\mathbf{i}, \mathbf{H}_{\mathrm{T}2} + \mathbf{h}_{\mathrm{T}2})} \tag{24}$$

The quantity Z can be interpreted as an impedance loading the first waveguide, and can be expressed as:

$$Z = \frac{1 + r_2 + p_2}{1 - r_2 + q_2} Z_{c2}. \tag{25}$$

With these notations the reflection coefficient r_1 can be expressed as:

$$r_1 = \frac{(1+q_1)Z - (1+p_1)Z_{c1}}{Z + Z_{c1}}. \tag{26}$$

In the theory of transmission lines the quantities p_j and q_j are assumed to equal zero, which is an important facility, because it is very difficult to determine them. The vectors **u** and **i** were exactly defined so that the absolute values of p_j and q_j should be as small as possible, which guarantees that the errors of the loading impedance and the reflection coefficient r_1 calculated on the basis of the transmission-line theory are small except in the case of $r_2 \approx -1$ or $r_2 \approx 1$. If $r_2 \approx -1$, the relative error of Z is large, but it does not cause large error in r_1, unless if Z and Z_{c1} are of the same order, i.e. $Z_{c1} \ll Z_{c2}$. Similarly, the error of r_1 is large, if $r_2 \approx 1$ and $Z_{c1} \gg Z_{c2}$, but otherwise it is small.

In the foregoing considerations the two waveguides were assumed to be of the same cross section. The relationships of the transmission-line theory are also used, if the two cross sections are different. The presented ideas and especially the interpretation of the voltage and current related to the qTEM mode guarantee that this approximation gives also satisfactory results with the characteristic impedance defined in this paper, if the two cross sections differ slightly.

Appendix I

At zero frequency the transversal electric field of a qTEM mode in a fixed cross section can be expressed as

$$\mathbf{E}_T = -\operatorname{grad} \varphi$$

in terms of a potential φ that equals zero along the outline L_0 of one conductor and equals a constant value U_1 along the outline L_1 of the other conductor. So the functional U defined by (2) must have this value U_1, from which the following relationship results after simple transformations:

$$U_1 = U_1 \oint_{L_1} (\mathbf{n} \times \mathbf{u})\, d\mathbf{L} + \int_A \varphi \operatorname{div} \mathbf{u}\, dA,$$

where $\mathbf{n}$ denotes a unit vector normal to the cross section A. This relationship is true for every possible φ only if

$$\operatorname{div} \mathbf{u} = 0 \qquad \oint_{L_1} (\mathbf{n} \times \mathbf{u})\, d\mathbf{L} = 1$$

Obviously, L_1 can denote here any closed curve enclosing one of the conductors and not only the outline of the conductor. If the region A is cut along a curve L_2 connecting the two conductors, the transversal magnetic field can be similarly expressed as gradient of a potential, which has a jump of the same value I_2 at every point of the curve L_2, if the curve is crossed. The functional I defined by (2) must have this value I_2, from which it follows that the vector $\mathbf{i}$ fulfils the conditions

$$\operatorname{div} \mathbf{i} = 0 \qquad \int_{L_2} (\mathbf{i} \times \mathbf{n})\, d\mathbf{L} = 1,$$

and it has no normal component along the boundary curves of the region A.

Appendix II

The vectors $\mathbf{e}_{T1}$ and $\mathbf{e}_{T2}$ figuring in Eqs. (5) are composed of the transversal electric fields of the higher modes. Either the electric or the magnetic field of these modes vanishes at zero frequency. If the magnetic field equals zero, the electric field can be expressed as

$$\mathbf{e}=(\mathbf{e}_T+e_z\mathbf{n})\exp(\pm\gamma_z),$$

where the unit vector n and the z axis are normal to the cross section, $\mathbf{e}_T$ and e_z do not depend on z, and $\mathbf{e}_T\mathbf{n}=0$. As at zero frequency curl $\mathbf{e}=0$,

$$\mp\gamma\mathbf{e}_T+\operatorname{grad} e_z=0.$$

Multiplying this equation by vector $\mathbf{u}^*$ and integrating it over the cross section, yields after some transformations:

$$\pm\gamma\int_A \mathbf{u}^*\mathbf{e}_T\,\mathrm{d}A=\int_A \operatorname{div}(e_z\mathbf{u}^*)\,\mathrm{d}A-\int_A e_z\operatorname{div}\mathbf{u}^*\,\mathrm{d}A.$$

In consequence of Gauss's theorem the first integral on the right-hand side equals zero, because $e_z=0$ along the boundary curves of the region A. The second integral equals also zero, if div $\mathbf{u}=0$. As in case of higher modes $\gamma>0$ at zero frequency, it is proved that the inner products $(\mathbf{u},\mathbf{e}_{T1})$ and $(\mathbf{u},\mathbf{e}_{T2})$ vanish at zero frequency, if div $\mathbf{u}=0$. It can be proved similarly that the inner products $(\mathbf{i},\mathbf{h}_{T1})$ and $(\mathbf{i},\mathbf{h}_{T2})$ vanish also at zero frequency, if div $\mathbf{i}=0$, and the vector $\mathbf{i}$ has no normal component along the boundary curves of the cross section.

Appendix III

The solutions of two extreme value problems are presented in this Appendix. Let us denote by $\mathscr{H}$ the Hilbert space of the two dimensional vectorial functions square integrable over the doubly connected region A, by $\mathscr{U}$ the linear subspace of $\mathscr{H}$ that contains the solenoidal vectors of $\mathscr{H}$ and by $\mathscr{I}$ the linear subspace of $\mathscr{U}$ containing all the vectors that have no normal component along the boundary curves of A. $\mathbf{v}$ is an element of the Hilbert space $\mathscr{H}$, the curl of which is defined in the whole region A. The elements $\mathbf{u}$ of the subspace $\mathscr{U}$ and the elements $\mathbf{i}$ of the subspace $\mathscr{I}$ have to be determined for which the quantities $s=|(\mathbf{u},\mathbf{v})|/\|\mathbf{u}\|$ and $t=|(\mathbf{i},\mathbf{v})|/\|\mathbf{i}\|$ are maximum.

Let us denote by $\mathbf{v}_U$ and $\mathbf{v}_u$ the projection of $\mathbf{v}$ in the subspace $\mathscr{U}$ and in the one dimensional linear subspace defined by an element $\mathbf{u}\in\mathbf{U}$, resp. Obviously

$$s=|(\mathbf{u},\mathbf{v})|/\|\mathbf{u}\|=\|\mathbf{v}_u\|\leqq\|\mathbf{v}_U\|$$

and $\|\mathbf{v}_u\| = \|\mathbf{v}_U\|$, if and only if $\mathbf{v}_u = \mathbf{v}_U$. Hence s is maximum, if $\mathbf{u}$ is element of the one dimensional linear subspace defined by $\mathbf{v}_U$. It will be proved that

$$\text{curl}\,\mathbf{v}_U = \text{curl}\,\mathbf{v},$$

the tangential components of $\mathbf{v}_U$ and $\mathbf{v}$ are equal along the boundary curves of region A, and the integrals of $\mathbf{v}_U$ and $\mathbf{v}$ are equal along an arbitrary curve that connects the two boundary curves of the doubly connected region A.

If $\mathbf{v}$ is given, the previous conditions with the equation $\text{div}\,\mathbf{v}_U = 0$ determine unambiguously the vectorial function $\mathbf{v}_U$. It follows from these conditions that the difference $\mathbf{v} - \mathbf{v}_U$ can be expressed as

$$\mathbf{v} - \mathbf{v}_U = \text{grad}\,\Psi$$

in terms of a function Ψ that equals zero along the boundary curves of A. Simple transformations lead to the following relationship:

$$(\mathbf{v} - \mathbf{v}_U, \mathbf{v}_U) = \int_A \text{div}\,(\Psi^* \mathbf{v}_U)\,\mathrm{d}A - \int_A \Psi^*\,\text{div}\,\mathbf{v}_U\,\mathrm{d}A.$$

The two integrals on the right-hand side vanish because of Gauss's theorem and the relationship $\text{div}\,\mathbf{v}_U = 0$, from which it follows that $\mathbf{v}_U$ is really the projection of $\mathbf{v}$ in the subspace $\mathscr{U}$.

Similarly, the quantity t is maximum if $\mathbf{i}$ is element of the one dimensional linear subspace defined by $\mathbf{v}_I$, the projection of $\mathbf{v}$ in the subspace $\mathscr{I}$. It can be proved similarly that

$$\text{curl}\,\mathbf{v}_I = \text{curl}\,\mathbf{v},$$

and the integrals of $\mathbf{v}_I$ and $\mathbf{v}$ around any closed curve in the region A are equal.

References

1. DENLINGER, E. J.: A Frequency Dependent Solution for Microstrip Transmission Lines, IEEE Trans. Microwave Theory Tech., vol MTT-19, 30, (1971).
2. KRAGE, M. K., HADDAD, G. I.: Frequency Dependent Characteristics of Microstrip Transmission Lines, IEEE Trans. Microwave Theory Tech., vol. MTT-20, 678, (1972).
3. BIANCO, B. et al.: Some Considerations about the Frequency Dependence of the Characteristic Impedance of Uniform Microstrips, IEEE Trans. Microwave Theory Tech., vol. MTT-26, 182, (1978).
4. GETSINGER, W. J.: Microstrip Characteristic Impedance, IEEE Trans. Microwave Theory Tech., vol. MTT-27, 293, (1979).
5. BIANCO, B. et al.: Comments on "Microstrip Characteristic Impedance", IEEE Trans. Microwave Theory Tech., vol. MTT-28, 152, (1980).
6. GETSINGER, W. J.: Response to Comments on "Microstrip Characteristic Impedance", IEEE Trans. Microwave Theory Tech., vol. MTT-28, 152, (1980).

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