SENSITIVITY ANALYSIS OF SWITCHED CAPACITOR NETWORKS

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Summary

In the paper simply derived formulas are presented for switched capacitor network sensitivities, and their useful interpretation as summed transfer function path-products is given. The extension of the nodal analysis is discussed for calculating all transfer functions necessary to evaluate the first and higher order sensitivities. At last the computer implementation of the presented technique in the program SCANSY is briefly mentioned.

Introduction

Recently a lot of attention has been devoted to switched capacitor (SC) networks because they allow filter implementation with low sensitivity in an integrated circuit. Since these circuits are often quite large, pen and paper analysis techniques are of limited use. The widely used nodal analysis (NA) and modified nodal analysis (MNA) allow simple implementation on computer [1,3–5]. However, the sensitivity analysis requires many computations even in the first order case [4, 5], using the sensitivities many informations about the network behaviour are available and even network optimization or approximate yield estimation can be performed.

In this paper a technique for general interpretation and simple determination of SC network sensitivities is presented. In Section 2 the definitions of the different frequency and z-domain transfer functions are detailed, then in Section 3 we derive the sensitivity formulae and give their interpretation. An extension of the NA is presented in Section 4 which allows an easy determination of all transfer functions for the sensitivity analysis.

The results stated in this paper are very general in the sense that they can be applied to SC networks with many phases, arbitrary duty cycles, with or without continuous coupling between input and output, in frequency or z-domain, for first or higher order sensitivities. The only restrictions are the linearity and ideality of the network elements.

Definitions

In the paper we consider SC networks containing ideal switches, capacitors and operational amplifiers. The switches are controlled by Boolean clock variables $\varphi_i(t)=0$ or 1. All switches driven by $\varphi_i(t)$ are open or closed at time t corresponding to $\varphi_i(t)=0$ or 1, respectively. The time is partitioned into time slots $\Delta_k=(t_{k-1}, t_k)$ such that the clock signals do not vary in Δ_k. We assume that the clock signals are periodic with N time slots in one period of duration T. The ensemble of the kth time slots in all periods is called the kth phase.

If any SC network defined above is excited by a piecewise-constant voltage source $u^i(t)=u_k^i$, $t\in\Delta_k$, then all node voltages in the network are also piecewise-constant signals $(v^j(t)=v_k^j,\ t\in\Delta_k)$. The constant values can be considered as samples in Δ_k, so we can define for any piecewise-constant signal

$$v^j(t)=\{v_1^j, v_2^j, \dots, v_N^j, v_{N+1}^j, \dots v_{k+rN}^j, \dots\}$$

N z-transforms (one for each phase) as

$$V_k^j(z)=Z\{v_{k+rN}^j\}=\sum_{r=0}^{\infty} v_{k+rN}^j z^{-r}, \qquad k=1,\dots,N \tag{1}$$

and analogous definitions Q_k^j, U_k^j, J_k^j for charge responces $q^j(t)$, voltage sources $u^j(t)$ and charge sources $j^j(t)$, respectively. The N z-transforms V_k^j of a signal can be collected in a vector V^j:

$$V^j=[V_1^j\, V_2^j \dots V_N^j]^{\mathrm{T}} \tag{2}$$

where $^{\mathrm{T}}$ denotes transpose and the upper index j (resp., lower index k) refers to the place (resp., the phase) of the signal.

With the above definitions the z-domain input-output relation can be expressed in the form of

$$V^o(z)=T^{oi}(z)U^i(z) \tag{3}$$

where $T^{oi}(z)$ is the z-domain voltage transfer function (VTF). In other words if non-zero voltage excitation is only applied in phase l and the output voltage is only observed in phase k, then the entry $T_{kl}^{oi}(z)$ of $T^{oi}(z)$ relates the z-transforms of the input sequence $U_l^i(z)$ to the output sequence $V_k^o(z)$. Different methods are available (see f.i. References [1–5]) to construct the z-domain equations of a SC network and to solve them for getting T^{oi} in (3).

The frequency domain transfer function $H(\omega)$ can be expressed by the z-domain $T^{oi}(z)$, but it depends on the applied source signal and the output observation. In all cases we assume bandlimited input signals, hence we only consider the baseband transmission without aliasing effect. Although the results can easily be extended for the aliasing case, for the sake of brevity, it is omitted here.

(i) Input sampled in phase 1 and held over a full period T, output sampled in phase k. If the input samples are taken from a sinusoidal excitation with pulsation ω then the samples of the output voltage in phase k fit also a sine wave with the same pulsation ω. The transfer function

$$H_k^{(s)}(\omega) = e^{-j\omega t_{k-1}} \sum_{l=1}^{N} T_{kl}^{oi}(e^{j\omega T}) \tag{4}$$

relates the complex phasors of these sine waves. The first term takes into account the time difference between the phases 1 and k, i.e. the sampling time instants.

(ii) Input and output as in (i) but output held over a time τ and passed through a smoothing filter. The transfer function

$$H_k^{(sh)}(\omega) = v(\omega) H_k^{(s)}(\omega) \tag{5}$$

with

$$v(\omega) = \frac{\tau}{T} \frac{\sin \omega\tau/2}{\omega\tau/2} e^{-j\omega\tau/2} \tag{6}$$

relates the Fourier transforms of the input and output signals of the whole system or in case of sinusoidal excitation it relates the complex input and output phasors at pulsation ω. $v(\omega)$ expresses the sample-and-hold effect.

(iii) Input as before, output directly passed through a smoothing filter. In this case the output can be considered as sampled in each phase k, $k = 1 \ldots N$ and held over $\tau_k = t_k - t_{k-1}$. The transfer function, having the same meaning as in (ii), has the form of

$$H^{(sh)}(\omega) = \sum_{k=1}^{N} v_k(\omega) e^{-j\omega t_{k-1}} \sum_{l=1}^{N} T_{kl}^{oi}(e^{j\omega T}) \tag{7}$$

with $v_k(\omega)$ as in Equ. (6) but τ substituted by τ_k.

(iv) Input and output piecewise-constant. The transfer function for phase k (output only observed in phase k) is given by

$$H_k^{(p)}(\omega) = v_k(\omega) \sum_{l=1}^{N} e^{-j\omega(t_{k-1} - t_{l-1})} T_{kl}^{oi}(e^{j\omega T}) \tag{8}$$

and for the overall transfer function (output as in (iii)) we have

$$H^{(p)}(\omega) = \sum_{k=1}^{N} H_k^{(p)}(\omega). \tag{9}$$

(v) Continuous input and output. In this case every signal can be partitioned [4] into a piecewise-constant component by sampling the signal at the end t_i^- of time slot i and holding backward in Δ_i, $i = 1 \ldots N$, and a remainder waveform having zero value at all t_l^-. Following this decomposition the transfer function for phase k is given by

$$H_k^{(c)}(\omega)=v_k^{(c)}(\omega)\sum_{l=1}^{N} e^{-j\omega(t_k-t_l)}T_{kl}^{oi}(e^{j\omega T})+$$

$$+\left[\frac{\tau_k}{T}-v_k^{(c)}(\omega)\right]T_{kk}^{oi}(\infty) \tag{10}$$

and for the overall transfer function we have

$$H^{(c)}(\omega)=\sum_{k=1}^{N} H_k^{(c)}(\omega) \tag{11}$$

with

$$v_k^{(c)}(\omega)=\frac{\tau_k}{T}\frac{\sin\omega\tau_k/2}{\omega\tau_k/2}e^{j\omega\tau_k/2} \tag{12}$$

expressing the backward-sample-and-hold effect.

Sensitivity formulas

In the SC network analysis every capacitor C can be characterized by its charge-voltage relation

$$Q^c=CMV^c \tag{13}$$

where V^c and Q^c are the capacitor voltage and incremental charge vectors, respectively, having values from the different phases, and the matrix M given by

$$M=\begin{bmatrix} 1 & & & -z^{-1} \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \tag{14}$$

implies that the charge Q_k^c in phase k depends upon the voltage difference between two consecutive phases, i.e. $V_k^c-V_{k-1}^c$. The upper right term z^{-1} shows that for $k=1$

$$V_{k-1}^c=z^{-1}V_N^c \tag{15}$$

i.e. the previous phase N belongs to the preceding period (z^{-1} corresponds to a one-period delay).

Concerning the input-output relation with respect to the capacitor C the signal flow graph [8] of Fig. 1a can be drawn. From the graph the z-domain VTF T^{oi} of (3) can be written as

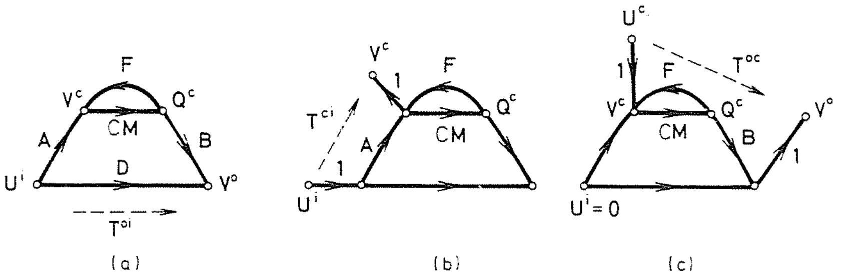

Fig. 1.(a) Signal flow graph of an SC network with respect to the capacitor C. (b) Transfer function T^{ci}. (c) Transfer function T^{oc}

$$T^{oi}=D+CB(1-CMF)^{-1}MA=D+CBP^{-1}MA \tag{16}$$

or as

$$T^{oi}=D+CBM(1-CFM)^{-1}A=D+CBMR^{-1}A \tag{17}$$

where the equivalence

$$P^{-1}M=MR^{-1} \tag{18}$$

can be verified by the identity

$$M(FM)^{-1}=(MF)^{-1}M. \tag{19}$$

Differentiating (16), applying the formula

$$\frac{\partial P^{-1}}{\partial C}=-P^{-1}\frac{\partial P}{\partial C}P^{-1} \tag{20}$$

for the derivative of an inverse matrix and using (18) we get the first order z-domain sensitivity as

$$S_c(z)=\frac{\partial T^{oi}(z)}{\partial C}=BP^{-1}PP^{-1}MA-CBP^{-1}(-MF)P^{-1}MA=$$

$$=BP^{-1}MR^{-1}A. \tag{21}$$

In the last expression two VTF's, shown in Figs 1b and 1c, can be recognized, i.e.

$$T^{oc}=CBP^{-1}M \qquad \text{and} \qquad T^{ci}=R^{-1}A. \tag{22}$$

They are the VTF's from the selected capacitor C to the output o and from the input i to the capacitor C, respectively. Substituting (22) into (21) we obtain

$$S_c(z)=\frac{1}{C}T^{oc}T^{ci}. \tag{23}$$

The Equ. (23) has the following interpretation. The two VTF's of (22) define a path between the input and output through the selected capacitor. Going backward on this path the product in (23) can be termed as a transfer-function-path-product (TFPP). Thus, the first order sensitivity is proportional to this TFPP.

The second order sensitivity can be obtained by differentiating (23):

$$S_{rs}^{(2)}=\frac{\partial^2 T^{oi}}{\partial C_r \partial C_s}=\frac{\partial}{\partial C_s}S_r=\frac{1}{C_r C_s}(T^{or}T^{rs}T^{si}+T^{os}T^{sr}T^{ri}). \tag{24}$$

Here again we can recognize two TFPP's corresponding to the two possible paths between the input and output through the selected capacitors, i.e. *i–s–r–o* and *i–r–s–o*. The second order sensitivity is proportional to the sum of these TFPP's corresponding to the possible paths defined above. The proportionality factor is 1 over $C_r C_s$, the product of the selected capacitor values.

Consecutive differentiations yield the following theorem:

The kth order sensitivity of the VTF $T^{oi}(z)$ with respect to the capacitors $C_1, C_2 \ldots C_k$ is the sum of TFPP's of all possible paths between the input and output through the selected capacitors divided by the product of the capacitor values. In a symbolic notation

$$S_{1\ldots k}^{(k)}=(C_1 C_2 \ldots C_k)^{-1} \sum_{\substack{\text{all}\\ \text{possible}\\ \text{paths}}} \text{TFPP}. \tag{25}$$

The theorem can be considered as an extension of the result in Reference [6] to the discrete time case for SC networks. A preliminary form of this theorem can be found in Reference [7].

The number of summed terms in (25) is $k!$ corresponding to the permutation of the k capacitors. Each term needs k matrix multiplications with N^3 multiplications of complex numbers. To determine all kth order sensitivities we need $\binom{n}{k}$ calculations detailed above, when n is the number of capacitors in the SC network. Therefore, a total sensitivity analysis in case of large N, n and k is a rather time consuming task. Nevertheless, in case of a 2-phase SC network and concerning only first order sensitivities we only need $8n$ complex multiplications.

The frequency domain sensitivities of the different VTF's defined in Equs. (4)–(12) can be expressed by the entries of the corresponding z-domain sensitivity. For example the sensitivity of $H_k^{(p)}(\omega)$ in (8) is given by

$$\frac{\partial \mathrm{H}_k^{(p)}(\omega)}{\partial C}=v_k(\omega)\sum_{l=1}^{N} e^{-j\omega(t_{k-1}-t_{l-1})}\left.\frac{\partial T_{kl}^{oi}(z)}{\partial C}\right|_{z=e^{j\omega T}} \tag{26}$$

and it needs the kth row of $S_c(z)$ in (23). All other calculations are straightforward and omitted here. The sensitivities of $T^{oi}_{kk}(\infty)$ in (10) can be obtained by well-known methods (f.i. by Ref. [6]) because $T^{oi}_{kk}(\infty)$ is the transfer function of the active C network of phase k.

Extension of nodal analysis

One of the most convenient methods for analyzing SC networks is the NA [1]. Following a similar way as in [1], on the base of Equ. (13) an N-phase nodal capacitance matrix can easily be built up and it is reduced according to the voltage force caused by the closed switches and the opamp inputs. Thus the z-domain equations are set up in the form of

$$J(z)=C(z)V(z) \tag{27}$$

or in detail

$$\begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{bmatrix} = \begin{bmatrix} C_{11} & & & -z^{-1}C_{1N} \\ -C_{21} & C_{22} & & \\ & \ddots & \ddots & \\ & & -C_{N,N-1} & C_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \tag{28}$$

where V_k and J_k are the node voltage and charge excitation vectors in phase k, respectively, the matrices C_{kl}'s are the reduced nodal capacitance matrices and the missing blocks are zero matrices. $C(z)$ in (28) has the same structure as the corresponding matrix in [3, 4] using MNA formulation but in (28) there are much less rows and columns.

Using the NA there arise two problems. First it can not handle voltage excitation. If we try to overcome this problem by some additional operations on the inverted matrix $D(z)=C^{-1}(z)$ there remains the second problem: for computing the inner VTF's for the sensitivity formulas of Equs. (23)–(25) we need to short-circuit the input port and this results in different network equations to be set up and to be solved. Although the MNA solves these problems, it needs putting additional voltage sources in series with all capacitors doubling consequently the number of the capacitive branch equations even in the second order sensitivity analysis. The advantages of the adjoint network concept in the MNA can be utilized only in the case of first order sensitivities.

We propose the following extension of the NA to avoid the above problems, by which we can convert any necessary voltage excitation into a charge excitation. First, we introduce the network of Fig. 2a as an input unit preceeding the SC network to be analyzed as shown in Fig. 2b. By inspection one can see that, connecting an uncharged unit capacitor in between the input and output in each phase, this network maps the charge excitation $j^{\hat{i}}(t)$ into the output voltage $v^i(t)= -j^{\hat{i}}(t)$ with the same value independently of the output load. Hence, any VTF in the original network excited by a voltage source $u^i(t)$ is equal to the corresponding charge-to-voltage transfer function (QVTF) in the extended network of Fig. 2b excited by a charge source $j^{\hat{i}}(t)= -u^i(t)$. The presence of the input network can be taken into account in the NA circuit description (28) by writing $C^{ii}_{kk}= -1$ and zero anywhere else into the ith row of each block row of $C(z)$. This extension of the NA is referred to as extended nodal analysis (ENA).

Our input network solves the short-circuiting problem as well. Because $j^{\hat{i}}=0$ implies $u^i=0$ the extended network without the input charge source is equivalent to the original network with short-circuited input port.

We have to solve yet the problem how to determine inner VTF's from a capacitor C using ENA. To do this we can partition the result in Equ. (21) as

$$S_c=(BP^{-1})M(R^{-1}A)=W^{oc}MT^{ci} \tag{29}$$

where from the graph of Fig. 1a W^{oc} is the QVTF from capacitor C to the output port. Comparing (23) and (29) we have for calculating the first term in (23)

$$\frac{1}{C}T^{oc}=W^{oc}M. \tag{30}$$

Resulting from the form of M in (14) the product $W^{oc}M$ can be calculated entry by entry as

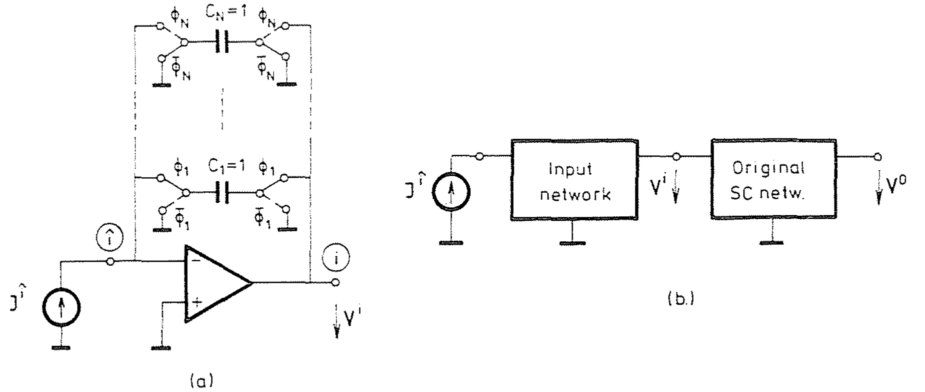

Fig. 2. (a) Input network. The composite switch pairs $\varphi_1 \dots \varphi_N$ connect uncharged unit capacitor in between the input and output in phases $1 \dots N$, respectively. (b) The use of the input network

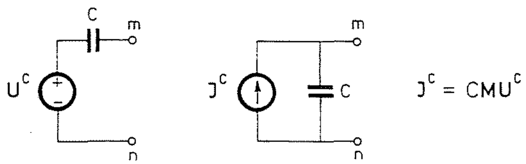

Fig. 3. Thevenin-Norton equivalence of voltage and charge sources in SC networks

$$(W^{oc}M)_{kl} = \begin{cases} W^{oc}_{kl} - W^{oc}_{k,l+1} & l=1\dots N-1 \\ W^{oc}_{kN} - z^{-1}W^{oc}_{k1} & l=N \end{cases} \tag{31}$$

avoiding any matrix multiplication.

We note here that Equ. (30) can be considered as the Thevenin-Norton equivalence in SC networks (see Fig. 3). Again from the form of M it states that a constant Thevenin voltage source U^c_k in phase k is equivalent with a constant Norton charge source $J^c_k = CU^c_k$, $J^c_{k+1} = -J^c_k$ in phases k and $k+1$.

Using the above detailed ENA and solving (27) we get

$$V(z) = D(z)J(z), \qquad D(z) = C^{-1}(z). \tag{32}$$

Any z-domain transfer function necessary for a simple network analysis and for a total sensitivity analysis can easily be read from the matrix $D(z)$. In detail:

$$\begin{aligned} T^{oi}_{kl} &= -W^{oi}_{kl} = -D^{oi}_{kl} \\ T^{ci}_{kl} &= -W^{ci}_{kl} = -(D^{mi}_{kl} - D^{ni}_{kl}) \\ W^{oc}_{kl} &= -(D^{om}_{kl} - D^{on}_{kl}) \\ W^{pq}_{kl} &= -(D^{m_p m_q}_{kl} - D^{m_p n_q}_{kl}) + (D^{n_p m_q}_{kl} - D^{n_p n_q}_{kl}) \end{aligned} \tag{33}$$

where the T's and W's are VTF's and QVTF's, respectively, and the capacitors C, C_p and C_q are connected between nodes m and n, m_p, and n_p, m_q and n_q, respectively.

Conclusion

In the paper simply derived formulas for the sensitivities of SC networks were presented and an interpretation easy to evaluate them was given. An extension of the NA was introduced by which each z-domain transfer function necessary for calculating any first or higher order sensitivity can directly be read from the inverted network matrix $D(z)$.

The presented technique has been implemented in the computer program SCANSY (*SC* network *an*alysis, *s*ensitivity and *y*ield calculation) in the case of 2-phase SC networks with 50% duty cycle and for first order sensitivities [9]. Among several examples the design of a PCM channel filter was checked by the program. Efficient improvement of the initial design could be arised only by investigating the sensitivity functions of the transmitter and receiver parts of the filter, such as correcting the inaccuracies of the LDI transformation in the design, decreasing the passband ripple and increasing the cutoff slope.

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