GENERAL EXPRESSION OF THE ACTION INTEGRAL IN ELECTRODYNAMICS

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Summary

The expression of the action integral is given in the paper for general media. Hence, Maxwell equations can be derived from a variational principle in this general case, too. The author shows invariance with respect to Lorentz transformation by rewriting the action integral in a four-dimensional form.

The significance of the action function of the electromagnetic field is that it provides basis for the derivation of the laws of classical electromagnetism, of Maxwell equations, from a variational principle. The action integral written for vacuum [4] has been generalized by I. Bárdi [1], [2] to include the case of linear media, and the way to compute electromagnetic fields on variational principles has been indicated.

In the present paper, the expression of the action integral is given for arbitrary media. This work, similarly to that of I. Bárdi, is concerned with the part of the action integral valid in lack of motion, the other parts are anyway not affected by the generalization.

In order to attain the above aim, the general expression of the magnetic and electric energy density is necessary. The magnetic energy density is

$$w_m = \int_{\boldsymbol{B}_0}^{\boldsymbol{B}} \boldsymbol{H}' \, \mathrm{d}\boldsymbol{B}' \tag{1}$$

with $\boldsymbol{B}$ denoting the flux density vector at a particular point in space, $\boldsymbol{B}_0$ is a fixed value of flux density, $\boldsymbol{H}'$ and $\boldsymbol{B}'$ are the values of magnetic field intensity and flux density corresponding to a state of magnetization between $\boldsymbol{B}_0$ and $\boldsymbol{B}$ of the particle at that point. Since the relationship between $\boldsymbol{H}$ and $\boldsymbol{B}$ in ferromagnetic medium is not single-valued, the concept of energy density in (1) is ambigous. It can be made unique by selecting in a definite manner one of the relationships between $\boldsymbol{H}'$ and $\boldsymbol{B}'$. A possibility is given in a paper of the author dating from 1962 [7] where w_m denotes, from the point of view of force calculations, the maximal energy density theoretically obtainable. Accordingly, $\boldsymbol{B}_0$ denotes the value of the flux density corresponding to $\boldsymbol{H}=\boldsymbol{0}$, and the relationship between $\boldsymbol{H}'$ and $\boldsymbol{B}'$ is characterized by a vector of magnetization

$M' = B' - \mu_0 H'$ of constant direction and monotonously decreasing absolute value during the [illegible]gnetization from B to B_0. Anyway, the manner of making the expression of energy density unique will be seen in the following to be indifferent at the variation of the action integral.

The electric energy density is chosen as

$$w_e = \int_{D_0}^{D} E' \mathrm{d}D' \tag{2}$$

where, similarly to the case of magnetic energy density, D_0 denotes the vector of displacement corresponding to $E = 0$, and the relationship between E' and D' is characterized by a monotonously decreasing and unidirectional vector of polarization $P = D' - \varepsilon_0 E'$. However, the quantity appearing in the action integral is not the electric energy density defined by (2) but the following quantity of the same dimension:

$$w_{e1} = \int_{0}^{E} D' \mathrm{d}E' . \tag{2a}$$

In case of linear (even anisotropic) medium, $w_{e1} = w_e$.

Now, the vector potential A and scalar potential φ are introduced as usual:

$$\operatorname{curl} A = B , \tag{3}$$

$$\operatorname{grad} \varphi = -\frac{\partial A}{\partial t} - E . \tag{4}$$

Thus, B and E, as expressed from (3) and (4), satisfy the Maxwell equations

$$\operatorname{div} B = 0$$

and

$$\operatorname{curl} E = -\frac{\partial B}{\partial t}$$

"automatically". The formulas (3) and (4) only define the vector and scalar potentials up to a gradient vector and a constant, respectively, this is the reason for the choice of a Lorentz gauge to define A and φ (this is not written here).

It will be shown that by varying A and φ, the first variation of the action integral

$$S = \int_{(t)} \int_{(V)} (AJ - \rho\varphi + w_{e1} - w_m) \,\mathrm{d}V \,\mathrm{d}t \tag{5}$$

vanishes if and only if the field quantities satisfy the Maxwell equations not written so far. On the right-hand side of (5), J denotes the conductive current density and ρ the volume charge density. To obtain the variation δS, the terms of the integrand are varied to yield:

$$\delta(\boldsymbol{A}\boldsymbol{J}-\rho\varphi)=\boldsymbol{J}\delta\boldsymbol{A}-\rho\delta\varphi\,,$$

$$\delta w_{e1}=\int_{\boldsymbol{E}}^{\boldsymbol{E}+\delta\boldsymbol{E}}\boldsymbol{D}'\,\mathrm{d}\boldsymbol{E}'=\boldsymbol{D}\delta\boldsymbol{E}=\boldsymbol{D}\delta\left(-\frac{\partial\boldsymbol{A}}{\partial t}-\operatorname{grad}\varphi\right)=$$

$$=-\boldsymbol{D}\frac{\partial(\delta\boldsymbol{A})}{\partial t}-\boldsymbol{D}\operatorname{grad}(\delta\varphi)=$$

$$=-\frac{\partial}{\partial t}(\boldsymbol{D}\delta\boldsymbol{A})+\frac{\partial\boldsymbol{D}}{\partial t}\delta\boldsymbol{A}-\operatorname{div}(\delta\varphi\boldsymbol{D})+\operatorname{div}\boldsymbol{D}\cdot\delta\varphi\,,$$

$$\delta w_m=\boldsymbol{H}\delta\boldsymbol{B}=\boldsymbol{H}\delta(\operatorname{curl}\boldsymbol{A})=\boldsymbol{H}\operatorname{curl}(\delta\boldsymbol{A})=$$

$$=\operatorname{div}(\delta\boldsymbol{A}\times\boldsymbol{H})+\delta\boldsymbol{A}\cdot\operatorname{curl}\boldsymbol{H}\,.$$

Substituting these into (5), the following is derived:

$$\delta S=\int_{(t)}\int_{(V)}\left[-\frac{\partial}{\partial t}(\boldsymbol{D}\delta\boldsymbol{A})-\operatorname{div}(\delta\varphi\boldsymbol{D})-\operatorname{div}(\delta\boldsymbol{A}\times\boldsymbol{H})\right]\mathrm{d}V\,\mathrm{d}t+$$

$$+\int_{(t)}\int_{(V)}\left[\left(\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}-\operatorname{curl}\boldsymbol{H}\right)\delta\boldsymbol{A}+(-\rho+\operatorname{div}\boldsymbol{D})\delta\varphi\right]\mathrm{d}V\,\mathrm{d}t\,.$$

By the extension of the first integral to the whole space the divergence terms vanish according to Gauss' theorem since the potentials decrease in the direction of the boundary in infinity according to $1/r$ and the field quantities according to $1/r^2$. The first term of the first integral is also zero since $\delta\boldsymbol{A}=\boldsymbol{0}$ at start and end of the time interval and only the start and end values remain after integrating with respect to time. Thus, if $\delta\boldsymbol{A}$ and $\delta\varphi$ are arbitrary, the variation δS of the action integral vanishes if and only if

$$\operatorname{curl}\boldsymbol{H}=\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}\quad\text{and}\quad\operatorname{div}\boldsymbol{D}=\rho\,,$$

and these are the Maxwell equations not written before.

According to (5), the action integral can be written in a four-dimensional form. Namely the 4-potential and the 4-current density are

$$\mathrm{A}=A+\frac{\varphi}{c}e_t\,,$$

and

$$\mathrm{J}=J+c\rho e_t$$

where c is the velocity of light in vacuum and e_t is the unit vector along the time axis in the inertial frame of reference used. Thus, e_t is perpendicular to the three-dimensional space vectors in this inertial frame of reference, and due to the indefinite metric of four-dimensional space-time:

$$e_t^2=-1\,. \tag{6}$$

Hence, the scalar product of the four-dimensional potential and current density is

$$\mathrm{AJ}=AJ-\varphi\rho\,, \tag{7}$$

thus, the first two terms in the integrand of (5) have been rewritten in a four-dimensional form. In order to rewrite the two other terms, the four-dimensional electric field tensor and excitation tensor are introduced:

$$\mathbb{F}=-B\times I-\frac{1}{c}E\circ e_t+\frac{1}{c}e_t\circ E,$$

$$\mathbb{G}=-H\times I-cD\circ e_t+e_t\circ cD$$

where the circle denotes the diadic product, I is the three-dimensional unit tensor (in the given inertial frame of reference) and $y\times I$ is a three-dimensional tensor satisfying

$$(y\times I)z=y\times z$$

for any z [6].

The tensor $\mathbb{G}\,\mathrm{d}\mathbb{F}$ is now formed:

$$\mathbb{G}\,\mathrm{d}\mathbb{F}=(-H\times I-cD\circ e_t+e_t\circ cD)\cdot$$
$$\cdot\left(-\mathrm{d}B\times I-\frac{1}{c}\mathrm{d}E\circ e_t+e_t\circ\frac{1}{c}\mathrm{d}E\right).$$

In view of

$$(-H\times I)(-\mathrm{d}B\times I)z=H\times(\mathrm{d}B\times z)=(Hz)\,\mathrm{d}B-(H\,\mathrm{d}B)z$$

and thus

$$(-H\times I)(-\mathrm{d}B\times I)=\mathrm{d}B\circ H-(H\,\mathrm{d}B)I,$$

as well as of the perpendicularity of $\boldsymbol{e}_t$ to space vectors and of (6):

$$\mathbf{G}\,\mathrm{dF}=\mathrm{d}\boldsymbol{B}\circ\boldsymbol{H}-(\boldsymbol{H}\,\mathrm{d}\boldsymbol{B})\boldsymbol{I}+\boldsymbol{e}_t\circ(\mathrm{d}\boldsymbol{B}\times c\boldsymbol{D})+\left(\frac{1}{c}\boldsymbol{H}\times\mathrm{d}\boldsymbol{E}\right)\circ\boldsymbol{e}_t-$$
$$-(\boldsymbol{D}\,\mathrm{d}\boldsymbol{E})\boldsymbol{e}_t\circ\boldsymbol{e}_t+\boldsymbol{D}\circ\mathrm{d}\boldsymbol{E}.$$

Now, the scalar invariant of G dF (the trace of its matrix) is formulated, taking into account the fact that the scalar invariant of a diad is the scalar product of the vectors forming it and that $\mathrm{Tr}(\boldsymbol{I})=3$:

$$\mathrm{Tr}\,(\mathbf{G}\,\mathrm{dF})=-2\boldsymbol{H}\,\mathrm{d}\boldsymbol{B}+2\boldsymbol{D}\,\mathrm{d}\boldsymbol{E}.$$

Hence, on the basis of (1) and (2a):

$$w_{el}-w_m=\frac{1}{2}\int_{\mathbf{F}_0}^{\mathbf{F}}\mathrm{Tr}\,(\mathbf{G}'\,\mathrm{dF}'),$$

and finally, combining this with formulas (5) and (7), the action integral is obtained as

$$S=\int_{(\Omega)}\left[\mathrm{AJ}+\frac{1}{2}\int_{\mathbf{F}_0}^{\mathbf{F}}\mathrm{Tr}\,(\mathbf{G}'\,\mathrm{dF}')\right]\mathrm{d}\Omega,$$

where $\mathrm{d}\Omega$ is the four-dimensional volume element.

Thus, the action integral (5) has been shown to be invariant with respect to Lorentz transformation, i.e. it is of general nature in this sense, too.

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