STATISTICAL THEORY OF QUANTIZATION: RESULTS AND LIMITS

I. KOLLÁR

Department of Measurement and Instrument Engineering,
Technical University, H-1521 Budapest

Summary

The statistical theory of quantization makes possible to design measurement procedures with quantized data. If carefully interpreted, this theory provides formulae to estimate bias and variance of quantized measurements and may show the means to reduce effectively the errors of quantized measurements. Limitations of the applicability of the theory are discussed too.

Introduction

Nowadays signal processing often means digital processing of continuous (analogue) signals. The first step of this procedure is to convert the signal into a form that can be handled by digital equipment. The conversion consists of two main steps: sampling in the time domain (simply called *sampling*) and sampling in the amplitude domain (called *quantization*).

These two operations mean a rather rough alteration of the signal (Fig. 1). So it is essential to formulate sufficient conditions for sampling rate, observation time, quantum size, etc. to provide that the converted signal (quantized time series) contain enough information in some sense.
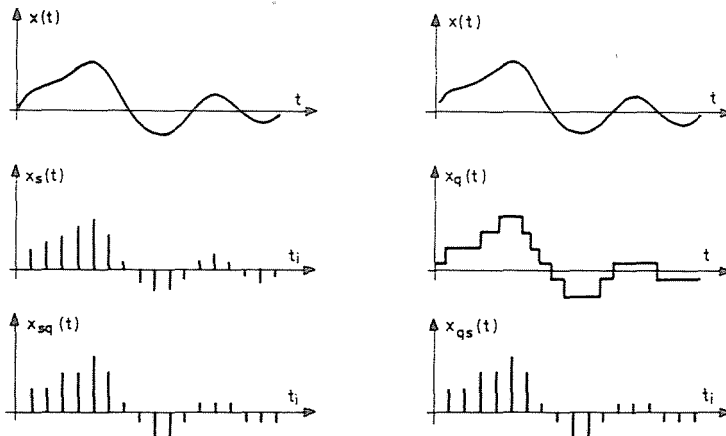

Fig. 1. The effect of sampling and quantization

It is easy to see that the above two operations may generally be investigated separately and that they are generally interchangeable (Fig. 1). Thus we shall treat them separately and shall refer to the other only if necessary.

Since sampling is a *linear* operation (it is interchangeable with multiplication by scalars and with addition), sampling can be rather easily handled. Its well-elaborated theory can be found in literature. Because of this fact only quantization will be dealt here in detail.

Quantization

Quantization is a *nonlinear* operation which transforms the continuous amplitude domain to a discrete one. Because of the nonlinearity, its theory is not elaborated in detail, still there are some results which can be used when designing measurement procedures. It will be tried to survey them in this paper.
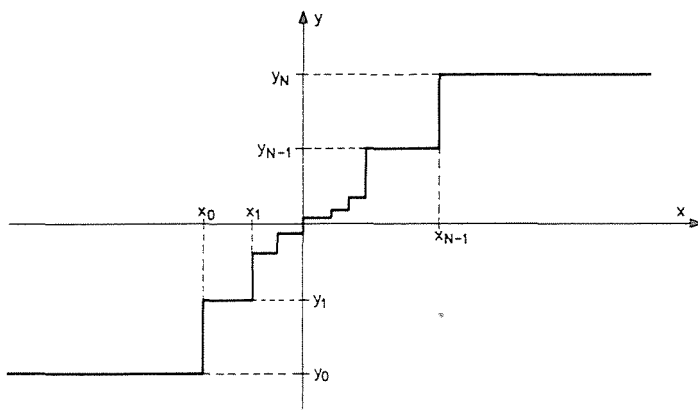

Fig. 2. Characteristics of a quantizer

In the following we will restrict our investigations to zero memory scalar quantizers with monotonous characteristics. This means that the quantizer does not use any information about formerly quantized values and processes only one sample at a time, performing the nonlinear operation (Fig. 2). We shall not deal with the design of (in some sense) optimal quantizers; only with the analysis of given (generally uniform) ones.

Because of the strong nonlinearity it is hopeless to arrive at general conclusions, but for *stochastic signals* surprisingly useful results may be

derived. So in the following we shall deal with the analysis of stochastic signals (please note that some questions of the quantization of deterministic signals are dealt with in [3]). Randomness is often not a really severe restriction, since:

— many fundamentally deterministic processes may be modelled by means of stochastic models like random-phase periodic signals, transients with slightly random initial conditions etc.;

— the quantum levels of real A/D converters are generally of slightly stochastic nature which can often be modelled by an additive random noise [4].
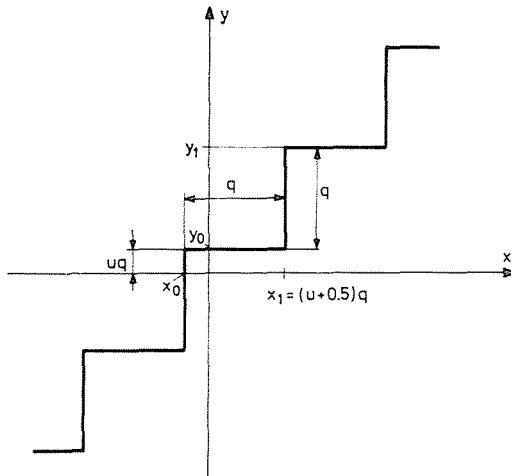

Fig. 3. The A/D-type quantizer

In the following sections we shall survey two approaches: the *white noise model* and the *characteristic function method*. In general we shall analyze the so-called A/D-type quantizer (Fig. 3):

$$x_i = uq + (i - 0.5)q$$
$$i = 0, \pm 1, \pm 2 \ldots \qquad (2.1)$$
$$y_i = uq + iq$$
$$x \to y_i \text{ if } x_i \leq x < x_{i+1}$$

By means of this quantizer, the general quantizer of Fig. 2 can be modelled as well (see Fig. 4). This modelling provides means to derive results for the non-uniform case, too.
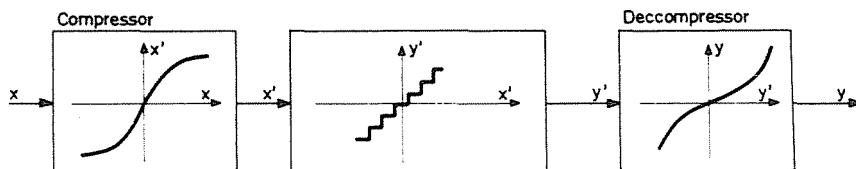
Fig. 4. Modelling a general quantizer by means of an A/D-type one

The Noise Model of the Quantization Error

Let us first consider a simple example. In Fig. 5 the quantization of a sine wave can be observed. If the quantization error $q(t)$ is considered, we may have the impression that it is more or less independent of the original signal, since it describes only signal variations relative to the next quantum level and does not take much account of the complete value of the signal. This brings the thought to model quantization by an additive noise (Fig. 6). Because of its linearity this is a widely used model.

The noise is generally assumed to be:

— independent of the signal;
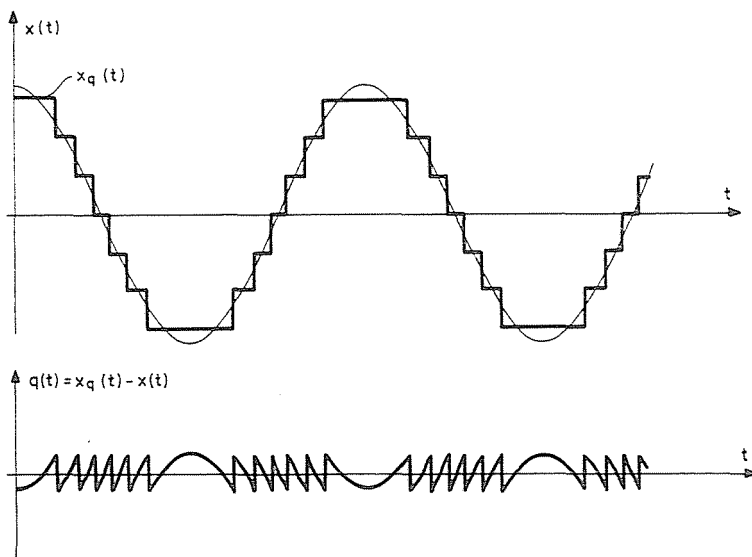— of uniform distribution;
— white.

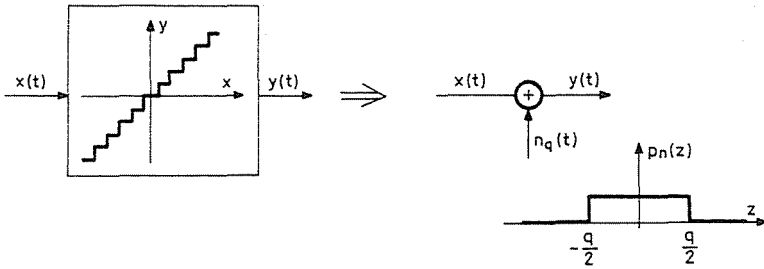*Fig. 5.* Quantization of a sine wave

Fig. 6. The noise model of quantization

What are the conditions for the validity of the above assumptions?

a) As it has already been stated above, if the quantum size is small enough compared to signal variations, independence will be approximately fulfilled.

   Note that there must be enough quantum levels to cover the whole amplitude domain of the signal.

b) In the sections of large signal variations $q(t)$ is very similar to a saw-tooth (see Fig. 5), with uniform distribution. So the condition is similar to the above one.

c) Quantization noise is usually not white, since the "saw-tooth" has continuous sections. But when the signal is sampled, aliasing occurs in the frequency domain, and the lags of $Q(f)$, which are summed up, may give an approximately white spectrum (Fig. 7). This occurs only when *sampling is not too dense*, that is, the greater q is, the smaller the sampling frequency f_s must be. Intuitively, the "frequency" of the "saw-tooth" prescribes a higher limit to sampling frequency [6].
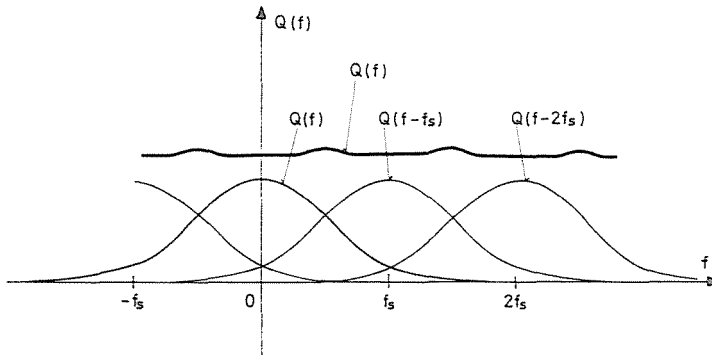

Fig. 7. The spectrum of the sampled quantization error

The assumptions under a) and b) are generally fulfilled, if

$$q \ll \sigma_x, \tag{2.2}$$

and if there are enough quantum levels to cover the amplitude domain of the signal (that is, truncation error is negligible). This model is called *fine quantization*. When the conditions are fulfilled, it gives rather good means to describe the effect of the quantizer.

Unfortunately the (2.2) condition is generally too severe, and still not exact. The prescriptions for measurement errors are generally more moderate and this means that we have to use more refined models to describe quantization.

The Characteristic Function Method

Stochastic signals can generally be described by their probability density functions. The probability density function of the quantized $y(t)$ signal can be easily expressed with that of $x(t)$ (see Fig. 8):

$$p_y(z) = \sum_{i=-\infty}^{\infty} P_i \delta(z - y_i),$$

$$P_i = \int_{x_{i-1}}^{x_i} p_x(z) dz. \tag{2.3}$$

The (2.3) expression makes possible to analyze the statistics of y, but this may be rather troublesome and the formula is not suitable to be analysed. However, Fig. 8 shows that quantization is a sort of sampling of $p_x(z)$. This gives the idea to try to analyse the Fourier transform pair of $p_y(z)$, the characteristic function
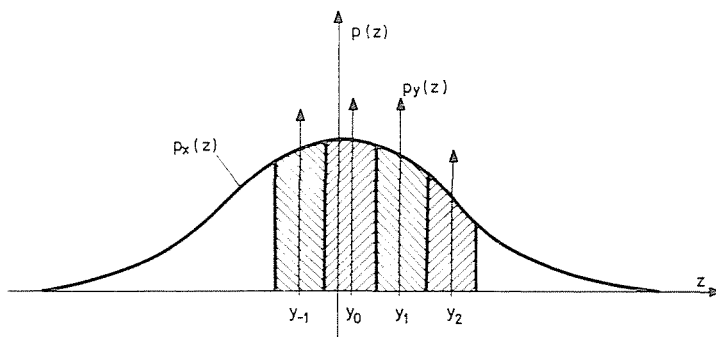

Fig. 8. Construction of the probability density function of a quantized signal

[1], [2], [5], [7], just as in case of analysing sampling. Omitting the troublesome derivation, we obtain for the uniform characteristics (2.1):

$$W_y(\alpha) = \int_{-\infty}^{\infty} p_y(z) e^{j\alpha z} dz = \sum_{k=-\infty}^{\infty} e^{+j2\pi k u} W_x \left(\alpha - \frac{2\pi k}{q} \right) \mathrm{sinc} \left(\frac{q\alpha}{2} - \pi k \right), \quad (2.4)$$

where

$$\mathrm{sinc}\,(x) = \begin{cases} \dfrac{\sin\,(x)}{x} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Studying (2.4), a theorem similar to the Nyquist sampling theorem can be formulated: when $W_x(\alpha)$ is α-bandlimited, that is,

$$W_x(\alpha) \equiv 0 \quad \text{for} \quad |\alpha| \geq \frac{\pi}{q}, \quad (2.5)$$

$W_x(\alpha)$ can be calculated from $W_y(\alpha)$, so there is no statistical information lost. Moreover, in the band

$$-\frac{\pi}{q} < \alpha < \frac{\pi}{q}$$

the characteristic function is equal to

$$W_y(\alpha) = W_x(\alpha) \, \mathrm{sinc} \left(\frac{q\alpha}{2} \right),$$

which is exactly the characteristic function of the sum of the original signal and an independent, uniform distribution noise.

In measurements usually moments of the signals are to be measured. Since

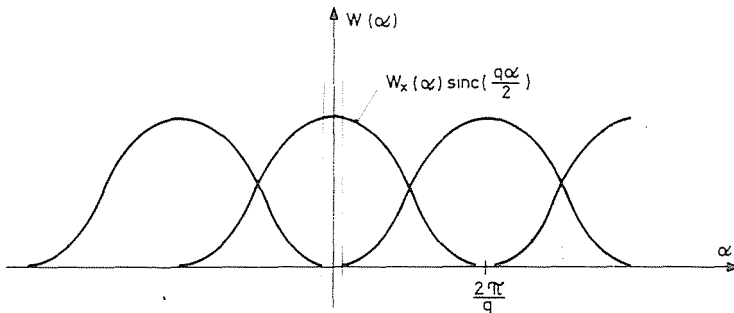$$E\{x^n\} = \frac{1}{j^n} \frac{d^n W_x(\alpha)}{d\alpha^n} \Big|_{\alpha = 0}, \quad (2.6)$$

Fig. 9. The condition of $E\{x\} = E\{y\}$

it is enough if $W_x(\alpha)$ can be restored in some environment of $\alpha = 0$ (see Fig. 9). So the quantization theorem is as follows: if

$$W_x(\alpha) \equiv 0 \quad \text{for} \quad |\alpha| > \frac{2\pi}{q} - \varepsilon, \quad \varepsilon > 0, \tag{2.7}$$

the moments of x can be calculated from the moments of y:

$$E\{x\} = E\{y\}$$

$$E\{x^2\} = E\{y^2\} - \frac{q^2}{12}$$

$$E\{x^3\} = E\{y^3\} - 3E\{y\}\frac{q^2}{12} \tag{2.8}$$

$$E\{x^4\} = E\{y^4\} - 6E\{y^2\}\frac{q^2}{12} - \frac{q^4}{80}$$

$$\vdots$$

The formulae of (2.8) are the so-called Sheppard-corrections. (Corrections appear because of the $\sin(x)/x$ factor in (2.4)). Note that in an environment of $\alpha = 0$, $W_y(\alpha)$ behaves just like the characteristic function of the additive, independent, uniform distribution noise model. Thus, considering the measurement of moments, (2.7) is a sufficient condition for the validity of the noise model (there exists a sufficient *and* as well necessary condition too, see (2.15)). The Sheppard-corrections can be calculated from the noise model as well.

Just like by sampling, there is no real signal which would fulfil the (2.5) condition (this would mean arbitrarily great signal amplitudes). However, some signals may approximately fulfil it, e.g. the Gaussian one:

$$W_x(\alpha) = e^{j\alpha\mu_x - \sigma_x^2 \frac{\alpha^2}{2}}. \tag{2.9}$$

As an example let us consider the distortion of the measurement of the mean in the case of a Gaussian signal [5]. From (2.4), (2.6) and (2.9)

$$E\{y\} = \mu_x + \frac{q}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}e^{-\frac{2\pi^2 i^2 \sigma_x^2}{q^2}}\sin\left[2\pi i\left(u + \frac{\mu_x}{q}\right)\right]. \tag{2.10}$$

This is a Fourier series as a function of μ_x. Since it converges very quickly (if $q < 3\sigma_x$, the error is less than 1.5% [5]), the maximum value of the relative
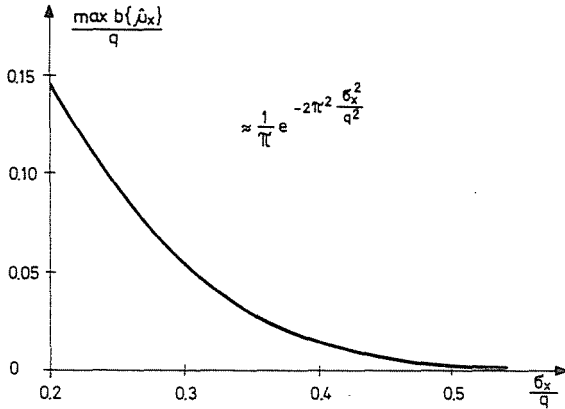
Fig. 10. The maximal distortion when measuring the mean of a Gaussian signal with quantized data

distortion may be expressed with the amplitude of the first term:

$$|\varepsilon_{r\,max}| = \frac{|b_{max}|}{q} \cong \frac{1}{\pi} e^{-\frac{2\pi^2\sigma_x^2}{q^2}}. \tag{2.11}$$

This function is plotted in Fig. 10. The diagram shows that the mean of Gaussian signals can be measured with small distortion even in the case of rough quantization.

Dithering

If a signal does not even approximately fulfil the (2.7) condition of the quantization theorem, or the required q would be too small, a special technique called dithering can be used.

Let us consider what happens when an additive, independent noise is given to the signal. Since

$$p_{x+n}(z) = \int_{-\infty}^{\infty} p_x(z-u)p_n(u)du, \tag{2.12}$$

the characteristic function is:

$$W_y(\alpha) = \sum_{k=-\infty}^{\infty} e^{+j2\pi ku} W_x\left(\alpha - \frac{2\pi k}{q}\right) W_n\left(\alpha - \frac{2\pi k}{q}\right) \text{sinc}\left(\frac{q\alpha}{2} - \pi k\right). \tag{2.13}$$

If $E\{n(t)\} = 0$, the additive noise does not change the mean of the continuous signal, but it may still help to fulfil (2.7). This technique is called dithering, $n(t)$ is the dither. $n(t)$ may be for example Gaussian.

However, not only (2.7) can provide the means of x and y being equal. If $n(t)$ is of uniform distribution in the $\left[-\dfrac{q}{2}, \dfrac{q}{2}\right]$ interval, (2.13) will be the following:

$$W_y(\alpha) = \sum_{k=-\infty}^{\infty} e^{+j2\pi ku} W_x\left(\alpha - \frac{2\pi k}{q}\right) \operatorname{sinc}^2\left(\frac{q\alpha}{2} - \pi k\right).$$

On basis of (2.6) it is easy to show that in this case

$$E\{y\} = E\{x\}, \tag{2.14}$$

independently of the distribution of x. To be more general, the uniform distribution noise model is valid, because $n(t)$ fulfils the *sufficient and necessary condition* of Sripad and Snyder [14]:

$$W\left(\frac{2\pi k}{q}\right) = 0 \quad \text{for all} \quad k = \pm 1, \pm 2\dots \tag{2.15a}$$

Condition (2.15a) directly follows from the expression of the probability density function of the quantization error:

$$p_{n_q}(z) = \frac{1}{q} + \frac{1}{q}\sum_{\substack{k=-\infty \\ k \neq 0}}^{k=\infty} W_x\left(\frac{2\pi k}{q}\right) e^{-j\frac{2\pi kz}{q}}, \quad -\frac{q}{2} \leq z < \frac{q}{2}. \tag{2.15b}$$

However, (2.15a) provides only the uniform distribution of the quantization error, and not the *independence* of $x(t)$ and $n(t)$ (see [3], [14]). For the latter (2.7) is sufficient.
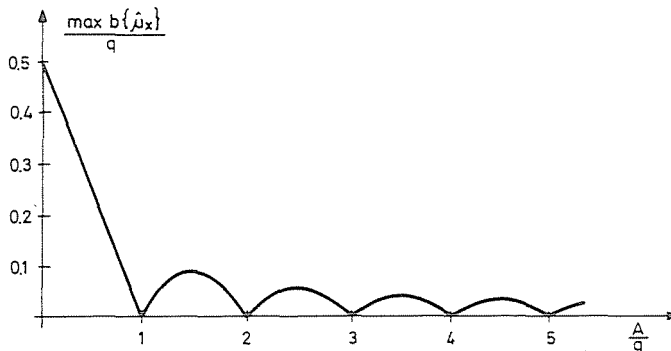

Fig. 11. The maximal distortion when measuring the mean of a uniform distribution signal with quantized data

Realization of the Dither

Let us emphasize that (2.14) is a theoretical result in the case when the peak-to-peak amplitude A of the dither equals exactly q. When this is not quite true, the distortion may rapidly rise with $|A - q|$ (see Fig. 11, [5]).

This is a reason why a simple uniform white noise is usually not used in practice. With a Gaussian dither for example the estimation of the mean would be less sensitive to alterations of the standard deviation of the dither (see Fig. 10).

However, there is a special case when the amplitude of the uniform distribution noise has nothing to do with q: the one-bit quantizer (comparator) with uniform distribution dither. This is the so-called stochastic-ergodic
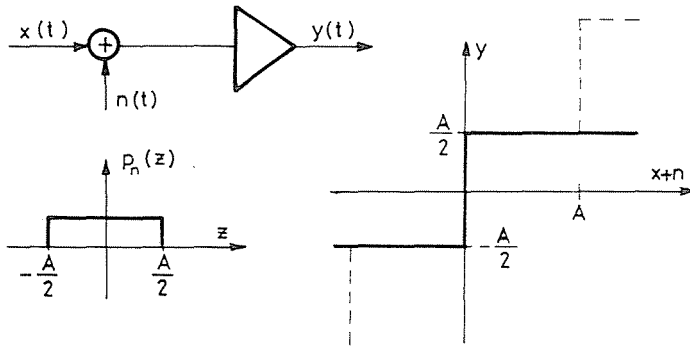

Fig. 12. The stochastic-ergodic converter

converter (Fig. 12, [8], [9]). In this case, if the inequality

$$|x(t)| < \frac{A}{2}$$

holds, $x(t) + n(t)$ will never surpass the next quantum level, that is, the comparator "simulates" a uniform quantizer with the parameter

$$q = A,$$

so the condition for the dither amplitude is automatically fulfilled. The accuracy of the measurement of the mean is determined by the accuracy of the knowledge of A.

A good example for a much more sophisticated realization of the dither can be observed in the HP3582A spectrum analyzer ([10], Fig. 13). In this instrument a sine wave of peak-to-peak amplitude $U_{pp} \cong 23q$ is used as a dither.
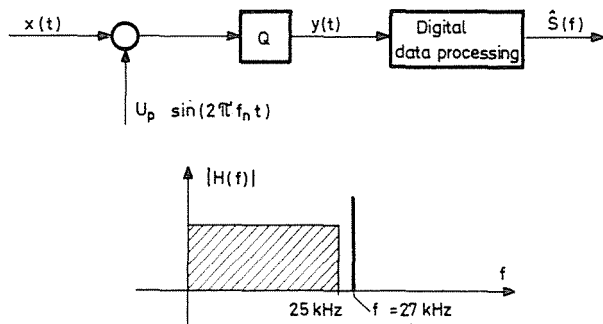
Fig. 13. Dithering in the HP3582A spectrum analyzer

As its frequency (27 kHz) is beyond the analyzer band, the main wave itself is filtered out (thus its variance is *eliminated* from the measurement) and its quantization error is a rather high quality uniform distribution white noise, since $U_{pp} \gg q$, and the sampling frequency is rather low related to the "bandlimit" of the quantization error. Moreover, since the sine wave makes use of several quantum levels, it helps to average their distortion.

Practical Limits of the Theoretical Results

With the characteristic function method we have proved that in the case of A/D-type quantizers the bias of the estimator of the mean may be effectively reduced, that is, measurements below the quantum size are possible. However this theory does not deal with two important viewpoints:

a) Although exact quantum levels may well describe the effect of roundings in some arithmetics, in A/D converters the quantum levels are not as exact as supposed by the theory. These levels not only have some variance, but their distortion is guaranteed only to be within e.g. ± 0.5 LSB. This fact often makes the resolution improvement illusory. What can still be done is to find a model for this distortion. If it may be described by some "overall" distortion function, the measured values may be corrected after (self-)calibration (see Fig. 14); if the distortion of the quantum levels is irregular, relatively large signal variations may help, since irregular deviations work against each other.

b) Dithering may reduce distortion even in the case of rough quantization, but the variance increases. The growth G of the variance of a sampled value depends on the dither and the signal itself, and is bounded by

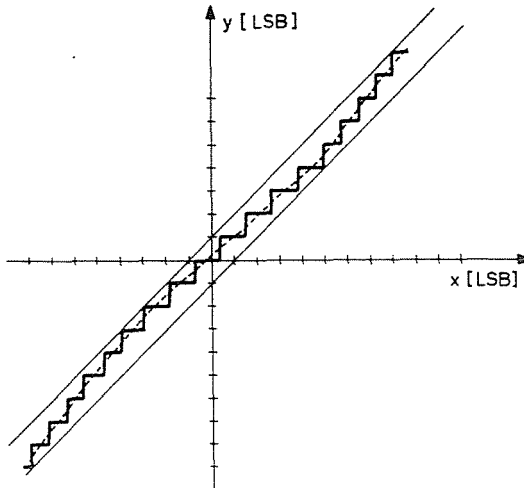$$0 \le G \le \left(\sigma_d + \frac{q}{2} \right)^2 .$$

Fig. 14. Deviation from the theoretical quantum levels in an A/D converter

G is zero e.g. if the signal equals the midpoint of a quantization interval with probability 1, and a dither of uniform distribution between $\left[-\dfrac{q}{2}, \dfrac{q}{2}\right]$ is used; G may be close to the upper limit in the case of a binary dither (which is a very clumsy choice).

If the dither and the signal fulfil certain conditions (e.g. $p_{x+d}(z)$ is relatively smooth and covers at least a few quantization intervals), the additive term in the variance may be well approximated by

$$G \approx \sigma_d^2 + \frac{q^2}{12},$$

or by $G \approx q^2/12$ if the dither is filtered out (see e.g. HP3582A).

This increase in the variance may not always be tolerated. Thus if enough averaging is not possible to overcome the variance, the use of high-resolution A/D converters cannot be avoided.

The Measurement of Second-order Moments

For higher-order moments the theory is rather similar to the previously presented one. In the case of non-uniform quantizers the general expression for the second-order moments is:

$$R_{y_a, y_b} = \sum_m \sum_n y_{a,m} y_{b,n} P_{m,n},$$

$$P_{m,n} = \int_{x_{a,m-1}}^{x_{a,m}} \int_{x_{b,n-1}}^{x_{b,n}} p_{x_a, x_b}(u, v)\, du\, dv.$$

$$(2.16)$$

The two-dimensional characteristic function may be defined as follows:

$$W_{x_a, x_b}(\alpha_a, \alpha_b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{x_a, x_b}(u, v) e^{j(\alpha_a u + \alpha_b v)} du dv. \qquad (2.17)$$

In case of uniform quantizers there is no distortion, if $W_{x_a, x_b}(\alpha_a, \alpha_b)$ is limited in both directions. Independent dithers may be used in both channels to decrease or to eliminate distortion. If the dithers are not independent, their joint moments appear in the result (see e.g. the expressions (2.8)).

However, for *Gaussian signals* some special and very effective techniques exist. For the investigation of the distortion of roughly quantized Gaussian processes the Price theorem [11] provides a theoretical background: if for the function g the inequality

$$|g(x_a, x_b)| < c \cdot e^{(x_a^d + x_b^d)}$$

holds for $c > 0$, $d < 2$, and x_a, x_b are of normal joint distribution,

$$\frac{\partial^n}{\partial \lambda^n} E\{g(x_a, x_b)\}\Big|_{\sigma_a, \sigma_b = \text{const}} = E\left\{\frac{\partial^{2n}}{\partial x_a^n \partial x_b^n} g(x_a, x_b)\right\}, \qquad (2.18)$$

where

$$\lambda = r \sigma_a \sigma_b.$$

From (2.18) the following expression may be derived:

$$E\{g(x_a, x_b)\} = \int_0^\lambda E\left\{\frac{\partial^2}{\partial x_a \partial x_b} g(x_a, x_b)\right\} d\lambda + E\{g(x_a, x_b)\}\Big|_{\lambda = 0}, \qquad (2.19)$$

and this expression can be evaluated in special cases of rough quantization.

According to theoretical results for zero mean Gaussian processes the correlation coefficient may be measured without dither even with rough quantizers, since the distortion can be calculated and so removed.
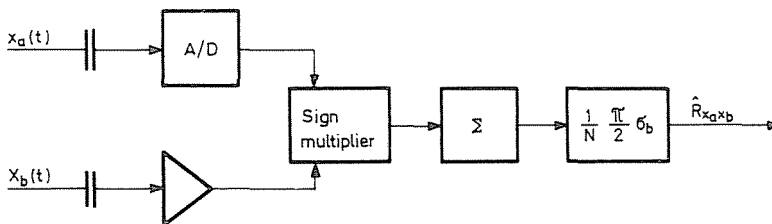

Fig. 15. Correlation measurement of Gaussian signals with a comparator in one of the channels (Relay-correlator)

In the literature [5], [12], [13] it is shown that for zero mean Gaussian processes

$$E\{x_a \operatorname{sign} x_b\} = \frac{2}{\pi} r_{ab} \sigma_a, \tag{2.20}$$

that is, with the equipment shown in Fig. 15 the correlation between x_a and x_b can be measured. The correction is simply a multiplicative factor, $\frac{\pi}{2} \sigma_b$; σ_b may be measured separately, if needed. As the factor is the same for any time delay between the samples $x_a = x_a(t)$, $x_b = x_b(t + \tau)$, the shape of the correlation will be correct even if σ_b is not known. The variance, *supposing independent measurements*, is as follows:

$$\operatorname{var}\{\hat{R}_{x_a, x_b}\} = \frac{1}{N} \operatorname{var}\{x_a \operatorname{sign} x_b\} = \frac{\sigma_a^2}{N}\left[1 - \left(\frac{2}{\pi}\right)^2 r_{ab}^2\right]. \tag{2.21}$$

In Fig. 16 the variance of the averaged correlation coefficient estimator

$$\hat{r}_{ab}^{(2)} = \frac{\pi}{2} \frac{1}{\sigma_a} \frac{1}{N} \sum_{i=1}^{N} (x_a \operatorname{sign} x_b)_i$$

is plotted as a function of r_{ab}:

$$\operatorname{var}\{\hat{r}_{ab}^{(2)}\} = \frac{1}{N}\left[\left(\frac{\pi}{2}\right)^2 - r_{ab}^2\right]. \tag{2.22}$$

It may be observed that the variance is comparable to that of the measurement with fine quantizers:

$$\hat{r}_{ab}^{(1)} = \frac{1}{\sigma_a \sigma_b} \cdot \frac{1}{N} \sum_{i=1}^{N} (x_a x_b)_i$$

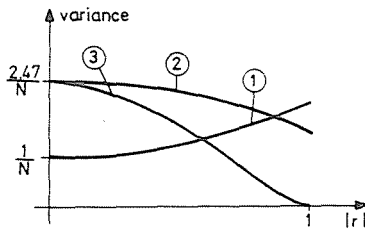$$\operatorname{var}\{\hat{r}_{ab}^{(1)}\} = \frac{1}{N}(1 + r_{ab}^2). \tag{2.23}$$

Fig. 16. The variance of different correlation coefficient estimators for Gaussian processes
1 — Correlating by means of 2 fine quantizers
2 — Correlating by means of a fine quantizer and a comparator
3 — Correlating by means of 2 comparators

Theory provides that the correlation function can be measured with two comparators as well [5, 12, 13]:

$$E\{\text{sign}(x_a)\,\text{sign}(x_b)\} = \frac{2}{\pi}\arcsin r_{ab}. \qquad (2.24)$$

This means that the circuit of Fig. 17 (the so-called polarity coincidence correlator for Gaussian signals) estimates r_{ab} approximately without distortion if N is great.
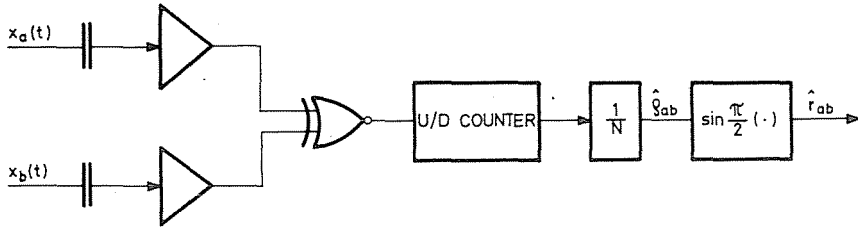

Fig. 17. Polarity coincidence correlator for Gaussian signals

For the estimator

$$\hat{r}_{ab}^{(3)} = \sin\left[\frac{\pi}{2}\frac{1}{N}\sum_{i=1}^{N}(\text{sign}(x_a)\,\text{sign}(x_b))_i\right]$$

the variance may be calculated with the binomial model of $\hat{\rho}$ [9]:

$$\text{var}\{r_{ab}^{(3)}\} = \frac{1}{N}\left[\left(\frac{\pi}{2}\right)^2 - \arcsin^2(r)\right][1-r^2]. \qquad (2.25)$$

It can observed in Fig. 16 that the polarity coincidence correlator is not only simpler and quicker than a common correlator, but its variance is also smaller in the case of: $|r_{ab}| > 0.59$. For the much more complicated case of correlated samples see [15].

References

1. WIDROW, B.: A Study of Rough Amplitude Quantization by Means of Nyquist Sampling Theory, Sc. D. dissertation, M.I.T. Elec. Engrg. Dept., Electronics Systems Lab., Cambridge, Mass., June 1956.
2. WIDROW, B.: Statistical Analysis of Amplitude-Quantized Sampled-Data Systems, Trans. AIEE, vol. 79. part II., Appl. and Ind., No. 52, pp. 555—568 (Jan. 1961).
3. DOBROWIECKI, T.: Modelling a Quantizer—Problems and Possible Approaches, Period. Polytechn. El. Eng. 28. 2—3, pp. 159—172. (1984).

4. PÁPAY, ZS.: Kvázistatisztikus modell az automatikus mérőszámgenerálásra (Quasi-Statistic Model for the Automatic Generation of a Quality Index of A/D Converters) C. Sc. Thesis, Budapest, 1978.
5. SZTIPÁNOVITS, J.: Optimális mérések kvantált adatokkal (Optimal Measurements with Quantized Data) C. Sc. Thesis, Budapest, 1979.
6. KOLLÁR, I.: Steady-State Value Measurements with Quantized Data, Proceedings of the second IMEKO TC8 Symposium, "Theoretical and Practical Limits of Measurement Accuracy". 10—12 May 1983, Budapest, Published by IMEKO Secretariat, Budapest, 1983. pp. 92—101.
7. KORN, G. A.: Random-Process Simulation and Measurements New York, McGraw-Hill, 1966.
8. WEHRMANN, W.: Einführung in die stochastisch-ergodische Impulstechnik (Introduction to the Stochastic-Ergodic Pulse Technique), Wien, R. Oldenburg Verlag, 1973.
9. KOLLÁR, I.: Pontos eredmények durva kvantálással: a sztochasztikus-ergodikus konverter és alkalmazásai (Exact Results by Rough Quantization: The Stochastic-Ergodic Converter and Its Applications) Mérés és Automatika, 30, 7, pp. 245—251 (1982).
10. PENDERGRASS, N. A.—FARNBACH, J. S.: A High Resolution, Low Frequency Spectrum Analyzer, Hewlett-Packard Journal, 29, No. 13, 2—13 (1978).
11. PAPOULIS, A.: Probability, Random Variables and Stochastic Processes, New York, McGraw-Hill, 1965.
12. HAGEN, J. B.—FARLEY, J. S.: Digital-Correlation Techniques in Radio Science, Radio Science, 8, No. 8—9, pp. 275—284 (1973).
13. SCHNELL, L. (ed.): Jelek és rendszerek méréstechnikája (Measurement Technology of Signals and Systems) In press. Műszaki Könyvkiadó, Budapest.
14. SRIPAD, A. B.—SNYDER, D. L.: A Necessary and Sufficient Condition for Quantization Errors to be Uniform and White, IEEE Transactions on Acoustics . . ., ASSP-25, No. 5, pp. 442—448 (1977).
15. VELTMAN, B. P. TH.: Quantisierung, Abtastfrequenz und statistische Streuung bei Korrelationsmessungen (Quantization, Sampling Frequency and Standard Deviation at Measurements of Correlation), Regelungstechnik, 14, 4, pp. 151—158 (1966).

István KOLLÁR   H-1521 Budapest