SENSITIVITY ANALYSIS OF LINEAR NETWORKS BY MODIFIED NODAL METHOD

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Summary

The paper presents a new method of obtaining first order differential sensitivities of transfer functions of linear networks, related to any component parameter. The procedure does not necessitate adjoint network analysis or symbolic treatment of parameters, it only uses results of the preliminary network analysis. The calculation of the network is done using modified nodal analysis. Methods of determining sensitivity functions are discussed both for the time and for the frequency domain approaches. The interactivity of the LINA program implementing the procedure increases the efficiency of the network analysis.

Introduction

In many cases of analyzing linear time-invariant networks it is not only required to determine time functions of certain voltages or currents, transfer functions, pole-zero patterns, and Bode plots, but also to compute various sensitivities. In the following, if we use the term "sensitivity function" (i.e., function of the complex frequency s), it is meant to denote a first-order differential sensitivity related to any specific parameter of any network component. For convenience, we shall consider open-circuit voltages and short-circuit currents to represent the network outputs.

Quite a number of methods of determining sensitivity functions are to be found in literature (see, e.g., [2]–[7]). Some of them require the symbolic treatment of component parameters to be applied, as, e.g., the method based upon Bode's bilinear (or biquadratic, resp.) form of transfer functions, or the signal flow graph method. Another group of methods requires modifications of the network topology or of component values with a subsequent analysis of the modified network. As examples, we mention the approximation by difference quotients, the adjoint network method, the calculation of sensitivity functions as products of transfer functions, etc.

This paper presents a new method of determining sensitivity functions using only results of the analysis performed on the network with its components at nominal parameter values. The procedure applies modified nodal analysis, which permits the investigation of networks consisting of components of any kind (not only admittances) whose explicit characteristics are known. Our procedure is further characterized by the step-by-step building up of the reduced set of equations [1], which comprises all network equations and may be formulated either in the time domain or directly in the frequency domain.

In section 1, we show how, starting from the time-domain equations, transfer functions and their sensitivity functions are obtained as ratios of two polynomials. Values of these expressions assumed at specific frequencies may be obtained by substituting $s=j\omega_k$.

Section 2 deals with the frequency domain approach. For some fixed frequency, the reduced set of equations with a complex coefficient matrix is built up; its solution yields one point of the transfer characteristics. Sensitivity values are obtained for the same frequency. Computation is repeated for all prescribed frequencies.

The LINA program package is based upon the above procedure. A large number of options and the interactive way of running the program make network analysis more efficient. The options of the program are described in section 3.

1. The computation of transfer and sensitivity functions from the time-domain equations

1.1 *The determination of the transfer function*

All N-terminal components of the linear, invariant network to be analyzed may be specified by the characteristics of the following types:

capacitive component:

$$\mathbf{i}_C = \boldsymbol{C}\dot{\mathbf{u}}_C \tag{1}$$

inductive component:

$$\mathbf{u}_L = \boldsymbol{L}\dot{\mathbf{i}}_L \tag{2}$$

resistive component:

$$\left.\begin{aligned} \mathbf{i}_y &= \boldsymbol{G}\mathbf{u}_y + \boldsymbol{N}\mathbf{i}_z + \mathbf{j} \\ \mathbf{u}_z &= \boldsymbol{M}\mathbf{u}_y + \boldsymbol{R}\mathbf{i}_z + \mathbf{v} \end{aligned}\right\} \quad \text{hybride type characteristics}$$

$$\left.\begin{aligned} \mathbf{u}_a &= \mathbf{0} \\ \mathbf{i}_a &= \mathbf{0} \end{aligned}\right\} \quad \text{ideal amplifier characteristics} \tag{3}$$

Some ports of a restive component specified by a hybride characteristic are of the y-type (i.e., voltages are independent variables, currents are dependent variables), other ports are of the z-type (i.e., currents are independent variables, and voltages dependent variables). The ideal amplifier is treated as a nullor: the input port is of the a-type (nullator), the output port is of the b-type (norator). R, C, and L components, as well as the independent and controlled sources, gyrators, etc., are special cases of the above general types of components.

Our aim is to obtain from the time-domain equations, transfer functions related to any input and output, resp., in the form of rational functions. As a partial result of the procedure, we obtain the normal form of the state equation of the network.

Applying the modified nodal analysis method, we build up [1] the set of equations (4), i.e., the reduced set of equations as shown below, by taking into account the parameters of the network components one by one:

$$\begin{bmatrix} \boldsymbol{G}^* & \boldsymbol{N}^* & \boldsymbol{B}^* & \boldsymbol{C}^* & \boldsymbol{0} \\ \boldsymbol{M}^* & \boldsymbol{R}^* & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_a^* & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_C & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{A}_L & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \mathbf{i}_z \\ \mathbf{i}_b \\ \dot{\mathbf{u}}_C \\ \dot{\mathbf{i}}_L \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_L & \boldsymbol{0} & \boldsymbol{B}_j \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1}_v & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{1}_c & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_C \\ \mathbf{i}_L \\ \mathbf{v} \\ \mathbf{j} \end{bmatrix} \tag{4}$$

$$\boldsymbol{P} \cdot \mathbf{f} = \boldsymbol{Q} \cdot \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

where $\boldsymbol{\varphi}$ represents the nodal voltage vector; $\mathbf{i}_z$, the z-branch currents; $\mathbf{i}_b$, the output currents of the ideal amplifiers; $\mathbf{x} = \begin{bmatrix} \mathbf{u}_C \\ \mathbf{i}_L \end{bmatrix}$ is the state variable vector, and $\mathbf{e} = \begin{bmatrix} \mathbf{v} \\ \mathbf{j} \end{bmatrix}$ is the excitation (input) vector. The block rows are, in turn, expressions of the following relations: current laws, z-type port characteristics, voltages of ideal amplifiers, voltages of capacitors, voltages of inductors. The $\boldsymbol{G}^*$ block and the $\boldsymbol{R}^*$ block of the $\boldsymbol{P}$ coefficient matrix are determined by resistive components with z-type and y-type characteristics, respectively; the $\boldsymbol{C}^*$ block contains capacitances, the $\boldsymbol{L}^*$ block inductances, finally, $\boldsymbol{N}^*$ consists of current gains, $\boldsymbol{M}^*$ of voltage gains. The remainder of $\boldsymbol{P}$ and the $\boldsymbol{Q}$ matrix consits of topological 0, $+1$, -1 elements only; and does not contain component parameters.

According to the rules described in [1], the building up of the set of equations is achieved by taking into account network components one by one, and adding their respective parameter values to certain elements of the $\boldsymbol{P}$ and $\boldsymbol{Q}$ (initially both zero) matrices.

Thus we get to know in which elements of the matrices the parameters may be looked for. This will be shortly used when computing sensitivities.

Consider, as an example, resistance G between nodes l and j: it will cause elements with indices (ll) and (jj) in the $\boldsymbol{G}^*$ block to change by $+G$, elements (lj) and (jl) to change by $-G$. Likewise, if the controlling (primary) port of a voltage-controlled voltage source is connected to nodes l and j, and its output (secondary) port is the k-th z-type port, then the voltage gain μ is added to the element indexed (kl) in block $\boldsymbol{M}^*$, subtracted from the element indexed (kj), and does not modify any other element of $\boldsymbol{M}^*$.

The set of equations (4) may be solved by inversion of $\boldsymbol{P}$, if the network is a regular one, i.e., if $\det \boldsymbol{P} \neq 0$:

$$\mathbf{f}=\boldsymbol{T}\cdot\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}; \qquad \boldsymbol{T}=\boldsymbol{P}^{-1}\boldsymbol{Q}=\begin{bmatrix}\boldsymbol{R} & \boldsymbol{S}\\ \boldsymbol{A} & \boldsymbol{B}\end{bmatrix} \tag{5}$$

From this, we get the normal form of the state equation for one single input $e=[\mathbf{e}]_i$ and output $y=[\mathbf{y}]_k$:

$$\begin{aligned}\dot{\mathbf{x}}&=\boldsymbol{A}\mathbf{x}+\mathbf{b}e\\ y&=\mathbf{c}^T\mathbf{x}+de\end{aligned} \tag{6}$$

As already mentioned in the introduction, we suppose the output to be an open-circuit voltage or a short-circuit current. (All of the voltages may be written in the φ_l or $\varphi_l-\varphi_j$ form, i.e., as open-circuit voltages; some currents are i_{z_l} or i_{b_l}, but the other currents are not represented in $\mathbf{f}$. In such cases, let us insert a z-branch with $R=0$ as its parameter value and we get, as the required current value, the current element in $\mathbf{i}_z$.) In this way, the output in fact does explicitly show up in $\mathbf{f}$. In other words, $\mathbf{c}^T$ and d related to $y=[\mathbf{f}]_l$ constitute the l-th row in the matrices $\boldsymbol{R}$, $\boldsymbol{S}$.

The Laplace transform of the (6) state equation yields the transfer function

$$W(s)=\frac{Y_k(s)}{E_i(s)}=\mathbf{c}^T(s\boldsymbol{I}-\boldsymbol{A})^{-1}\mathbf{b}+d \tag{7}$$

In order to invert the matrix $(s\boldsymbol{I}-\boldsymbol{A})$, whose elements are polynomial ratios, we apply the Souriau-Frame algorithm:

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}=\frac{\operatorname{adj}(s\boldsymbol{I}-\boldsymbol{A})}{\det(s\boldsymbol{I}-\boldsymbol{A})}=\frac{\sum_{i=0}^{n-1}\boldsymbol{H}_{n-i}s^i}{\sum_{i=0}^{n}q_{n+1-i}s^i} \tag{8}$$

where n is the order of the network. Substituting this into (7), we get

$$W(s)=\frac{N(s)}{D(s)}=\frac{\sum_{i=0}^{n} k_{n+1-i}s^i}{\sum_{i=0}^{n} q_{n+1-i}s^i}; \tag{9}$$

$$k_{n+1-i}=\mathbf{c}^T\boldsymbol{H}_{n-i}\mathbf{b}+dq_{n+1-i}, \qquad i=0, 1, \ldots, n.$$

The coefficients $\boldsymbol{H}_{n-i}$, q_{n+1-i} can be computed using recursive formulae [2]. Eqn. (9) yields the transfer function as a ratio of two polynomials. Points of the transfer function are obtained by substituting specific frequency values:

$$W(j\omega_k)=W(s)|_{s=j\omega_k} \tag{10}$$

1.2 *Determination of the sensitivity functions*

Our aim is now to obtain, for any $W(s)$ transfer function and any parameter h of any network component the (absolute) sensitivity function

$$S_h^W(s)=\frac{\partial W}{\partial h} \tag{11}$$

as a ratio of two polynomials.

Knowing $W(s)$, h and $S_h^W(s)$ it is easy to express the semirelative and relative sensitivity functions. The parameter h may denote a capacitance value, or the transfer parameter of a controlled source, the voltage ratio of an ideal transformer, etc.

The method to be described below does not require the symbolic expression for $W(s, h)$, nor does it necessitate the analysis of an adjoint network to be performed, nor an iterated network analysis. It only uses the results of the network computation as described above.

Remembering $\boldsymbol{P}\cdot\boldsymbol{P}^{-1}=\boldsymbol{I}$, we get

$$\frac{\partial \boldsymbol{P}^{-1}}{\partial h}=-\boldsymbol{P}^{-1}\frac{\partial \boldsymbol{P}^{-1}}{\partial h}\boldsymbol{P}^{-1} \tag{12}$$

and, using (5)

$$\frac{\partial \boldsymbol{T}}{\partial h}=\frac{\partial \boldsymbol{P}^{-1}\boldsymbol{Q}}{\partial h}=\boldsymbol{P}^{-1}\frac{\partial \boldsymbol{Q}}{\partial h}+\frac{\partial \boldsymbol{P}^{-1}}{\partial h}\boldsymbol{Q}=-\boldsymbol{P}^{-1}\frac{\partial \boldsymbol{P}}{\partial h}\boldsymbol{T} \tag{13}$$

since $\boldsymbol{Q}$ does not contain component parameters. (The partial derivatives are numerical values assumed at the nominal value of the parameter h.) In analogy to (12), we get

$$\frac{\partial(s\boldsymbol{I}-\boldsymbol{A})^{-1}}{\partial h}=(s\boldsymbol{I}-\boldsymbol{A})^{-1}\frac{\partial \boldsymbol{A}}{\partial h}(s\boldsymbol{I}-\boldsymbol{A})^{-1} \tag{14}$$

Let us write down the absolute sensitivity function (11) which, in fact, is the partial derivative of expression (7) with respect to h, and apply (14):

$$S_h^W=\frac{\partial W}{\partial h}=\frac{\partial d}{\partial h}+\frac{\partial \mathbf{c}^T}{\partial h}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\mathbf{b}+$$

$$+\mathbf{c}^T(s\boldsymbol{I}-\boldsymbol{A})^{-1}\frac{\partial \mathbf{b}}{\partial h}+\mathbf{c}^T(s\boldsymbol{I}-\boldsymbol{A})^{-1}\frac{\partial \boldsymbol{A}}{\partial h}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\mathbf{b} \tag{15}$$

Using expressions (8) and (9), we get

$$S_h^W(s)=\frac{\partial W}{\partial h}=\frac{R(s)}{D^2(s)};$$

$$R(s)=D(s)\left[s^n\frac{\partial d}{\partial h}+\sum_{i=0}^{n-1}\left(q_{n+1-i}\frac{\partial d}{\partial h}+\frac{\partial \mathbf{c}^T}{\partial h}\mathbf{z}_{n-i}+\mathbf{v}_{n-i}^T\frac{\partial \mathbf{b}}{\partial h}\right)s^i\right]+$$

$$+\left[\sum_{i=0}^{n-1}\left(\mathbf{v}_{n-i}^T\frac{\partial \boldsymbol{A}}{\partial h}\right)s^i\right]\left[\sum_{i=0}^{n-1}\mathbf{z}_{n-i}s^i\right] \tag{16}$$

where

$$D(s)=\sum_{i=0}^{n}q_{n+1-i}s^i \tag{17}$$

$$\left.\begin{aligned}\mathbf{z}_{n-i}&=\boldsymbol{H}_{n-i}\mathbf{b}\\ \mathbf{v}_{n-i}^T&=\mathbf{c}^T\boldsymbol{H}_{n-i}\end{aligned}\right\}\qquad i=0,1,\dots,n-1 \tag{18}$$

The matrices $\boldsymbol{H}_{n-i}$ have already been determined in the course of computing (8).

Expression (16) yields the absolute sensitivity function as the ratio of two polynomials of order not higher than $2n$. The polynomial of order $2n$ in the denominator simply equals the square of the denominator $D(s)$ of the transfer function. The numerator is seen to be the sum of two polynomials, one of them being the product of $D(s)$ and a polynomial with scalar coeffiecients; the other is the product of two polynomials with row vector and column vector coefficients, respectively. In the majority of cases of practical interest, it is required to find sensitivities of a specific transfer function relative to several component parameters, so it is important to notice that among the polynomials showing up in expression (16), it is only two which depend upon the specific meaning of parameter h.

The partial derivatives $\frac{\partial \boldsymbol{A}}{\partial h}, \frac{\partial \mathbf{b}}{\partial h}, \frac{\partial \mathbf{c}^T}{\partial h}$ and $\frac{\partial d}{\partial h}$ in the numerator polynomials of the sensitivity functions $R(s)$ (cf eqn (16)) are determined as follows: First, matrix $\frac{\partial \boldsymbol{T}}{\partial h}$ is calculated according to eqn (13). Note that $\boldsymbol{P}^{-1}$ and $\boldsymbol{T}$ are already known (5). When building up matrix $\frac{\partial \boldsymbol{P}}{\partial h}$ we start from a zero matrix and, substituting $h=1$, repeatedly apply those rules for foming the $\boldsymbol{P}$ coefficient matrix, which relate to parameter h. Thus, all of the elements of the matrix $\frac{\partial \boldsymbol{P}}{\partial h}$ will assume $+1$, -1, or zero values. Referring to the two examples in sec. 1.1: if $h=G$, then the (ll) and (jj) elements of the matrix $\frac{\partial \boldsymbol{P}}{\partial h}$ will equal 1; the (lj) and (jl) elements, -1; the other elements are zero. If, however, $h=\mu$, the (kl) elements of $\frac{\partial \boldsymbol{P}}{\partial h}$ will assume the value 1; (kj) elements, -1; the other elements are zero. So, calculating the matrix

$$\frac{\partial \boldsymbol{P}}{\partial h} = \begin{bmatrix} \frac{\partial \boldsymbol{R}}{\partial h} & \frac{\partial \boldsymbol{S}}{\partial h} \\ \frac{\partial A}{\partial h} & \frac{\partial \boldsymbol{B}}{\partial h} \end{bmatrix} \tag{19}$$

according to (13), (cf. notation (5)) the partial derivatives $\frac{\partial A}{\partial h}, \frac{\partial \mathbf{b}}{\partial h}, \frac{\partial \mathbf{c}^T}{\partial h}, \frac{\partial d}{\partial h}$ may be read out from it just as we obtained A, $\mathbf{b}$, $\mathbf{c}^T$, d from the matrix $\boldsymbol{T}$.

2. Calculation of transfer characteristics and sensitivities from the frequency domain equations

Another way of analyzing linear networks is to construct the network equations in the frequency domain. Starting this way we directly obtain points of the transfer characteristics.

Let the linear invariant network contain components whose characteristics are of the form

$$\left.\begin{aligned} \mathbf{I}_y &= \boldsymbol{Y}\mathbf{U}_y + \boldsymbol{N}\mathbf{I}_z + \mathbf{J} \\ \mathbf{U}_z &= \boldsymbol{M}\mathbf{U}_y + \boldsymbol{Z}\mathbf{I}_z + \mathbf{V} \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \mathbf{U}_a &= \mathbf{0} \\ \mathbf{I}_a &= \mathbf{0} \end{aligned}\right\} \quad \text{ideal amplifier} \tag{20}$$

where each of the matrix and vector elements is a complex number for any given frequency. Special cases of the general components defined in this way are R, L, C components, independent and controlled sources, etc.

Applying the modified nodal analysis method, we arrive at the set of linear equations (the reduced set of equations) (21) by step-by-step addition of network component parameters [1]:

$$\begin{bmatrix} Y^* & N^* & B^* \\ M^* & Z^* & 0 \\ A^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi} \\ \mathbf{I}_z \\ \mathbf{I}_b \end{bmatrix} = \begin{bmatrix} \mathbf{J}^* \\ \mathbf{V}^* \\ 0 \end{bmatrix} \tag{21}$$

$$P \cdot \mathbf{F} = \mathbf{E}^*$$

where matrix P, for a specific frequency, contains complex numbers, $\mathbf{\Phi}$ nodal voltages, $\mathbf{I}_z$ currents of z-ports, $\mathbf{I}_b$ the output currents of ideal amplifiers. Finally, $\mathbf{J}^*$ and $\mathbf{V}^*$ are vectors constituted from complex amplitudes of excitation quantities.

In the case of current source excitation let us choose, for convenience, as a reference point, the node connected to the more negative point of the current source. In this case, as in the case of voltage source excitation, the $\mathbf{E}^*$ vector will contain only one non-zero element; let this element be E_i^*.

For the output $Y=[\mathbf{F}]_k$ and the excitation E_i^*

$$W(j\omega_r)=\frac{Y}{E_i^*}=-[P^{-1}]_{ki} \tag{22}$$

In order to determine the value of the transfer characteristics assumed at the frequency ω_r, we need one of the complex elements of matrix P^{-1}. The sensitivity at the frequency ω_r of $W(j\omega)$ with respect to an arbitrary component parameter h may be obtained in analogy to (12) as

$$S_h^W(j\omega_r)=\left.\frac{\partial W}{\partial h}\right|_{\omega_r}=\left[P^{-1}\frac{\partial P}{\partial h}P^{-1}\right]_{ki} \tag{23}$$

Computation of the sensitivity requires knowledge of the k-th row and the i-th column of the matrix P^{-1}, as well as $\frac{\partial P}{\partial h}$. The latter matrix is formed—just as in the case of time domain calculations—by applying the rules related to the step-by-step building up of the coefficient matrix P of the set of equations (21). However, $\frac{\partial P}{\partial h}$ may now also contain complex numbers. Treating, as an example, capacitors as y-ports, their admittance $j\omega_r C$ will appear in the Y^*

block, and the corresponding elements of $\frac{\partial \boldsymbol{P}}{\partial h}$ will be complex numbers

$$\frac{\partial j\omega_r C}{\partial C} = j\omega_r \tag{24}$$

When performing the operations (23), it may be useful to remember that $\frac{\partial \boldsymbol{P}}{\partial h}$ contains only few non-zero elements and that it is only one element of the product which must be determined.

From the frequency domain equations we directly get single points on the transfer characteristics and the sensitivity functions. The computation has to be repeated for each prescribed frequency. As compared with the time domain approach, we now have to handle smaller matrices, which, however, contain complex elements.

3. The LINA program package

The LINA program [8] for the analysis of time invariant linear networks has been written by the author of this paper in the BASIC-PLUS language. It can be run in the time-sharing system of the PDP 11/45 computer operating at the Department for Measuring Instruments and Techniques of Budapest Technical University. Its purpose is to yield manifold information about the network subject to analysis and to enhance the efficiency of the user's work by making maximal use of the possibilities resulting from interactivity.

The program works according to the method described in section 1. It yields the normal form of the state equation and uses it to compute, as polynomial ratios, transfer functions related to output voltages or currents specified by the user for various kinds of excitation. Furthermore, it determines poles and zeroes according to the Bairstow method. It then computes the Bode-plot in any frequency interval specified by the user, displays it graphically or by plotter, and lists its values.

Following the procedure and method described in the preceding sections, the LINA program also computes the relative sensitivity functions with respect to any R, L or C one-port component value for any transfer function specified by the user.

The network may contain any component of quite a number of sorts [8], but in its present state, it yields only sensitivity functions relative to R, L and C parameters. It is planned to extend the options of the program to the determination of sensitivity functions relating to the parameters of all other components, too. We also mention that the program—over and above the computations which were the topics of the foregoing description,—also determines sensitivity functions related to transfer functions of currents of any capacitor or resistor, as outputs, without insertion of series short circuits. Details about this are omitted here. It also should be noted that for all resistors, the $S_R^{(r)W}(s)$ relative sensitivity functions are determined, even if the G prameter values are specified for the resistors. The LINA program prints out, for all frequencies specified by the user, sensitivity values in the form of complex numbers and plots their reals and imaginary parts.

4. Example

Consider the following simple network (a Sallen-Key second order high pass filter, cf. Fig. 1).

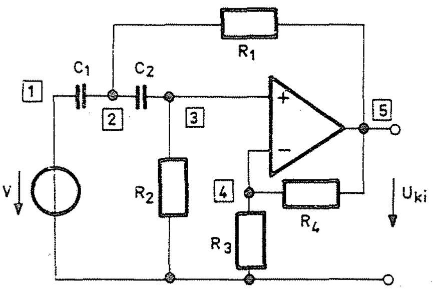

Fig. 1. The network subject to analysis ($R_1=R_2=R_3=R_4=10$ kΩ, $C_1=C_2=0.1$ μF)

(a) First, we follow the procedure described in section 1. The time domain set of equations (4) is built up, according to the rules given in [1], by considering the network components one by one and modifying the original zero matrix elements, taking into account the location of the pertaining component in the network:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0 & -0.1 & 0 & 0 & -0.1 & 0.1 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0.2 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & -0.1 & 0.2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ i_z \\ i_b \\ \dot{u}_{c1} \\ \dot{u}_{c2} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{c1} \\ u_{c2} \\ v \end{bmatrix}$$

After inverting the coefficient matrix $\boldsymbol{P}$, we can write down $\boldsymbol{T}=\boldsymbol{P}^{-1}\boldsymbol{Q}$. This being done, the matrices of the state equation (5) may be read out from $\boldsymbol{T}$:

$$\boldsymbol{A}=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}; \qquad \mathbf{b}=\begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

for the output $y=\varphi_5$

$$\mathbf{c}^T=[-2 \quad -2]; \qquad d=2.$$

Using the Souriau-Frame algorithm, we get, for the quantities in Eqns (8) and (9):

$$q_1=1$$

$$\boldsymbol{H}_1=\boldsymbol{E}$$

$$q_2=-\operatorname{sp} \boldsymbol{H}_1\boldsymbol{A}=1$$

$$\boldsymbol{H}_2=\boldsymbol{H}_1\boldsymbol{A}+q_2\boldsymbol{E}=\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$q_3=-\frac{1}{2}\operatorname{sp} \boldsymbol{H}_2\boldsymbol{A}=1$$

$$k_1=d=2$$

$$k_2=\mathbf{c}^T\boldsymbol{H}_1\mathbf{b}+dq_2=0$$

$$k_3=\mathbf{c}^T\boldsymbol{H}_2\mathbf{b}+dq_3=0$$

so the transfer function in its polynomial ratio form is seen to be equal

$$W(s)=\frac{\Phi_5(s)}{V(s)}=\frac{2s^2}{s^2+s+1}$$

Our next task is to obtain the relative sensitivity function $S_{R_1}^{(r)W}(s)$. The matrix

$\frac{\partial \boldsymbol{P}}{\partial G_1}$ related to resistor R_1 between nodes 2 and 5, which contains only four non-zero elements, is obtained by repeatedly applying part of the rules for forming matrix $\boldsymbol{P}$:

$$\frac{\partial \boldsymbol{P}}{\partial G_1} = \begin{array}{c} \\ 2 \\ \\ 5 \\ \\ \end{array} \left[\begin{array}{ccccc} & 2 & & 5 & \\ & \vdots & & \vdots & \\ \dots & 1 & \dots & -1 & \dots \\ & \vdots & & \vdots & \\ \dots & -1 & \dots & 1 & \dots \\ & \vdots & & \vdots & \end{array} \right]$$

Using the notations and expressions of section 2, the computation of the sensitivity function is perfomed in the following steps:

$$\frac{\partial \boldsymbol{T}}{\partial G_1} = -\boldsymbol{P}^{-1} \frac{\partial \boldsymbol{P}}{\partial G_1} \boldsymbol{T}; \qquad \text{hence}$$

$$\frac{\partial A}{\partial G_1} = \begin{bmatrix} 10 & 20 \\ 0 & 0 \end{bmatrix}; \qquad \frac{\partial \mathbf{b}}{\partial G_1} = \begin{bmatrix} -10 \\ 0 \end{bmatrix};$$

$$\frac{\partial \mathbf{c}^T}{\partial G_1} = [0 \quad 0]; \qquad \frac{\partial d}{\partial G_1} = 0$$

According to formulae (16), (17) and (18)

$$\mathbf{z}_1 = \boldsymbol{H}_1 \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad \mathbf{z}_2 = \mathbf{H}_2 \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_1^T = \mathbf{c}^T \boldsymbol{H}_1 = [-2 -2]\,; \mathbf{v}_2^T = \mathbf{c}^T \boldsymbol{H}_2 = [0 -2]$$

$$D^2(s) = (s^2 + s + 1)^2 = s^4 + 2s^3 + 3s^2 + 2s + 1$$

$$R(s) = (s^2 + s + 1)20s + [-20 - 40]s \left| \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} s \right| = 20s^3 - 20s^2$$

So the sensitivity function is

$$S_{G_1}^{W}(s) = \frac{20s^3 - 20s^2}{s^4 + 2s^3 + 3s^2 + 2s + 1}.$$

Using this expression, the relative sensitivity function related to R_1 is easily obtained:

$$S_{R_1}^{(r)W}(s) = -\frac{G_1}{W} S_{G_1}^{W}(s) = \frac{-2s^3 + 2s^2}{2s^4 + 2s^3 + 2s^2}.$$

If, e.g., $\omega_r = 1 \frac{\text{krad}}{s}$, then $S_{R_1}^{(r)W}(j) = -1 - \text{j}$. Sensitivity functions related to any other parameter R_i, C_i may be calculated in a similar way.

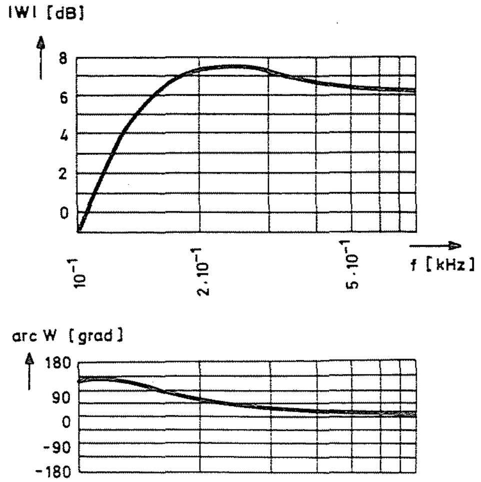

Fig. 2. Bode plot of network shown in fig. 1

Part of the results of a network analysis case as obtained with the LINA program are shown below.

Graphs printed out on the plotter by the program are sown in Figs 2, 3, and 4.

Let us now determine the values of the network transfer characteristics and the sensitivity function $S^W_{R_1}$, for the frequency $\omega_r = 1\,\frac{\text{krad}}{s}$, using the method described in section 2. First, step by step addition of network component parameters yields the frequency domain equations (24):

$$\begin{bmatrix} 0.1j & -0.1j & 0 & 0 & 0 & 1 & 0 \\ -0.1j & 0.1+0.2j & -0.1j & 0 & -0.1 & 0 & 0 \\ 0 & -0.1j & 0.1+0.1j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & -0.1 & 0 & 0 \\ 0 & -0.1 & 0 & -0.1 & 0.2 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ I_z \\ I_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -V \\ 0 \end{bmatrix}$$

$$\boldsymbol{P} \cdot \mathbf{F} = \mathbf{E}^*$$

Inverting the complex element matrix $\boldsymbol{P}$ related to the fixed frequency ω_r, we get, according to (22):

$$W(j\omega_r) = W(j) = \frac{\Phi_5}{V} = -[\boldsymbol{P}^{-1}]_{56} = 2j$$

Inspection of the topology of the network directly suggests

$$\frac{\partial \boldsymbol{P}}{\partial G_1} = \begin{matrix} \\ 2 \\ 5 \\ \\ \end{matrix} \begin{matrix} & 2 & & 5 & \\ & \vdots & & \vdots & \\ \ldots & 1 & \ldots & -1 & \ldots \\ \ldots & -1 & \ldots & 1 & \ldots \\ & \vdots & & \vdots & \end{matrix}$$

According to formula (23), the sensitivity equals

$$s^W_{G_1}(j\omega_r) = \left[\boldsymbol{P}^{-1} \frac{\partial \boldsymbol{P}}{\partial G_1} \boldsymbol{P}^{-1} \right]_{56} = -20 + 20j\,.$$

Hence the relative sensitivity equals

$$S^{(r)W}_{R_1}(j\omega_r) = -\frac{G_1}{W(j\omega_r)} S^W_{G_1}(j\omega_r) = -1 - j\,.$$

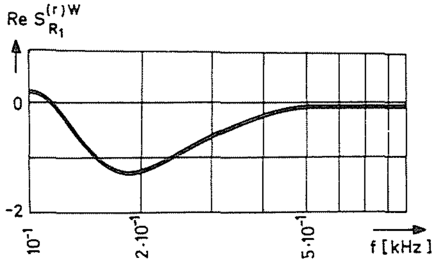

Fig. 3. Frequency dependence of the real part of the $S^{(r)W}_{R_1}$ sensitivity function

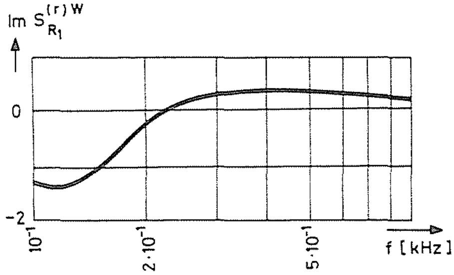

Fig. 4. Frequency dependence of the imaginary part of the $S^{(r)W}_{R_1}$ sensitivity function

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TRANSFER FUNCTION:

EXCITATION: F1
RESPONSE: U5

	COEFFICIENTS OF DENOMINATOR	COEFFICIENTS OF NOMINATOR
S^2	1	2
S^1	1	0
S^0	1	0

SENSITIVITY FUNCTIONS:

INVESTIGATED TRANSFER FUNCTION: W(S)=U5(S)/F1(S)

COEFFICIENTS OF SENSITIVITY FUNCTIONS DENOMINATORS:

S^4	2
S^3	2
S^2	2
S^1	0
S^0	0

COEFFICIENTS OF NUMERATORS:

	(R)W(S) S R1	(R)W(S) S C2	(R)W(S) S R2
S^4	0	0	0
S^3	-2	2	4
S^2	2	2	2
S^1	0	0	0
S^0	0	0	0

SENSITIVITY VALUES:

INVESTIGATED TRANSFER FUNCTION: W(S)=U5(S)/F1(S)

FREQUENCY	COMPONENT(H)	(R)W(S) S H
.1	R1	.276494 -J 1.32522
.1	C2	1.31394 -J .325923
.1	R2	1.83266 +J .173727
.159155	R1	-1 -J .999999
.159155	C2	.999999 -J 1
.159155	R2	2 -J 1
.2	R1	-1.12731 -J .276241
.2	C2	.52232 -J 1.03649
.2	R2	1.34713 -J 1.41662
.5	R1	-.211648 +J .279232
.5	C2	.112944E-1 -J .350197
.5	R2	.122766 -J .664912
1	R1	-.512851E-1 +J .154917
1	C2	.657866E-3 -J .163184
1	R2	.266294E-1 -J .372234

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