MEASURABILITY OF SET-VALUED FUNCTIONS

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Summary

In the paper to the definition of measurability of set-valued functions such a σ-algebra is presented by which the well-known properties of the real functions will furtheron hold, e.g. Lusin's and Egorov's theorems. The first two theorems discuss the correlation of the various definitions of measurability applied in the mathematical literature. The difficulty of proof lies in the fact that the Ω space does not inherit the properties of basic space Z, thus the image space of the examined functions is not a compact metric space.

Introduction

Because of its connections with control theory, differential inequalities and implicite differential equations, the theory of generalized differential equations with a set-valued right-hand side ($\dot{x} \in F(t, x)$) has a long history. Recently the fundamental existence theorem of Filippov [1]; [2] has been extended to equations with measurable right-hand side, see Hermes [3], Blagodatskih [5]. Blagodatskih's results have some connections with ordinary differential equations, too. This extension is due to Plis's generalization of Lusin's theorem to set-valued functions (see [4]). It would be very useful to extend somme basic theorems on real functions to measurable set-valued functions, in order to obtain new results in the original problems. Unfortunately, the measurability concept of set-valued functions — although it is a straight forward generalization of measurability — is not convenient enough even in proving very simple statements. For instance, to verify the measurability of continuous functions, or that of the limit of measurable functions, a rather sophisticated argument is needed. The aim of this paper is to replace the usual definition of measurability of set-valued functions by a new one, on the basis of which several theorems of real analysis can be extended to set-valued functions without any essential modification of the proofs. We prove that this definition coincides with the usual one, for example, if we consider compact valued functions in a locally compact space.

$\sum$ — measurability and $\cap$ — measurability

Let (Z, d) be a metric space. If A and B are bounded, nonempty subsets of Z, the Hausdorff metric ρ is defined as: $\rho(A, B)$ is the lower limit of non-negativ numbers λ such that A is contained in a λ neighbourhood of B, and B is contained in a λ neighbourhood of A. $\rho(A, B)=0$, if and only if the closure of A equals the closure of B; i.e. $\bar{A}=\bar{B}$. Therefore we can say that the nonempty bounded sets A, B are equivalent if $\bar{A}=\bar{B}$, and the nonempty bounded, closed set $\bar{A}$ represents the class of sets which are equivalent to A. Let A and B be nonempty, bounded, closed subsets of Z, then

$$\rho(A, B)=\inf\{\lambda;\ A\subset S_{B,\lambda} \quad \text{and} \quad B\subset S_{A,\lambda}\}$$

where

$$S_{A,\lambda}=\{x;\ x\in Z,\ d(A, x)\leqq\lambda\}$$

and

$$\mathrm{d}(A, x)=\inf_{y\in A}\mathrm{d}(x, y).$$

We shall denote the set of all nonempty bounded closed subsets of Z with topology induced by the Hausdorff metric ρ, by Ω, the set of all nonempty compact subsets of Z by Ω^*. For $C\in\Omega$ $(C\in\Omega^*)$ the C centered closed ball $\mathscr{G}_{C,\lambda}$ of radius λ in Ω (in Ω^*) is defined as:

$$\mathscr{G}_{C,\lambda}=\{A;\quad A\in\Omega \quad (A\in\Omega^*),\quad \rho(A, C)\leqq\lambda\},$$

and the smallest σ-algebra containing all the closed balls in Ω (in Ω^*) is denoted by $\sum$ (by $\sum^*$). Naturally $\Omega^*\subset\Omega$ and $\sum^*\subset\sum$.

The space (Ω^*, ρ) has analogue properties as the space (Z, d), see [6]. For example, Ω^* is separable if and only if Z is such. Therefore, in this case the σ-algebra Σ^* contains the open subsets of Ω^*. Indeed, every open set of a separable space is a countable union of open balls. But every open ball of a metric space is a countable union of closed balls.

Let $\{T, \Sigma_T\}$ be a measurable space.

Definition 1. The function F: $T\mapsto\Omega$ (Ω^*) is called *measurable with respect to Σ (to Σ^*)*, if $F^{-1}(\mathscr{A})\in\Sigma_T$, for every $\mathscr{A}\in\Sigma$ $(\mathscr{A}\in\Sigma^*)$, where

$$F^{-1}(\mathscr{A})=\{t;\ t\in T,\ F(t)\in\mathscr{A}\}.$$

Suppose that $F^{-1}(\mathscr{G})\in\Sigma_T$, for every closed ball $\mathscr{G}$ in Ω (in Ω^*). For any $\lambda\geqq 0$ the set

$$\{t;\ t\in T,\ F(t)\in\mathscr{G}_{C,\lambda}\}=\{t;\ t\in T,\ \rho(F(t), C)\leqq\lambda\}$$

with fixed $C \in \Omega$ ($C \in \Omega^*$) is measurable if, and only if the real-valued function ρ_C: $T \mapsto E^1$, $\rho_C(t) = \rho(F(t), C)$ is measurable. This assertion shows the equivalence of Definitions I and I′.

Definition I′. The function F: $T \mapsto \Omega$ (Ω^*) is called *measurable with respect to* Σ (*to* Σ^*), if the real-valued functions

$$\rho_C: T \mapsto E^1, \quad \rho_C(t) = \rho(F(t), C)$$

are measurable for every $C \in \Omega$ ($C \in \Omega^*$).

The notation $\mathcal{M}_\Sigma$ ($\mathcal{M}_{\Sigma^*}$) will be used to denote the set of measurable functions F: $T \mapsto \Omega$ (Ω^*) with respect to Σ (to Σ^*). We say, that the elements of $\mathcal{M}_\Sigma$ (of $\mathcal{M}_{\Sigma^*}$) are Σ — *measurable* (Σ^* — *measurable*). Of course, the Σ — measurable functions are Σ^* — measurable, i.e. $\mathcal{M}_\Sigma \subset \mathcal{M}_{\Sigma^*}$. In many applications (see [3], [4], [5]) another definition of measurability is used.

Definition II. The function F: $T \mapsto \Omega$ is called $\cap$ — *measurable*, if for each $A \in \Omega$ the set

$$T_A^1 = \{t;\, t \in T,\, F(t) \cap A = \emptyset\} \in \Sigma_T.$$

We shall denote the set of $\cap$ — measurable functions by $\mathcal{M}_\cap$.

In this section we will study the equivalence of Definitions I and II under separability of Z.

Lemma 1. Let Z be a separable metric space, and let F: $T \mapsto \Omega$ be $\cap$ — measurable, then

$$T_A^2 = \{t;\, t \in T,\, F(t) \subset A\} \in \Sigma_T \qquad \text{for each } A \in \Omega. \tag{1}$$

Proof. Since Z is a separable space, every open set $Z \setminus A = A^c$ is a countable union of closed balls A_i of Z. For arbitrary $A \in \Omega$;

$$T_A^2 = \{t;\, t \in T,\, F(t) \cap A^c = \emptyset\} =$$

$$= \left\{t;\, t \in T,\, F(t) \cap \bigcup_{i=1}^{\infty} A_i = \emptyset\right\} =$$

$$= \bigcap_{i=1}^{\infty} \{t;\, t \in T,\, F(t) \cap A_i = \emptyset\} \in \Sigma_T.$$

For compact valued functions this condition is both necessary and sufficient.

Lemma 2. Let Z be a separable metric space, then the function F: $T \mapsto \Omega^*$ is $\cap$ — measurable if and only if relation (1) holds.

Proof. If (1) is valid, then

$$T_A^1 = \{t;\, t \in T,\, F(t) \subset A^c\} =$$

$$= \bigcup_{n=1}^{\infty} \{t;\, t \in T,\, F(t) \subset (S_{A,\,n} \setminus \operatorname{Int} S_{A,\,\frac{1}{n}})\} \tag{2}$$

Indeed, let $t_0 \in T$ be such that $F(t_0) \subset A^c$. From the compactness of $F(t_0)$ it follows that

$$\inf_{x \in F(t_0)} d(x, A) = \lambda_1 > 0.$$

On the other hand,

$$\sup_{x \in F(t_0)} d(x, A) = \lambda_2 < \infty.$$

There exists a positive integer n such, that $\frac{1}{n} < \lambda_1 \leqq \lambda_2 < n$, that is

$$F(t_0) \subset (S_{A,n} \setminus \operatorname{Int} S_{A,\frac{1}{n}})$$

and it is easy to see that (2) really holds. The set (2) is an element of Σ_T, because

$$(S_{A,n} \setminus \operatorname{Int} S_{A,\frac{1}{n}}) \in \Omega.$$

The converse statement follows from Lemma 1.

Lemma 3. Let Z be a separable metric space, and let function $F\colon T \mapsto \Omega$ be $\cap$ — measurable, then

$$T_A^3 = \{t;\, t \in T,\, A \subset F(t)\} \in \Sigma_T, \quad \text{for each } A \in \Omega. \tag{3}$$

Proof. For arbitrary $A \in \Omega$,

$$T_A^3 = \{t;\, t \in T,\, D \subset F(t)\}, \tag{4}$$

where D is a countable dense set in A. Suppose $D \subset F(t)$, then for each $\varepsilon > 0$ and $a \in A$ we have an $x \in D$ such that $\rho(a, x) < \varepsilon$, whence

$$\rho(a, F(t)) \leqq \rho(a, x) + \rho(x, F(t)) < \varepsilon,$$

that is $\rho(a, F(t)) = 0$. Since $F(t)$ is a closed set, this implies $a \in F(t)$, i.e.

$$T_A^3 \subset \{t;\, t \in T,\, D \subset F(t)\}.$$

Since the opposite inclusion clearly holds, this proves (4). From (4) we get

$$T_A^3 = \{t;\, t \in T,\, x \in F(t) \text{ for each } x \in D\} =$$

$$= \bigcap_{x \in D} \{t;\, t \in T,\, \{x\} \cap F(t) \neq \emptyset\} \in \Sigma_T.$$

Theorem 1. Let Z be a separable metric space, $F\colon T \mapsto \Omega$ be $\cap$ — measurable, then F is Σ — measurable; i.e. $\mathcal{M}_\cap \subset \mathcal{M}_\Sigma$.

Proof. In the case of an arbitrary $A \in \Omega$, and a $\lambda \geqq 0$,

$$\{t;\, t \in T,\, \rho(A, F(t)) \leqq \lambda\} =$$

$$= \{t;\, t \in T,\, A \subset S_{F(t),\lambda}\} \cap$$

$$\cap \{t;\, t \in T,\, F(t) \subset S_{A,\lambda}\} \in \Sigma_T$$

in view of Lemmas 1. and 3.; since

$$\{t : t \in T, S_{F(t),\lambda} \cap A = \emptyset\} = \{t : t \in T, F(t) \cap S_{A,\lambda} = \emptyset\},$$

$$\text{and} \quad S_{A,\lambda} \in \Omega\text{, i.e. } S_{F(t),\lambda} \quad \text{is} \quad \cap\text{-measurable.}$$

The next Lemma can be found in a generalized form in [6]. For the sake of convenience we are giving here an independent proof.

Lemma 4. If (Z, d) is separable, then (Ω^*, ρ) is separable, too.

Proof. Let $D \subset Z$ be a countable dense set in Z, and consider the set $\mathscr{D}$ of finite subsets of D. $\mathscr{D}$ is countable and we prove that it is dense in Ω^*. Indeed, if $A \in \Omega^*$, then for each $\varepsilon > 0$ we have an open covering

$$A \subset \bigcup \{K_{x,\varepsilon}; \quad x \in D \quad \text{and} \quad K_{x,\varepsilon} \cap A \neq \emptyset\}.$$

Since A is compact, there exists such a finite subset $B \subset D$, that

$$A \subset \bigcup_{x \in B} K_{x,\varepsilon} \subset S_{B,\varepsilon},$$

where $K_{x,\varepsilon} \cap A \neq \emptyset$, that is $x \in S_{A,\varepsilon}$, i.e. $B \subset S_{A,\varepsilon}$. Thus for each $A \in \Omega^*$ and $\varepsilon > 0$ there exists a $B \in \mathscr{D}$ such that $\rho(A, B) < \varepsilon$ and this means that $\mathscr{D}$ is dense in Ω^*.

Theorem 2. Let Z be a separable metric space and let $F : T \mapsto \Omega^*$, then F is Σ^*-measurable if and only if it is $\cap$-measurable, i.e.

$$\mathscr{M}_{\cap} = \mathscr{M}_{\Sigma^*}.$$

Proof. From Lemma 3 we have $\mathscr{M}_{\cap} \subset \mathscr{M}_{\Sigma} \subset \mathscr{M}_{\Sigma^*}$, it is enough to show that $\mathscr{M}_{\Sigma^*} \subset \mathscr{M}_{\cap}$. Let A be an arbitrary element of Ω. Then

$$T_A^1 = \{t; \quad t \in T, \quad d(F(t), \quad A) > 0\} \tag{5}$$

where $d(A, B)$ denotes the ordinary distance of the sets $A, B \in \Omega$, i.e.

$$d(A, B) = \inf_{x \in A} d(x, B) = \inf_{x \in A} \inf_{y \in B} d(x, y). \tag{6}$$

We shall prove that the real-valued function $d : \Omega^* \mapsto E^1$, $d(X) = d(X, A)$ is measurable with respect to Σ^*. In this case the composite function $d(F(t)) = d(F(t), A)$ is measurable with respect to Σ_T, that is the set of (5) is an element of Σ_T, which makes complete the proof. Of course, a continuous real function defined on Ω^* is measurable, because the open sets of the separable metric space Ω^* generate Σ^*, and the continuous inverse image of an open set is open. It only remains to verify the continuity of the function d. It is done as follows. Let ε be an arbitrary positive number, and $\rho(X, Y) < \varepsilon$. This means that

$$X \subset S_{Y,\varepsilon} \quad \text{and} \quad Y \subset S_{X,\varepsilon}.$$

Thus for every $y \in Y$ there exists $x_y \in X$, such that $d(x_y, y) < \varepsilon$.

For each $a \in A$, $x \in X$, $y \in Y$ it holds

$$d(a, x) \leq d(a, y) + d(y, x),$$

whence

$$d(A, X) \leq d(a, y) + d(y, x).$$

Set $x = x_y$, then

$$d(A, X) \leq d(a, y) + \varepsilon \tag{7}$$

for each $a \in A$, $y \in Y$, and similarly,

$$d(A, Y) \leq d(a, x) + \varepsilon \tag{8}$$

for each $a \in A$, $x \in X$.

Exploiting Definition (6), from (7) and (8) we get

$$d(X, A) \leq d(Y, A) + \varepsilon,$$

$$d(Y, A) \leq d(X, A) + \varepsilon$$

whence

$$|d(X, A) - d(Y, A)| < \varepsilon, \quad \text{if} \quad \rho(X, Y) < \varepsilon;$$

i.e. the function d is continuous for each $A \in \Omega$.

If Z is a locally compact metric space, then Z is separable, and each closed and bounded set is compact; i.e. $\Omega = \Omega^*$ (see [7]). Therefore, all the measurability concepts discussed in this section are equivalent to each other, further on, the statement of Lemma 2 characterizes measurability, as well.

Applications

We are now in a position to extend some basic theorems of real analysis on continuous functions and on sequences of measurable functions to set-valued functions. When we speak of convergence of set-valued functions, it is always meant the convergence in the Hausdorff metric. If Z is complete, then Ω^* is complete, too (see [6]), thus Cauchy sequences converge. Apart from Corollary 2 we suppose that T is a measure space, i.e. (T, Σ_T, μ). As it is usually done, when speaking about convergence almost everywhere (a.e), we assume that (T, Σ_T, μ) is complete; i.e. all subsets of a zero-set are measurable.

Corollary 1. Assume that T is a topological space such that the open sets of T are contained in Σ_T. Let Z be a separable metric space, and let $F: T \mapsto \Omega^*$ be continuous, then it is Σ^*-measurable.

Proof. Because of the separability of Ω^*, the open sets belong to Σ^*, and generate it. Since the continuous inverse image of an open set is open again, and the sets with a measurable inverse image form a σ-algebra, this proves the statement.

Corollary 2. Suppose F_m is Σ-measurable (Σ^*-measurable), and there exists $F: T \mapsto \Omega(\Omega^*)$, such that

$$\rho(F_m(t), \quad F(t)) \to 0 \qquad \text{a.e.,} \quad \text{as} \quad m \to \infty;$$

then F is Σ-measurable (Σ^*-measurable).

Proof. Since ρ is continuous,

$$\rho(A, F(t)) = \lim \rho(A, F_m(t)) \qquad \text{a.e.,} \quad \text{as} \quad m \to \infty,$$

for each $A \in \Omega (A \in \Omega^*)$. However, $\rho(A, F_m(t))$ is a measurable real-valued function, that is $\rho(A, F(t))$, as a limit of measurable real valued functions, is measurable, too (with respect to Σ or Σ^*, respectively). Consequently, Definition I′ implies the statement.

Theorem 3. Let (T, Σ_T, μ) be finite, and $\lim F_m(t) = F(t)$ a.e. as $m \to \infty$, where F_m are Σ^*-measurable. Then there exists a partition $T_0, T_1, T_2, \dots$ of T into disjoint measurable sets in such a way that F_m converges uniformly on each $T_i, i = 1, 2, \dots$ and $\mu(T_0) = 0$.

Proof. Via Definition I′ we reduce the statement to Egorov's theorem. Observe that ρ as a function on $\Omega^* \times \Omega^*$ is continuous in its own topology, whence the measurability of $\rho_m(t) = \rho(F_m(t), F(t))$ follows immadiately. However, $\lim \rho_m(t) = 0$ a.e. on T, thus we have a partition $T_0, T_2, T_2, \dots$ (see [7]) such that $\lim \rho_m(t) = 0$ uniformly on each T_i, $i = 1, 2, \dots$, and $\mu(T_0) = 0$. This proves the above generalization of Egorov's theorem.

From now one Z will always denote a separable space, and we consider only compact valued functions. In this situation the concepts of $\cap$-measurability and Σ^*-measurability coincide (see Theorem 2.), thus we may use the term "measurable function".

Definition III. A measurable function is *simple (elementary)* if its range is a finite (countable) subset of Ω^*.

Theorem 4. Every measurable function can be uniformly approximated by a sequence of elementary functions.

Proof. In view of Lemma 4. there is a countable dense set $\mathscr{D}$ in Ω^*. Let $B_1, B_2, \dots, B_n, \dots$ be an array of the elements of $\mathscr{D}$. For each compact set C of Ω^* and for each $\varepsilon > 0$ there exists a $B_n \in \mathscr{D}$ such that $\rho(B_n, C) < \varepsilon$. Given a measurable function $F: T \mapsto \Omega^*$, define $F_\varepsilon(t) = B_n$ if n is the first indice for which $\rho(B_n, F(t)) < \varepsilon$. Since F takes on only compact values, $F_\varepsilon(t)$ is defined for each $t \in T$, *and* $\rho(F_\varepsilon(t), F(t)) < \varepsilon$. On the other hand, the set

$$\begin{aligned} T_n = \{t; \quad t \in T, \quad F(t) = B_n\} = \\ = \{t; \quad t \in T, \quad \rho(F(t), \quad B_n) \leq \varepsilon \quad \text{and} \quad \rho(F(t), \quad B_k) > \varepsilon, \\ \text{if} \quad k < n\} \in \Sigma_T; \end{aligned}$$

in view of Definition I′, i.e. F_ε is measurable; the proof is complete.

For simple functions we have a little bit weaker approximation theorem.

Theorem 5. Let (T, Σ_T, μ) denote a finite measure space, then each measurable function F is the limit of simple functions μ-almost everywhere.

Proof. In view of Theorem 4., for each m we have an elementary function F_m with $\rho(F_m(t), F(t)) < \frac{1}{m}$ for each $t \in T$. Denote $A_1, A_2, \ldots, A_n, \ldots$ the possible values of F_m and set $F'_m(t) = F_m(t)$ if $F_m(t) = A_n$ $n \leq N$ and $F'_m(t) = A_{N+1}$ if $F_m(t) = A_n$ $n > N$, where N is so large that

$$\sum_{n=N+1}^{\infty} \mu(\{t;\quad t \in T,\quad F_m(t) = A_n\}) < \frac{1}{m}.$$

Then F'_m is a sequence of simple functions such that

$$\mu(\{t;\quad t \in T,\quad \rho(F_m(t), F(t)) > \varepsilon\}) < \frac{1}{m}$$

if $\varepsilon > \frac{1}{m}$, that is F'_m goes to F in μ. However, $\rho_m(t) = \rho(F'_m(t), F(t))$ goes to zero also in μ, thus we can select a subsequence m_k such that $\lim \rho_{m_k}(t) = 0$ μ a.e., as $m_k \to \infty$. This means that $\lim F_{m_k} = F$ μ a.e., as $m_k \to \infty$.

As a consequence we prove a generalization of Plis's result (see [4]) on Lusin's theorem of set-valued functions.

Theorem 6. Let T be a complete, separable metric space and μ a finite measure on the σ-algebra Σ_T of its Borel sets. If $F: T \mapsto \Omega^*$ is measurable and $\varepsilon > 0$, then there exists a compact set $T^* \subset T$ such that $\mu(T \setminus T^*) < \varepsilon$, and F is continuous on T^*.

Proof. From Theorems 3 and 5 we have such a $T' \subset T$ and a sequence F_m of simple functions that $\mu(T \setminus T') < \frac{\varepsilon}{2}$, and $\lim F_m(t) = F(t)$ uniformly on T', as $m \to \infty$. Denote $A_1^m, A_2^m, \ldots, A_{K_m}^m$ the possible values of F_m and define

$$T_i^m = \{t;\quad t \in T'\ F_m(t) = A_i^m\} \quad i = 1, 2, \ldots, K_m.$$

Since μ is finite, and T complete, there exist compact sets $T_i^{*m} \subset T_i^m$ with $\mu(T_i^m \setminus T_i^{*m}) < \varepsilon \cdot K_m^{-1} \cdot 2^{-m-1}$ (see [8]); then F_m is continuous on the compact set $\bigcup_{i=1}^{Km} T_i^{*m} = T^{*m}$, $T^{*m} \subset T'$ and $\mu(T' \setminus T^{*m}) < \varepsilon \cdot 2^{-m-1}$. Therefore $T^* = \bigcap_{m=1}^{\infty} T^{*m}$ is compact, $\mu(T \setminus T^*) \leq \mu(T' \setminus T^*) + \mu(T \setminus T') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Further, F_m converges uniformly on T^* to F which proves the statement.

A. Plis has proved this theorem in the special case when Z is a compact metric space and his argument works only in locally compact spaces.

In a forthcoming paper several consequences of this theorem in the theory of set-valued integrals will be disscused.

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