APPLICATION OF THE BUBNOV–GALERKIN METHOD TO NONLINEAR STATIONARY MAGNETIC FIELD PROBLEMS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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Summary

A numerical procedure based on the Bubnov–Galerkin method is presented for the approximate solution of a nonlinear stationary magnetic field problem with inhomogeneous boundary conditions. Neumann-type boundary conditions are included in the operator equation describing the boundary value problem. Their incorporation is presented both with the aid of a special functional Hilbert space and by the use of generalized functions. The Dirichlet-type boundary conditions are reduced to ones of Neumann type.

1. Introduction

The Galerkin-type projection methods are widely employed for the approximate solution of the differential equations of mathematical physics [6, 7, 8, 9, 10]. These methods have also been applied to electromagnetic field problems both in global element [11] and finite element [3, 13] procedures. Nonlinear differential equations are also readily treated by the methods. However, Galerkin-like procedures have only been applied to problems with homogeneous boundary conditions (or to ones reduced to such problems), or they have sometimes been extended to inhomogeneous boundary conditions without sufficient explanation.

The aim of this paper is to extend the Bubnov–Galerkin method with global approximation to problems with inhomogeneous boundary conditions. Special effort has been made to ensure that the approximating functions don't have to satisfy any boundary conditions. The procedure is presented for nonlinear stationary magnetic field problems. The reason for selecting a nonlinear case is that such problems cannot be reduced to ones with homogeneous boundary conditions by constructing functions satisfying the inhomogeneous conditions. A stationary problem has been chosen where the Bubnov–Galerkin method coincides with the Ritz's process applied to the minimum of a functional [1]. It must be stressed, however, that the convergence of the Bubnov–Galerkin method is assured for a broader class of problems, and the results of this paper can easily be extended to non-stationary case.

2. The Bubnov–Galerkin method

Consider the operator equation

$$P(u)-f=0 \tag{1}$$

where P is an operator, in general nonlinear, acting in a Hilbert space H. The range of definition of P is $D(P)$, and the solution $u_0 \in D(P)$ of (1) is sought, if $f \in H$ is kown. Under very loose conditions on P (e.g. $D(P)$ is a linear set dense in H, and P can be written as

$$P=A+K \tag{2}$$

with K bounded and the inverse of A completely continuous), the following Bubnov–Galerkin process is convergent [6, 7, 10].

Let an algebraic base $\{\varphi_i\} \subset D(P)$ be chosen in H, and the approximate solution of (1) be sought as

$$u_n=\sum_{k=1}^{n} a_k \varphi_k, \tag{3}$$

with a_k $(k=1,2,\ldots,n)$ being numerical coefficients. These latter are determined from the condition that after the substitution of (3) into (1) the left-hand side of (1) is orthogonal to the elements $\varphi_1, \varphi_2, \ldots, \varphi_n$:

$$\left(P\left(\sum_{k=1}^{n} a_k \varphi_k\right), \varphi_i\right)=(f, \varphi_i), \qquad i=1,2,\ldots,n \tag{4}$$

where (,) denotes the scalar product in H.

The solution of the set of algebraic equations (4) yields the coefficients a_k which, in turn, give an approximate solution of the form (3). If n is increased, these latter tend to the exact solution u_0 of (1). If e.g. the Frechet derivative A' of the operator A in (2) exists and is positive definite, the convergence takes place in the energy norm of A' [9].

3. Formulation of the electromagnetic field problem

The stationary magnetic field in ferromagnetic medium is obtained by the solution of the Maxwell equations

$$\mathbf{curl}\,\mathbf{H}=\mathbf{J}\,, \tag{5}$$

$$\operatorname{div}\mathbf{B}=0\,, \tag{6}$$

$$\mathbf{H}=\nu(|\mathbf{B}|)\mathbf{B}\,. \tag{7}$$

H is the magnetic field intensity, **B** is the magnetic flux density and **J** is the current density which is assumed to be given. $\nu(|\mathbf{B}|)$ is the reluctivity of the medium, and with the hystheresis neglected it is a one-to-one function of the absolute value of **B**.

On satifying (6) by the introduction of the vector potential **A**:

$$\mathbf{B}=\mathbf{curl}\,\mathbf{A}\,, \tag{8}$$

(5) and (7) yield the following differential equation:

$$\mathbf{curl}\,[\nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A}]=\mathbf{J}\,. \tag{9}$$

To determine the stationary magnetic field in a bounded region Ω, appropriate boundary conditions are necessary. Two types of boundary conditions are considered.

In one case, the tangential component of the magnetic field intensity is known on a part S_1 of the boundary:

$$\mathbf{H}\times\mathbf{n}|_{S_1}=\nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A}\times\mathbf{n}|_{S_1}=\mathbf{h} \tag{10}$$

where **h** is known, **n** is the normal unit vector of S_1 and the left-hand side of (10) is the tangential component of **H** rotated in the tangential plane by 90 degrees. This will be called Neumann boundary condition.

The other type of boundary conditions is obtained, if the normal component of magnetic flux density is given on some other part S_2 of the boundary. This boundary condition can be formulated for the vector potential as

$$\mathbf{n}\times\mathbf{A}|_{S_2}=\mathbf{a} \tag{11}$$

where **a** is known and the left-hand side of (11) is the tangential component of the vector potential rotated in the tangential plane by 90 degrees. This will be called Dirichlet boundary condition.

It can be shown that if the union of S_1 and S_2 is the boundary of the region Ω, the solution in Ω of the differential equation (9) with the boundary conditions (10) and (11) is unique for the magnetic flux density [1].

Our aim is to present the application of the Bubnov–Galerkin method to the solution of the boundary value problem (9), (10), (11).

4. Treatment of the Neumann boundary condition

4.1. Statement of the problem

Initially, it will be presumed that $S_2 = \emptyset$, i.e. the tangential component of the magnetic field intensity is given on the whole of the boundary. Accordingly, the differential equation (9):

$$\mathbf{curl}\,[\nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A}] = \mathbf{J} \quad \text{in } \Omega \tag{9a}$$

has to be solved in a bounded region Ω with the boundary condition

$$\nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A} \times \mathbf{n} = \mathbf{h} \quad \text{on } S \tag{10a}$$

where S is the boundary of Ω.

In order to apply the Bubnov–Galerkin method, an operator equation equivalent to the boundary value problem has to be formulated. This means that both the differential equation and the boundary condition have to be included in the equation.

A usual way of application in linear case is to reduce the problem to one with homogeneous boundary conditions by the construction of functions satisfying the inhomogeneous conditions [8]. (This is usually not an easy task, especially at involved geometrical layouts, but a possibility is found in [12], and its electrodynamic application in [5].) In case of homogeneous boundary conditions the range of definition of the operator can be chosen as the set of sufficiently differentiable functions satisfying the boundary conditions. The operator-equation is then the differential equation.

In nonlinear case, this approach is not possible since superposition is not valid. An alternative possibility is to allow the range of definition to contain all functions with sufficient derivatives, and to include the boundary conditions in the operator equation. This inclusion will first be carried out by the introduction of a suitable function-space, and then the concept of generalized functions (distributions) will be employed.

4.2. Formulation in a functional Hilbert space

In order to carry out the above inclusion let us define a Hilbert space as follows. The elements of H are pairs of vector functions of which one is square integrable in Ω and the other is square integrable in S the boundary of Ω:

$$\{\mathbf{u}_1, \mathbf{u}_2\} \in H \quad \text{if} \quad \mathbf{u}_1 \in \mathbf{L}_2(\Omega) \quad \text{and} \quad \mathbf{u}_2 \in \mathbf{L}_2(S)\,. \tag{12}$$

$\mathbb{L}_2(\Omega)$ and $\mathbb{L}_2(S)$ are the Hilbert spaces of the vector functions square integrable in Ω and S, respectively. The scalar product in H is defined as

$$(\{\mathbf{u}_1, \mathbf{u}_2\}, \{\mathbf{v}_1, \mathbf{v}_2\}) = \int_\Omega \mathbf{u}_1, \mathbf{v}_1 \, \mathrm{d}\Omega + \oint_S \mathbf{u}_2 \mathbf{v}_2 \, \mathrm{d}S . \tag{13}$$

It is shown in Appendix A that H is in fact a Hilbert space. Evidently, if $\mathbf{u}$ is a function in $\mathbb{L}_2(\Omega)$, it is also an element of H.

The operator of the boundary value problem (9a), (10a) is defined in H. Its range of definition $D(P)$ is formed by the vector functions twice differentiable in Ω and once in S. This is a linear set dense in H. The operator equation is:

$$P(\mathbf{A}) \equiv \{\mathbf{curl}\,[\nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A}],\ \nu(|\mathbf{curl}\,\mathbf{A}|)\,\mathbf{curl}\,\mathbf{A} \times \mathbf{n}|_S\} = \\ = \{\mathbf{J}, \mathbf{h}\} . \tag{14}$$

It can easily be proved that the Frechet derivative of $P(\mathbf{A})$ exists and is positive [1] which is sufficient for the convergence of the Bubnov–Galerkin process [9]. The approximate solution of (14) is sought as

$$\mathbf{A}_n = \sum_{k=1}^{n} a_k \boldsymbol{\varphi}_k \tag{15}$$

where $\boldsymbol{\varphi}_k$ are chosen from an entire set in $D(P)$. The Galerkin equation corresponding to (4) can be written on the basis of (13) and (14). Using the Gauss theorem it is of the form:

$$\int_\Omega \nu(|\mathbf{curl}\,\mathbf{A}_n|)\,\mathbf{curl}\,\mathbf{A}_n\,\mathbf{curl}\,\boldsymbol{\varphi}_i \,\mathrm{d}\Omega = \int_\Omega \mathbf{J}\boldsymbol{\varphi}_i \,\mathrm{d}\Omega + \oint_S \mathbf{h}\boldsymbol{\varphi}_i \,\mathrm{d}S , \\ i = 1, 2, \ldots, n . \tag{16}$$

This is a system of nonlinear algebraic equations for the coefficients a_k ($k = 1, 2, \ldots, n$). Its solution yields an approximation of the form (15) for the function solving the boundary value problem. The sequence of these approximations converges to the exact solution in the energy norm of the Frechet derivative of P. This means that

$$\int_\Omega |\mathbf{curl}\,\mathbf{A}_n - \mathbf{curl}\,\mathbf{A}_0|^2 \,\mathrm{d}\Omega \to 0 \tag{17}$$

where $\mathbf{A}_0$ denotes the solution of the boundary value problem [1].

It is noted that the result (16) can be obtained by the application of the Ritz process to the minimum problem of an appropriate energy functional [1]. However, the Bubnov–Galerkin method lends itself more easily to generalization since its convergence is assured for a broader class of problems [7, 10].

4.3. Formulation with the aid of generalized functions

In the following, an alternative formulation of the operator equation corresponding to the boundary value problem (9a), (10a) will be presented. To this end, we introduce two generalized functions (distributions). One of them is denoted by Θ_Ω and defined as

$$\langle \Theta_\Omega, \varphi \rangle = \int_\Omega \varphi \, \mathrm{d}\Omega \,, \qquad \varphi \in D \,. \tag{18}$$

D is the set of functions with an infinite number of derivatives and compact support in the three-dimensional Euclidean space, $\langle \Theta_\Omega, \varphi \rangle$ denotes the result of the application of the functional Θ_Ω to the function φ. Θ_Ω can be identified with the function which equals 1 in Ω and vanishes outside it. The other distribution denoted by δ_S is defined by

$$\langle \delta_S, \varphi \rangle = \oint_S \varphi \, \mathrm{d}S \,, \qquad \varphi \in D \,. \tag{19}$$

δ_S is a Dirac distribution concentrated on the surface S. It is closely related to the derivatives of Θ_Ω, namely

$$\mathbf{grad}\, \Theta_\Omega = -\mathbf{n}\delta_S \tag{20}$$

where $\mathbf{n}$ is the outer normal unit vector of S. The rules governing the operations with distributions are summarized in Appendix B.

The boundary value problem is now formulated as

$$\Pi(\mathbf{A}) \equiv \Theta_\Omega \, \mathbf{curl}\, [\nu(|\mathbf{curl}\, \mathbf{A}|)\, \mathbf{curl}\, \mathbf{A}] + \\ + \delta_S \nu(|\mathbf{curl}\, \mathbf{A}|)\, \mathbf{curl}\, \mathbf{A} \times \mathbf{n} = \Theta_\Omega \mathbf{J} + \delta_S \mathbf{h} \,. \tag{21}$$

This is clearly equivalent to (9a), (10a). The operator $\Pi(\mathbf{A})$ on the left-hand side of (21) can also be written as (cf. (20) and Appendix B):

$$\Pi(\mathbf{A}) \equiv \mathbf{curl}\, [\Theta_\Omega \nu(|\mathbf{curl}\, \mathbf{A}|)\, \mathbf{curl}\, \mathbf{A}] \,. \tag{22}$$

This can be interpreted as the curl of a function which coincides with the magnetic field intensity in Ω and vanishes outside it. The right-hand side of (21) is seen as the sum of the volume current density $\mathbf{J}$ and a surface current density $\mathbf{h}$. Therefore, (21) is a generalization of the Maxwell equation (5) for the case when the magnetic field intensity is not continuous along a surface.

The operator Π defined in (21) acts from the space D to its dual space D' consisting of functionals identified with distributions. For this case, a natural formulation of the Bubnov–Galerkin method is the following.

The approximate solution of (21) is sought in the form (15), but with the functions $\boldsymbol{\varphi}_k$ chosen from D. The unknown coefficients in (15) are now determined from the condition that the functionals on the two sides of (21) yield the same result on application to the functions φ_i ($i=1, 2, \ldots, n$) after (15) has been substituted:

$$\langle \Theta_\Omega \, \mathbf{curl}\, [\nu(|\mathbf{curl}\, \mathbf{A}_n|)\, \mathbf{curl}\, \mathbf{A}_n], \varphi_i \rangle + \langle \delta_S \nu(|\mathbf{curl}\, \mathbf{A}_n|)\, \mathbf{curl}\, \mathbf{A}_n \times \mathbf{n}, \varphi_i \rangle =$$
$$= \langle \Theta_\Omega \mathbf{J}, \varphi_i \rangle + \langle \delta_S \mathbf{h}, \varphi_i \rangle\,, \qquad i=1, 2, \ldots, n\,. \tag{23}$$

Using (18), (19) and the Gauss theorem, (23) yields the same set of algebraic equations as (16).

This result is significant, bacause it permits the use of the Dirac function δ_S defined in (19) to describe inhomogeneous Neumann boundary conditions in a form easily added to the differential equation. This makes the direct application of the Bubnov–Galerkin process to the modified differential equation possible.

5. Treatment of the Dirichlet boundary condition

Let us now return to the boundary value problem (9), (10), (11). The Dirichlet boundary condition (11) cannot directly be included in the operator equation. An alternative treatment is presented now [1, 2].

Let a set of functions $\{\xi_i\}$ be chosen on S_2 the surface with Dirichlet boundary condition prescribed on it. Let ξ_i be of tangential direction on the surface, and such functions will be called tangential vector functions. The function set $\{\xi_i\}$ should be entire in L_2 sense in the space of tangential vector functions on S_2. Hence, the tangential component of the field intensity rotated by 90 degrees can be written on S_2 as

$$\mathbf{H} \times \mathbf{n}|_{S_2} = \nu(|\mathbf{curl}\, \mathbf{A}|)\, \mathbf{curl}\, \mathbf{A} \times \mathbf{n}|_{S_2} = \sum_{j=1}^{\infty} \lambda_j \xi_j\,. \tag{24}$$

An approximation of the left-hand side of (24) is derived by using a partial sum of the infinite sum:

$$\nu(|\mathbf{curl}\, \mathbf{A}|)\, \mathbf{curl}\, \mathbf{A} \times \mathbf{n}|_{S_2} \approx \sum_{j=1}^{m} \lambda_j \xi_j\,. \tag{25}$$

Now, the approximate problem (9), (10), (25) is the same as (9a), (10a) with the exception that a sum with unknown coefficients λ_j appears in (25) instead of the known function $\mathbf{h}$. Accordingly, the Galerkin system of algebraic equations corresponding to (16) is

$$\int_\Omega \nu(|\mathbf{curl}\, \mathbf{A}_n|)\, \mathbf{curl}\, \mathbf{A}_n\, \mathbf{curl}\, \varphi_i \, d\Omega - \sum_{j=1}^{m} \lambda_j \int_{S_2} \xi_j \varphi_i \, dS =$$
$$= \int_\Omega \mathbf{J} \varphi_i \, d\Omega + \int_{S_1} \mathbf{h} \varphi_i \, dS\,, \qquad i=1, 2, \ldots, n\,. \tag{16a}$$

A_n here is an approximation of the form (15) of the vector potential. This can be obtained by the solution of (16a). The function thus derived approximately satisfies (9) and (10), and includes the coefficients λ_j $(j=1,2,\ldots,m)$ as free parameters. To select the parameters λ_j and to obtain a solution satisfying (11) as well as possible, the function $(\mathbf{n}\times A_n|_{S_2}-\mathbf{a})$ is chosen to be orthogonal on S_2 in L_2 sense to the functions $\mathbf{n}\times\xi_j$ $(j=1,2,\ldots,m)$:

$$\int_{S_2}(\mathbf{n}\times A_n-\mathbf{a})(\mathbf{n}\times\xi_j)\,\mathrm{d}S=0\,, \tag{26}$$

or in an equivalent form:

$$\int_{S_2}A_n\xi_j\,\mathrm{d}S=\int_{S_2}(\xi_j\times\mathbf{a})\,\mathrm{d}S\,,\qquad j=1,2,\ldots,m\,. \tag{26a}$$

This is again an application of the Galerkin method.

It m is chosen to fulfil the inequality

$$m<n \tag{27}$$

the solution of the algebraic equations (16a), (26a) for the coefficients a_k $(k=1,2,\ldots,n)$ and λ_j $(j=1,2,\ldots,m)$ yields an approximation (15) which approximately satisfies (9), (10) and (11).

Appendix A

It is shown in the following that the function-set defined in (12) is a real Hilbert space with the scalar product (13). It is to be shown that H is linear, (13) is in fact a scalar product with the norm generated by it satisfying Schwarz's inequality, and that H is entire in the sense of the norm defined in accordance with (13).

1. The linearity of H is evident if multiplication by numbers and addition is defined in the natural way.

2. The linearity and commutativity of the scalar product (13) is evident. Also, if

$$(\{\mathbf{u}_1,\mathbf{u}_2\},\{\mathbf{u}_1,\mathbf{u}_2\})=\int_{\Omega}|\mathbf{u}_1|^2\,\mathrm{d}\Omega+\oint_{S}|\mathbf{u}_2|^2\,\mathrm{d}S=0\,,$$

then $\{\mathbf{u}_1,\mathbf{u}_2\}=\emptyset$ where $\emptyset$ is the zero element of H i.e. the pair formed by the zero elements of $\mathbb{L}_2(\Omega)$ and $\mathbb{L}_2(S)$.

3. To show Schwarz's inequality, let $(\ ,\ )_{\Omega}$ and $(\ ,\ )_S$ denote the scalar products in $\mathbb{L}_2(\Omega)$ and $\mathbb{L}_2(S)$. With this:

$$(\{\mathbf{u}_1,\mathbf{u}_2\},\{\mathbf{v}_1,\mathbf{v}_2\})=(\mathbf{u}_1,\mathbf{v}_1)_{\Omega}+(\mathbf{u}_2,\mathbf{v}_2)_S\,.$$

Thus:

$$|(\{\mathbf{u}_1, \mathbf{u}_2\}, \{\mathbf{v}_1, \mathbf{v}_2\})|^2 = |(\mathbf{u}_1, \mathbf{v}_1)_\Omega|^2 + |(\mathbf{u}_2, \mathbf{v}_2)_S|^2 +$$
$$+2|(\mathbf{u}_1, \mathbf{v}_1)_\Omega||(\mathbf{u}_2, \mathbf{v}_2)_S| \leqq$$
$$\leqq \|\mathbf{u}_1\|_\Omega^2\|\mathbf{v}_1\|_\Omega^2 + \|\mathbf{u}_2\|_S^2\|\mathbf{v}_2\|_S^2 + 2\|\mathbf{u}_1\|_\Omega\|\mathbf{v}_1\|_\Omega\|\mathbf{u}_2\|_S\|\mathbf{v}_2\|_S \leqq$$
$$\leqq \|\mathbf{u}_1\|_\Omega^2\|\mathbf{v}_1\|_\Omega^2 + \|\mathbf{u}_2\|_S^2\|\mathbf{v}_2\|_S^2 + \|\mathbf{u}_1\|_\Omega^2\|\mathbf{v}_2\|_S^2 + \|\mathbf{u}_2\|_S^2\|\mathbf{v}_1\|_\Omega^2 =$$
$$= (\|\mathbf{u}_1\|_\Omega^2 + \|\mathbf{u}_2\|_S^2)(\|\mathbf{v}_1\|_\Omega^2 + \|\mathbf{v}_2\|_S^2) = \|\{\mathbf{u}_1, \mathbf{u}_2\}\|^2\|\{\mathbf{v}_1, \mathbf{v}_2\}\|^2 .$$

This is the Schwarz inequality. The same in $\mathbb{L}_2(\Omega)$ and in $\mathbb{L}_2(S)$ as well as the inequality between the arithmetic and geometric means has been used here.

4. H is entire, because it follows from

$$\|\{\mathbf{u}_{1n} - \mathbf{u}_{1m}, \mathbf{u}_{2n} - \mathbf{u}_{2m}\}\|^2 = \|\mathbf{u}_{1n} - \mathbf{u}_{1m}\|_\Omega^2 + \|\mathbf{u}_{2n} - \mathbf{u}_{2m}\|_S^2 \to 0$$

that

$$\|\mathbf{u}_{1n} - \mathbf{u}_{1m}\|_\Omega \to 0 \qquad \text{and} \qquad \|\mathbf{u}_{2n} - \mathbf{u}_{2m}\|_S \to 0 ,$$

and this implies that the sequence $\{\mathbf{u}_{1n}\}$ is convergent in $\mathbb{L}_2(\Omega)$ and $\mathbf{u}_{2n}$ in $\mathbb{L}_2(S)$, thus by the definition of H, $\{\{\mathbf{u}_{1n}, \mathbf{u}_{2n}\}\}$ is convergent in H.

Appendix B

Some rules of operations on distributions are summarized in the following.

If a distribution T can be identified with a locally integrable function, then

$$\langle T, \varphi \rangle = \int_{R^3} T\varphi \, \mathrm{d}x , \qquad T \in D' , \quad \varphi \in D .$$

It T_1 and T_2 are two distributions, their sum is defined by

$$\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle , \quad \varphi \in D .$$

If $a \in C^\infty$ and $T \in D'$, their product is defined by

$$\langle aT, \varphi \rangle = \langle T, a\varphi \rangle , \quad \varphi \in D .$$

The partial derivative of a distribution with regard to the variable x_1 is defined as

$$\left\langle \frac{\partial T}{\partial x_1}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_1} \right\rangle, \quad T \in D' , \quad \varphi \in D .$$

The above can be extended to vector distributions in a natural way. The definitions of the operations **grad**, div and **curl** for distributions are the following:

$$\langle \mathbf{grad}\, T, \boldsymbol{\varphi} \rangle = -\langle T, \operatorname{div} \boldsymbol{\varphi} \rangle\,, \quad T \in D', \quad \boldsymbol{\varphi} \in D\,.$$

$$\langle \operatorname{div} \mathbf{T}, \varphi \rangle = -\langle \mathbf{T}, \mathbf{grad}\, \varphi \rangle, \quad \mathbf{T} \in D', \quad \varphi \in D\,.$$

$$\langle \mathbf{curl}\, \mathbf{T}, \boldsymbol{\varphi} \rangle = \langle \mathbf{T}, \operatorname{curl} \boldsymbol{\varphi} \rangle, \quad \mathbf{T} \in \mathbf{D}', \quad \boldsymbol{\varphi} \in \mathbf{D}\,.$$

A more detailed summary can be found e.g. in [4].

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