CAPACITIVE SLIT DISCONTINUITY IN A PARALLEL-PLATE GUIDE*

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Summary

The paper introduces the integral equation network of a bisected grating for multimode excitation. The method is fundamental for solving a wide class of electromagnetic problems.

Given a parallel plate waveguide with a zero thickness plate attached to it in a symmetrical manner as shown in Fig. 1. The guide and the discontinuity (the plate) are perfectly conducting.

The excitation is such that the only field component in the y direction is an electric field, which explains the words "capacitive slit" in the title.

The excitation is an E mode and due to the geometry no other mode is present. Observe further that the field components are independent of the x coordinate.

A general E mode excitation can be decomposed into two: in the even symmetric excitation (with respect to E_y) the electric field E_y is the same at z and $-z$, H_x at z and at $-z$ are of opposite sign, but of equal magnitude, hence H_x is zero at $z=0$ in the aperture region. This case is shown in Fig. 1. The odd

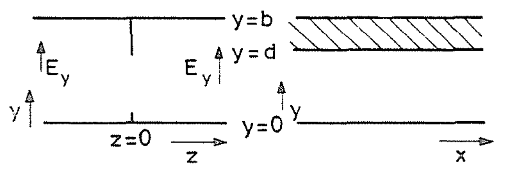

Fig. 1. Parallel plate waveguide with a zero thickness perfectly conducting discontinuity at $z=0$

symmetric case on the other hand is a trivial reflection problem since E_y equals zero at $z=0$ both on the discontinuity plate and in the aperture; this case is not treated.

* Dedicated to Professor Károly Simonyi on the occasion of his Seventieth Birthday

Due to symmetry, it is appropriate to treat half the structure only. We write for the transverse (to z) fields at $z=0$

$$E_y(y) = -\sum_{n=0}^{\infty} \frac{\varepsilon_n}{b} V_n(0) \cos\frac{n\pi y}{b} \tag{1}$$

$$H_x(y) = \sum_{n=0}^{\infty} I_n \cos\frac{n\pi y}{b}. \tag{2}$$

where ε_n is defined by

$$\varepsilon_n = \begin{cases} 1 & (\text{for} \quad n=0) \\ 2 & (\text{for} \quad n \geq 1) \end{cases}. \tag{3}$$

At the discontinuity plate ($z=0$) the electric field is zero for $d<y<b$, and the magnetic field H_x is zero for $0<y<d$.

We have in general the modes $0, 1, \ldots, N$ excited, while the $n>N$ modes are not excited. The z dependence of the excitation is given by

$$I_n(z) = \begin{cases} I_n \cos\kappa_n z - jY_n V_n(0) \sin\kappa_n z & (\text{for} \quad 0<n\leq N) \\ -Y_n V_n(0) e^{j\kappa_n z} & (\text{for} \quad n>N) \end{cases}. \tag{4}$$

Thus for a non-excited wave the scattered term, looking to the left, sees an admittance $-Y_n \cdot Y_n$ is given for the normalization used in (1) and (2) by

$$Y_n = \frac{\varepsilon_n}{b}\frac{\omega\varepsilon_0}{\kappa_n} = \frac{\varepsilon_n}{b}\frac{\omega\varepsilon_0}{\sqrt{\kappa_0^2 - \left(\frac{n\pi}{b}\right)^2}}. \tag{5}$$

We shall need the low frequency $\left(\text{i.e. } \kappa_0 \ll \frac{n\pi}{b}\right)$ limit of the previous expression; we denote it by Y_{ns}. Noting that the imaginary part of κ_n can never be positive we write

$$Y_{ns} = \frac{jz\omega\varepsilon_0}{n\pi} = \frac{Y_{1s}}{n}. \tag{6}$$

We shall proceed to formulate the integral equation for the discontinuity. For a given excitation the electric field $E_y(y)$ is the only unknown. It is equivalent, see Eq (1), to find $V_n(0)$; for brevity we shall denote the latter by V_n. To express V_n in terms of E_y multiply both sides of Eq (1) by $\cos\frac{n\pi y}{b}$ and integrate. Observe that $E_y=0$ for $d<y<b$:

$$V_n = -\int_0^d E_y(y) \cos\frac{n\pi y}{b}\, \mathrm{d}y. \tag{7}$$

We write for the aperture region ($0<y<d$), where H_x is zero, using Eqs (2) and (4) and adding to both sides of (2) the term $\sum_{n=1}^{\infty} Y_{ns}V_n \cos\frac{n\pi y}{b}$

$$I_0+\sum_{n=1}^{N}(I_n+Y_{ns}V_n)\cos\frac{n\pi y}{b}+\sum_{N+1}^{\infty}(Y_{ns}-Y_n)V_n\cos\frac{n\pi y}{b}=\sum_{n=1}^{\infty}Y_{ns}V_n\cos\frac{n\pi y}{b}. \tag{8}$$

We define $\hat{I}_n$ by

$$\hat{I}_n=\begin{cases} I_0 & \text{for } n=0 \\ I_n+Y_{ns}V_n & \text{for } n=1,\ldots,N \\ (Y_{ns}-Y_n)V_n & \text{for } n>N \end{cases} \tag{9}$$

and the kernel $K(y, y')$ is defined by

$$K(y,y')=-\sum_{n=1}^{\infty}\frac{1}{n}\cos\frac{n\pi y}{b}\cos\frac{n\pi y'}{b}=\frac{1}{2}\ln\left[2\left|\cos\frac{\pi y}{b}-\cos\frac{\pi'}{b}\right|\right]. \tag{10}$$

The identity contained in (10), namely that the sum is identical to the simple logarithmic format, is due to Julian Schwinger [1].

Using the information contained in (6) and (7), the definitions presented in (9) and (10), we rewrite the integral equation (8) in the form

$$\Sigma\hat{I}_n\cos\frac{n\pi y}{b}=Y_{1s}\int_0^d K(y,y')E_y(y')\,dy'. \tag{11}$$

The problem is shown in network terms in Fig. 2. The figure is incomplete in the sense that the coupling between the transmission lines has not yet been

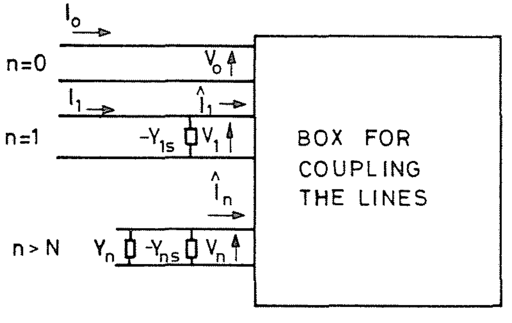

Fig. 2. Equivalent network of the bisected discontinuity. The box shows schematically that the transmission lines are coupled. The manner of the coupling has to be determined. (2)

determined. The transmission line $n=0$ and the line $n=1$ are both explicitly shown. The relation $\hat{I}_1 = I_1 + Y_1 V_1$ (see Eq (9)) is also evident from the figure.

We define a measure of the field $E_n(y)$ for each mode n by

$$E_y(y') = \Sigma Z_{1s} \hat{I}_n E_n(y') \tag{12}$$

where $Z_{1s} Y_{1s} = 1$. We substitute Eq (12) into (11) and compare the coefficients for each n

$$\cos \frac{n\pi y}{b} = \int_0^d K(y, y') E_n(y') \, dy' . \tag{13}$$

In order to deal with the problem in a network manner, we define network sufficients Z_{mn} by the relation

$$V_m = \sum_{n=0}^{\infty} Z_{mn} \hat{I}_n . \tag{14}$$

We recognize, of course, that by computing Z_{mn}, we can find the coupling of the lines, and thereby the box in Fig. 2. can be completed. Substitute (12) into (7)

$$V_m = -Z_{1s} \sum_{n=0}^{\infty} \left[\hat{I}_n \int_0^d E_n(y') \cos \frac{m\pi y'}{b} \, dy' \right]. \tag{15}$$

Comparing our last two equations, we get

$$\frac{Z_{mn}}{Z_{1s}} = - \int_0^d E_n(y') - \cos \frac{m\pi y'}{b} \, dy' . \tag{16}$$

Now our problem is stated mathematically in Eqs (10), (13) and (16). The latter equation makes it possible to cast the results in network terms and incorporate the discontinuity as a part of the microwave system.

To solve the integral equation, we must have orthogonal field in the aperture.

We define new variable φ by the transformation

$$\cos \frac{\pi y}{b} = \alpha \cos \varphi + \beta$$

$$\alpha = \sin^2 \frac{\pi d}{2b} \tag{17}$$

$$\beta = \cos^2 \frac{\pi d}{2b}$$

The new variable φ varies from zero to π as y varies from zero to d. Thus the kernel Eq (10) in terms of φ takes the form, using (17):

$$K(\varphi, \varphi') = \frac{1}{2} \ln \sin^2 \frac{\pi d}{2b} + \frac{1}{2} \ln (2 \,|\cos \varphi - \cos \varphi'\,|)$$

$$= \frac{1}{2} \ln \sin^2 \frac{\pi d}{2b} - \sum_{\nu=1}^{\infty} \frac{1}{\nu} \cos \nu\varphi \cos \nu\varphi' . \tag{18}$$

We write now $\cos \frac{n\pi y}{b}$ in Fourier series of $\cos \varphi$:

$$\cos \frac{n\pi y}{b} = \sum_{\nu=0}^{\infty} a_{n\nu} \cos \nu\varphi . \tag{19}$$

We have to evaluate the Fourier coefficients for each value of n. For example, for $n=1$ (see Eqs (19) and (17)) we get

$$\cos \frac{\pi y}{b} = \alpha \cos \varphi + \beta = a_{10} + a_{11} \cos \varphi + a_{12} \cos 2\varphi + \dots$$

$$\begin{aligned} &a_{10} = \beta \\ &a_{11} = \alpha \\ &a_{1\nu} = 0 \qquad (\text{for} \quad \nu \geq 2) . \end{aligned} \tag{20}$$

We get the table for the relevant coefficients:

Table 1

Fourier coefficients $a_{n\nu}$

ν \ n	0	1	2
0	1	β	$\alpha^2 + 2\beta^2 - 1$
1	0	α	$4\alpha\beta$
2	0	0	α^2
3	0	0	0

Note the triangular form of the table, i.e. $a_{n\nu} = 0$ for $\nu > n$.

Let us define a measure for the field in the n-th mode in terms of φ by

$$E_n(y)\,\mathrm{d}y = E_n(\varphi)\,\mathrm{d}\varphi . \tag{21}$$

We rewrite now the integral equation (13) in terms of the variable φ using Eqs. (18) (19) and (21):

$$\sum_{\nu=0}^{\infty} a_{n\nu} \cos \nu\varphi = \int_0^{\pi} \left\{ \frac{1}{2} \ln \sin^2 \frac{\pi d}{2b} - \sum_{\nu=1}^{\infty} \frac{1}{\nu} \cos \nu\varphi \cos \nu\varphi' \right\} E_n(\varphi')\,\mathrm{d}\varphi' \quad (22)$$

where $0 \leq \varphi < \pi$.

Equating coefficients we obtain:

$$\begin{aligned} a_{n0} &= \frac{1}{2} \ln \sin^2 \frac{\pi d}{2b} \int_0^{\pi} E_n(\varphi')\,\mathrm{d}\varphi' \\ a_{n\nu} &= -\frac{1}{\nu} \int_0^{\pi} E_n(\varphi') \cos \nu\varphi'\,\mathrm{d}\varphi'\,; \quad \nu \geq 1\,. \end{aligned} \quad (23)$$

It is not necessary to evalute $E_n(\varphi)$ explicitly to obtain the network coefficients Z_{mn} in terms of the Fourier coefficients $a_{n\nu}$. The latter can be expressed in terms of geometry.

To proceed we substitute into (16) and use (19) (21) and (23) to get

$$\frac{Z_{mn}}{Z_{1s}} = -\int_0^{\pi} E_n(\varphi') \sum_{\nu=0}^{\infty} a_{m\nu} \cos(\nu\varphi')\,\mathrm{d}\varphi' =$$

$$= -\sum_{\nu=0}^{\infty} \int_0^{\pi} E_n(\varphi') \cos \nu\varphi'\,\mathrm{d}\varphi' = -\frac{2a_{n0}a_{m0}}{\ln \alpha} + \sum_{\nu=1}^{\infty} \nu a_{m\nu} a_{n\nu}\,. \quad (24)$$

Using Eq (24) and Table 1, we indicate the most relevant (i.e. the lowest order) network coefficients. Denoting for shortness $-\dfrac{2}{\ln \alpha}$ by A we get

Table 2

Normalized network coefficients Z_{mn}/Z_{1s}

$n \backslash m$	0	1
0	A	βA
1	βA	$\beta^2 A + \alpha^2$

If the dominant mode is the only excited one and if this is the only propagated one, than in the lowest order of approximation for Eq (14) [see also Fig. 2] we get (using Eq. (6) for Z_{1s}):

$$
\begin{aligned}
V_0 &= Z_{00} I_0 = Z_{1s} A I_0 \\
Z_{00} &= \frac{1}{j\omega C_b} = \frac{\pi}{j2\omega\varepsilon_0 \ln \operatorname{cosec} \dfrac{\pi d}{2b}} .
\end{aligned}
\tag{25}
$$

We defined in (25) the capacitor C_b for a bisected structure (see Fig. 3.a′); for the total structure the capacitor is $2C_b$.

If we assume that two lowest modes ($n=0$ and $n=1$) are excited then we write [see Eq (14)] approximately [3]:

$$
\begin{aligned}
V_0 &= Z_{00} I_0 + Z_{01} \hat{I}_1 = Z_{00} I_0 + \beta Z_{00} \hat{I}_1 \\
V_1 &= Z_{10} I_0 + Z_{11} \hat{I}_1 = \beta Z_{00} I_0 + \beta^2 (Z_{00} + \alpha^2 Z_{1s}) \hat{I}_1 .
\end{aligned}
\tag{26}
$$

We rewrite the last equation in the form

$$
\frac{V_1 - \alpha^2 Z_{1s} \hat{I}_1}{\beta} = Z_{00} I_0 + \beta Z_{00} \hat{I}_1 . \tag{27}
$$

Obviously, the network representation Fig. 3 (b) and Eqs (26) (27) are identical. The network refers to the bisected structure.

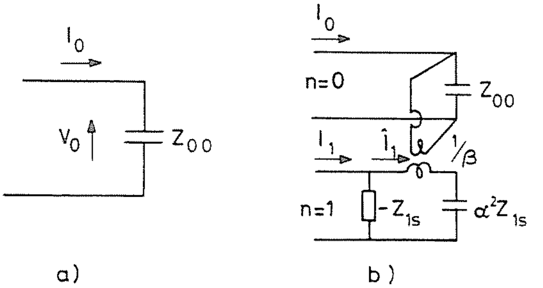

Fig. 3. Equivalent network for the bisected grating a) dominant mode incident b) two modes incident

Though we have achieved our main objective of finding the effect of the discontinuity on the propagating modes, for completeness, let us calculate the electric field E_y at the discontinuity. Let me remark in passing, that the knowledge of the field is less relevant for the behavior of the total system, than the previous calculation.

To proceed, let me restrict myself to the case when the effect of the higher ($n \geq 1$) modes is negligible.

The electric field $E_y(y)$ has been given by Eq (12) as a function of $E_n(y)$. The latter is given in terms of $E_n(\varphi)$ by Eq (21). We write $E_n(\varphi)$ in Fourier cosine series:

$$E_n(\varphi)=\sum_{\nu=0}^{\infty} d_{n0}\cos\nu\varphi\,. \tag{28}$$

Due to the previously mentioned triangular character of table 1, we need only d_{00}; its value, as evident from Eq (23) is

$$d_{00}=\frac{2a_{00}}{\pi\ln\sin^2\dfrac{\pi d}{2b}}\,. \tag{29}$$

Taking differentials of Eq (17) we get

$$\pi\sin\frac{\pi y}{b}\,\mathrm{d}y=\alpha b\sin\varphi\,\mathrm{d}\varphi\,. \tag{30}$$

In order to express $\sin\varphi$ in terms of y, we use Eq (17) to get

$$\sin\varphi=\frac{2\sin\dfrac{\pi y}{2b}}{\sin^2\dfrac{\pi d}{2b}}\left\{\sin^2\frac{\pi d}{2b}-\sin^2\frac{\pi y}{2b}\right\}^{1/2}\,. \tag{31}$$

Thus for this case, we get from Eqs (12), (21), (28) to (31):

$$E_y(y)=-\frac{V_0}{b}\,\frac{\cos\dfrac{\pi y}{2b}}{\left\{\sin^2\dfrac{\pi d}{2b}-\sin^2\dfrac{\pi y}{2b}\right\}^{1/2}}\,. \tag{32}$$

Observe that for $y=0$ the electric field is

$$E_y(0)=-\frac{V_0}{b}\,\frac{1}{\sin\dfrac{\pi d}{2b}}\,. \tag{33}$$

The factor of $-\dfrac{V_0}{b}$ represents the dominant mode electric field for the static field in the absence of the discontinuity. Thus the field in the aperture at $y=0$ is increased by the factor $\dfrac{1}{\sin\dfrac{\pi d}{2b}}$. This factor increases as the width of the aperture decreases.

References

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