CLASSICAL ELECTRON MOTION IN ELECTROMAGNETIC FIELDS*

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Summary

While wondering what might be a suitable topic for my contribution to this issue, I came across a beautiful experiment described in a recent issue of Physics Today [1]. A group at the University of Washington managed to trap a single electron in a Penning trap for no less than 10 months. During the time of its captivity the electron, residing in a d.c. magnetic field and driven by microwave radiation, exhibited bistability and hysteresis. Details of the physics will be given shortly.

I have learned about electromagnetic fields and particle motion in such fields, as a student in Budapest, working under the supervision of Professor Simonyi, at the Technical University and the Central Research Institute for Physics (KFKI). Charged particle motion in electromagnetic fields plays a role in particle accelerators, plasma physics and free electron lasers to mention just a few examples. While the fields are described by the linear Maxwell equations, the particle motion is intrinsically nonlinear hence it can give rise to subtle effects many of which are still unexplored.

In the following three simple examples the motion of a single charged particle in electromagnetic waves will be described. The first involves bistability and hysteresis, the second particle acceleration and the third chaotic motion.

Bistability and Hysteresis

Consider an electron gyrating in a uniform magnetic field B_0. The electron is prevented from moving along this field, by the electric field of the Penning trap or by making B_0 a magnetic mirror configuration [2] A circularly polarized electromagnetic wave propagating along B_0 interacts with the electron. This interaction has a resonance near the cyclotron frequency $\Omega = qB_0/m$ (q, m are charge and mass respectively), except that m is a function

* Dedicated to Professor Károly Simonyi on the occasion of his Seventieth Birthday

of particle velocity $m=\gamma m_0$, where $\gamma=(1-v^2/c^2)^{-1/2}$ describes the relativistic mass increase.

Kaplan [3] has shown that even for $v/c \ll 1$ the relativistic effect is important and gives rise to bistability and hysteresis. It turns out that the approximations and the expansions of Ref. 3 are unnecessary, and a very simple physical picture can be presented from which the exact solution follows immediately.

The electron, also suffers radiation damping as well as damping due to the currents induced in the resistive wall, and possibly collisions with other particles. This will be represented by the damping force $-\Gamma v$. When the microwave source is switched on and after the transients died out the particle will gyrate on a circular orbit with the angular velocity ω, where ω is the wave frequency. Since the wave is circularly polarized the electric and magnetic fields rotate together with the electron. This situation is illustrated on Fig. 1. For the

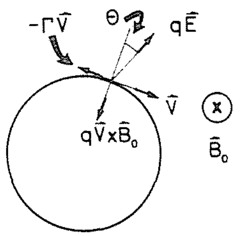

Fig. 1. Forces acting on gyrating particle

asimuthal forces one finds

$$qE \sin \Theta = \Gamma v \tag{1}$$

while the resultant radial force is responsible for the circular motion

$$m\omega v = qvB_0 - qE \cos \Theta \, . \tag{2}$$

Rearranging and squaring these equations gives

$$v^2(\omega/\sqrt{1-v^2/c^2} - \Omega_0)^2 = (qE/m_0)^2 \cos^2 \Theta \tag{3}$$

$$(\Gamma v/m_0)^2 = (qE/m_0)^2 \sin^2 \Theta \tag{4}$$

where $\Omega_0 = qB_0/m_0$. Adding (3) and (4) gives immediately

$$(v/c)^2[\Gamma^2/m_0^2 + (\omega/\sqrt{1-(v/c)^2} - \Omega_0)^2] = (qE/m_0c)^2 \, . \tag{5}$$

This is our fundamental equation from which v/c is to be determined. In order to see that three solutions are possible consider the $\Gamma=0$ limit. With $v/c=\beta$ this becomes

$$f(\beta) = \beta(\omega/\sqrt{1-\beta^2} - \Omega_0) = \pm qE/m_0c \, . \tag{6}$$

The function $f(\beta)$ starts out as a linear function with a negative slope for $\omega < \Omega_0$, for $\beta \ll 1$, and becomes large positive for $\beta \to 1$ and large negative for $\beta \to -1$. If the constant qE/m_0c is not too large it intersects the $f(\beta)$ curve in three points, provided $\omega < \Omega_0$. More quantitatively $f(\beta)$ has extrema where $f'(\beta)=0$ giving

$$f' = \omega/(1-\beta^2)^{3/2} - \Omega_0 = 0 \tag{7}$$

so

$$(\omega/\Omega_0)^{2/3} = 1-\beta^2 . \tag{8}$$

Substituting back this value of $\beta = \beta_0$ into (6) gives the condition for three roots.

$$|\, qE/m_0\Omega_0 c \,| < |f(\beta_0)/\Omega_0| = [1-(\omega/\Omega_0)^{2/3}]^{3/2} . \tag{9}$$

When Γ is finite one solves (5) numerically. Introducing the dimensionless variables $y=\beta^2$, $G=(\Gamma/m_0\Omega_0)^2$, $K=(qE/m_0c\Omega_0)^2$ and $x=\omega/\Omega_0$ the result of such a calculation is shown in Fig. 2. It shows a resonance curve tilted to the left

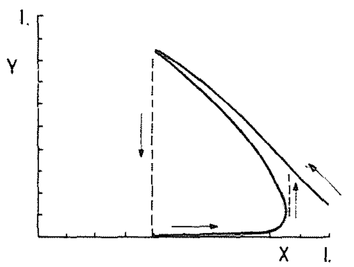

Fig. 2. Hysteresis curve $y=\beta^2$ versus $x=\omega/\Omega_0$, with field strength parameter $K=10^{-3}$ and damping parameter $G=10^{-3}$

due to the relativistic nonlinearity. There is evidently a range of $\omega/\Omega_0 < 1$ where three different circular orbits exist. It turns out upon closer examination that the middle orbit is unstable. This leaves two stable orbits hence we have bistability.

Which orbit will the electron chose? This depends on the initial conditions. Suppose the wave frequency is slowly varied in time. Starting from the right the particle climbs up the resonance curve as the frequency is reduced, increasing its kinetic energy. After the peak is reached it falls down to the lower curve. If now the frequency is increased it moves to the right on the lower curve until it is forced to jump up again. The result is hysteresis. What happens to the energy difference as the electron jumps down to the lower curve? Evidently it is radiated away. This effect may form the basis of a new kind of free electron laser, where energy is pumped continuously into electrons from a microwave source, to be radiated away in an instantaneous burst. [4]

Finally a remark about the physical basis of the bistability. When the forces qE and $qv \times B_0$ are opposite as shown in Fig. 1 the particle velocity is small since $mv\omega$ balances the sum of these forces. The other possibility is that these forces are in the same direction (for $\Gamma \to 0$) giving rise to a large gyration velocity.

One can find other configurations as well that produce bistability and hysteresis. A linearly polarized wave propagating perpendicular to B_0 with the electric field vector oscillating in the B_0 direction produces similar effects if the electron has an average drift velocity along B_0. The physics is somewhat more involved and the mathematics less transparent [5].

Particle Acceleration

Since the wave magnetic field B is perpendicular to E on Fig. 1 in the $\Gamma \to 0$, ($\Theta \to 0$) limit, B is collinear with v so $v \times B = 0$. One may set however $E // v$ producing particle acceleration perpendicular to B_0 as well as a parallel acceleration along B_0 due to $qv \times B$. There is of course the question whether the phase relationship necessary for acceleration can be maintained over many oscillation periods. It turns out that this is possible [6, 7], and acceleration schemes based on this effect have been proposed more then 20 years ago [6]. With the availability of high power lasers this subject may be revived [8, 9]. In the following a different derivation of this process is given [10].

We set up the uniform magnetic field B_0 in the z z direction, the circularly polarized wave field as $E = E_0[\hat{x} \cos(kz - \omega t) - \hat{y} \sin(kz - \omega t)]$, $B = k \times E/\omega = \hat{z} \times E/c$, and the charge of the electron $q = -e$. The equations of motion are

$$\mathrm{d}P/\mathrm{d}t = -e[E + v \times (B + \hat{z}B_0)] = -e[E + v \times (\hat{z} \times E/c + \hat{z}B_0)] \tag{10}$$

where $P = m_0 \gamma v$ is the electron momentum. Energy conservation requires

$$d/\mathrm{d}t(m_0 c^2 \gamma) = -eE \cdot v\,. \tag{11}$$

The $\hat{z}$ component of (10) gives

$$\mathrm{d}P_z/\mathrm{d}t = -e/cE \cdot v\,. \tag{12}$$

According to (10) and (11) the rate of change of energy is just c times the rate of change of z momentum. For each wave photon absorbed by the electron its energy increases by $\Delta\varepsilon = \hbar\omega$, and its forward momentum by $\Delta P_z = \hbar k = \hbar\omega/c$.

A constant of motion

$$\gamma(\omega - kv_z) = a = \gamma_0(\omega - kv_{z0}) \tag{13}$$

follows from (10) and (11), where γ_0 and v_{z0} are the values at $t = 0$ when the electron is injected.

To analyze the transverse components of (10) it is convenient to introduce $v^- = v_x - iv_y$ and $E^- = E_x - iE_y = E_0 \exp[i(kz-\omega t)]$, to find

$$d/\mathrm{d}t(m_0\gamma v^-) + im_0\Omega\gamma v^- = -eE^-(1-v_z/c) = -(ea/\omega)(E^-/\gamma) \qquad (14)$$

where $\Omega = \Omega_0/\gamma$, and (13) has been used to obtain the right hand side of (14).

To integrate this equation one writes it as

$$d/\mathrm{d}t[\gamma v^- e^{i\int_0^t \Omega \mathrm{d}t}] = -ea/m_0\,\omega e^{-\int_0^t \Omega \mathrm{d}t}(E^-/\gamma) \qquad (15)$$

and integrates

$$\gamma v^- = -eaE_0/m_0\,\omega e^{-\int_0^t \Omega \mathrm{d}t} \int_0^t e^{i(kz'-\omega t' + \int_0^{t'} \Omega \mathrm{d}t'')}/\gamma\, \mathrm{d}t'. \qquad (16)$$

However from (13)

$$kz = k\int_0^t v_z\, \mathrm{d}t' = \int_0^t (\omega - a/\gamma)\, \mathrm{d}t' \qquad (17)$$

and the integral in (16) can be evaluated

$$\int_0^t \frac{e^{i(\Omega_0-a)\int_0^{t'} \gamma^{-1}\mathrm{d}t''}}{\gamma(t')}\mathrm{d}t' = \frac{1}{i(\Omega_0-a)}\int_0^t \left\{\left[\frac{\gamma(t')}{i(\Omega_0-a)}\right]^{-1} \exp\int_0^{t'} \frac{i(\Omega_0-a)}{\gamma(t'')}\mathrm{d}t''\right\}\mathrm{d}t' =$$

$$= \frac{1}{i(\Omega_0-a)}\left[e^{i(\Omega_0-a)\int_0^t \frac{\mathrm{d}t'}{\gamma}} - 1\right]. \qquad (18)$$

Substitute this back into (16) to find

$$\gamma v^- = eaE_0 i/m_0\omega(\Omega_0-a)\,[e^{-ia\int_0^t \mathrm{d}t'/\gamma} - e^{-i\Omega_0\int_0^t \mathrm{d}t'/\gamma}]. \qquad (19)$$

From (13) $\int_0^t a/\gamma\, \mathrm{d}t' = \omega t - kz$, so

$$\gamma v^- = \frac{eaE_0 i}{m_0\omega(\Omega_0-a)} e^{i(kz-\omega t)}\,[1 - e^{-i(\Omega_0-a)\int_0^t \frac{\mathrm{d}t'}{\gamma}}]. \qquad (20)$$

Finally from (11)

$$m_0c^2 d\gamma/\mathrm{d}t = -e\,\mathrm{Re}\,(E^* v^-) = (e^2aE_0^2/m_0\omega)\frac{\sin[(\Omega_0-a)\int_0^t \mathrm{d}t'/\gamma]}{(\Omega_0-a)\gamma} \qquad (21)$$

or

$$d/\mathrm{d}t(\gamma^2/2) = (eE_0/m_0\omega c)^2 a\omega \frac{\sin\left[(\Omega_0 - a)\int_0^t \mathrm{d}t'/\gamma\right]}{\Omega_0 - a}. \tag{22}$$

In general as γ and $\int \mathrm{d}t'/\gamma$ change with time the right hand side oscillates and the electron gains and loses energy. Consistent acceleration occurs when $a = \Omega_0$, and

$$d/\mathrm{d}t(\gamma^2/2) = (eE_0/m_0\omega c)^2 a\omega \int_0^t \mathrm{d}t'/\gamma \tag{23}$$

or

$$d^2/\mathrm{d}t^2(\gamma^2) = 2(eE_0/m_0\omega c)^2 a\omega/\gamma \tag{24}$$

and the particle consistently gains energy as well as forward momentum, at a rate proportional to $\gamma \sim t^{2/3}$.

If the particle is to be accelerated from rest, the resonance condition $a \approx \Omega_0$ requires $\omega \approx \Omega_0$. For even a relatively low frequency laser beam, like CO_2 with ω of the order of 10^{14}/sec this would require B_0 to be a many megagauss field, and is therefore not feasible. If one injects the electron with a high velocity $v_{z0} = c - \delta$, $a = \omega/\sqrt{2c/\delta - 1} = \omega/\sqrt{2c/\delta}$, so $a \ll \omega$ for $\delta/c \ll 1$. One can now match a with Ω_0, and acceleration is possible. The particle experiences the laser field Doppler downshifted to its own cyclotron frequency.

Chaotic Electron Motion

In the two problems just discussed the equations of motion could be solved analytically with relative ease. Problems of that type, (and practically all examples treated in textbooks) are called integrable. There has been growing awareness in the last few decades, that most problems involving equations of motion of particles do not fall into this category and are in fact non-integrable. A famous old example of this class of problems is the three body problem of classical mechanics. Since the publication of the Kolmogorov, Arnold, Moser (KAM) theorem [11], and the appearance of a number of papers using high speed computers to integrate equations of motions numerically, a much clearer understanding of such problems has emerged [12].

Integrable problems are easy to define. An n degree of freedom system, with a Hamiltonian $H(q_1 \ldots q_n,\ p_1 \ldots p_n)$ is integrable if there exist n independent constants of motion $F_i(q_1 \ldots q_n, p_1 \ldots p_n)$, "in involution", namely the Poisson bracket expressions for any pair $[F_i, F_j] = 0$. The Kepler problem (two body problem) has three degrees of freedom, and the energy E, one component of the angular momentum say L_z, and the angular momentum square L^2 are constants of motion in involution.

All other Hamiltonian problems are nonintegrable. The trouble is that there is no general method for finding the constants of motion, so after one found $n-1$ of them one is still not sure whether the last one exists or not. Numerical integration of the equations is of course always possible, and the results reveal with reasonable certainty (sufficient for the physicist if not for the mathematician), whether the problem is integrable. The phase space of nonintegrable equations always contains chaotic regions. Chaos is usually defined by sensitive dependence on initial conditions, namely the phase space trajectories of any two particles in this region, started with arbitrarily close initial conditions, diverge exponentially with time. In fact the trajectories in this region behave in a manner similar to ergodic behavior postulated in statistical mechanics.

Chaotic regions are bounded by toroidal KAM surfaces. These are nonchaotic trajectories, similar to many found in integrable systems, covering the surface of an n torus. While there can be large chaotic regions, there are an infinity of very small ones as well. In fact one can find chaotic regions arbitrarily close to KAM tori.

The minimum number of phase space dimensions for a nonintegrable system is three. ($H(q, p)$ where $n=1$ has the total energy as a constant). Instead of representing the trajectories in this space, one looks at the intersections of these with a plane. On this surface of section plot, introduced first by Poincaré, a trajectory is typically represented by a sequence of points, or if the trajectory is a KAM torus, a closed line made up of an infinity of points. The points corresponding to a chaotic trajectory will be scattered all over the chaotic domain.

A charged particle moving in the field of two or more waves in general represents a nonintegrable problem [13, 14]. Such is the case for a wave in a cavity resonator. Here we take the simple case of a plane polarized wave bouncing between two parallel mirrors resulting in a standing wave (the sum of two traveling waves).

Let us take the planes of the mirrors parallel to the $x-y$ plane and let the wave be polarized in the y direction. The field can be characterized by a vector potential $A=\hat{y}A(z, t)=\hat{y}a_0 \sin \omega t \sin kz$. The Hamiltonian of the charged particle is

$$H=c\sqrt{m_0^2c^2+P_x^2+(P_y-qA)^2+P_z^2}\,. \tag{25}$$

Since H does not depend on x and y, P_x and P_y are constants. To simplify matters we chose $P_x=P_y=0$, without sacrificing generality since one can always get rid of such constants by a Lorentz transformation. We have now $H(z, P_z, t)$ and no apparent constant of motion. Phase space is the three dimensional z, P_z, t space of the trajectories and since A is periodic in time we chose a $t=\text{const}$ surface for the surface of sections as specified later. The equations of motion are Hamilton's canonical equations

$$\dot{z}=\partial H/\partial P_z=cP_z/\sqrt{m_0^2c^2+q^2A^2+P_z^2} \tag{26}$$

$$\dot{P}_z=-\partial H/\partial z=-q^2A\,\partial A/\partial z(c/\sqrt{m_0^2c^2+q^2A^2+P_z^2})\,. \tag{27}$$

These equations could have been obtained just as well from (10). It is convenient to introduce the dimensionless quantities $P=P_z/m_0c$, $A_0^2=$ $=q^2a_0^2/m_0^2c^2$, and the new time $\omega t\to\pi t$ and coordinate $kz\to\pi z$. This gives

$$\dot{z}=P/\sqrt{1+A_0^2\sin^2\pi t\sin^2\pi z+P^2} \tag{28}$$

$$\dot{P}=-\frac{(\pi/2)A_0^2\sin 2\pi z\sin^2\pi t}{\sqrt{1+A_0^2\sin^2\pi t\sin^2\pi z+P^2}}\,. \tag{29}$$

This system of equations is periodic in z and t with period one. One also notes two symmetries: $t\to-t$, $z\to-z$, as well as $t\to-t$, $P\to-P$ leave the equations invariant. Consequently the z as well as the P axis are symmetry lines of the mapping. There is one parameter, the field intensity $A_0^2\equiv K$.

We proceed now as follows: After fixing the parameter K we pick some z, P as an initial condition and integrate numerically from $t=0$ to $t=1$. This gives some new z, P value, a point on this "time one map". Further integration for a unit time interval gives the second point and so on. The result is seen in Fig. 3.

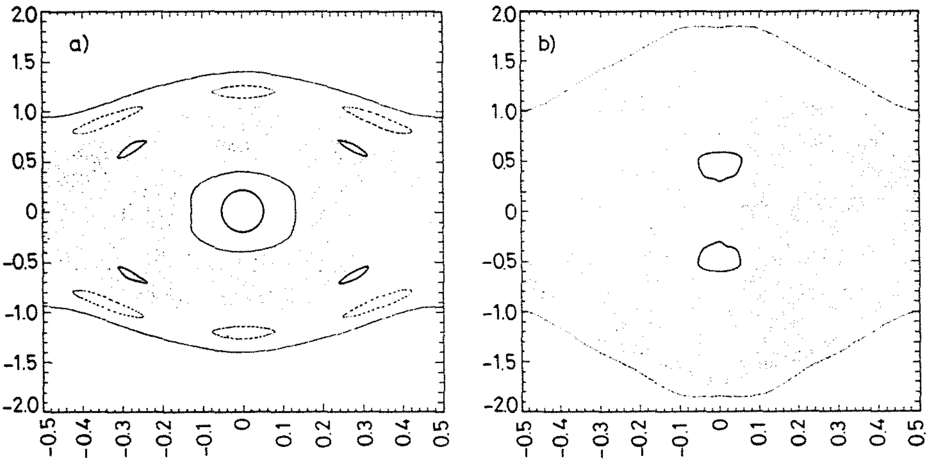

Fig. 3. Surface of section plots for Eqs (28) and (29). 2000 iterations were used for chaotic orbits, 500 for regular ones

a. $K=1$. Seed points: $z=0$, $P=.2$ and .4 gives circles, $z=.3$, $P=\pm.55$ period two circles, $z=0$, $P=\pm1.14$ period three circles, $z=0$, $P=\pm1.4$ KAM line. Chaotic trajectory originates from $z=0$, $P=-.5$.

b. $K=2$. Seed points: $z=0$, $P=\pm.4$ period two circles, $z=0$, $P=\pm1.84$ KAM lines, $z=0$, $P=1$ chaos

Due to the periodicity in z all points can be exhibited on the $-.5<z<.5$ interval. These surfaces of section plots are a good representation of the trajectories. Chaos and KAM lines, the intersection of KAM tori with the $t=$const. plane, are clearly visible. Some KAM lines surround islands in the sea of chaos. These islands are associated with stable periodic orbits.

Figure 3a shows the map for $K=1$. This map was constructed from the following "seeds" (initial conditions). From $z=0$, $P=.2$ and $z=0$, $P=.4$ originate two KAM circles (topological circles) inside the island formed about the fixed point $Z=0$, $P=0$. The seeds at $z=.3$, $P=\pm.55$ produce KAM circles for two sets of period two islands formed around period two stable fixed points of the map. There are two period three island chains from the seed points $z=0$, $P=\pm 1.14$. Two KAM lines originate from $z=0$, $P=\pm 1.4$. The chaotic points roughly filling the area between islands, and bounded by KAM lines, come from a single seed $z=0$, $P=-.5$.

The map for $K=2$ is shown in Fig. 3b. Chaos is much more widespread, and the chaotic points all come from the single seed $z=0$, $P=1$. There are again two sets of period two islands generated now from $z=0$, $P=\pm.4$. The KAM lines started from $z=0$, $P=\pm 1.84$. As K is further increased chaos spreads to larger areas of phase space.

The largest island in Fig. 3a corresponds to the "fixed point" of the map at $z=0$, $P=0$. It is clear from (28) and (29) that $z=P=0$ is a solution for all times. As long as it is stable, nearby points map into nearby points and there is an island around the origin. The stability of this point can be determined analytically. For $z\ll 1$, $P\ll 1$ one expands (28), (29) to find

$$\dot{z}=P \tag{30}$$

$$\dot{P}=-\pi^2 A_0^2 \sin^2 \pi t \cdot z \tag{31}$$

or

$$\ddot{z}+((\pi A_0)^2/2)\cdot[1-\cos 2\pi t]z=0\,. \tag{32}$$

This is Mathieu's equation whose stability properties are well known. One finds from tabulated values [15] that stable solutions are possible for $A_0^2=K<1.316$. So one expects the presence of the island for $K=1$, while at $K=2$ the island is inundated by the chaotic sea as illustrated in Fig. 3b. The other islands submerge similarly when the periodic orbits generating them destabilize [16–18].

One concludes that the motion of a charged particle in the field of a standing wave is non-integrable and has a complexity comparable to that of the three body problem.

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