TRIANGULAR DECOUPLING THROUGH LINEAR STATE VARIABLE FEEDBACK

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Summary

The paper studies the problem of triangular decoupling through l.s.v.f. using frequency-domain approaches. The interactor concept and the matrix fraction right description form are used to resolve the questions concerned with this problem and in turn an algorithm is given for triangularization of any right invertible proper system by using l.s.v.f. alone. The paper describes the use of triangular decoupling as an intermediate step for exact decoupling of systems having unstable decoupling zeros.

1. Introduction

The triangular decoupling concept was introduced by Morse and Wonham [3]. They used the geometric approach to solve the triangular decoupling problem (TDP) by the use of linear state variable feedback (l.s.v.f.) control law. Algebraically the problem using the time-domain approach was already attacked by many authors. Wang [6], was the first who proved that the necessary and sufficient conditions for the existence of l.s.v.f. laws for triangular decoupling are equivalent to the conditions of invertibility, i.e., an invertible system can always be triangularized by using l.s.v.f. alone. More recently, alternative forms and generalization were given by Furta and Kamiyam [2], and Descusse and Lizarzaburu [1], respectively. It is important to note that the time-domain approach is based mainly upon the Silverman's inversion algorithm [4], and Silverman-Payne structure algorithm [5].

In the frequency-domain, Wolovich [7], suggested an algorithm for TDP by using l.s.v.f. In our opinon a more theoretical material is needed to modify and satisfy this algorithm. Actually we shall use the interactor idea, which is the frequency-domain version of Silverman work used to resolve TDP in the time-domain, for such a purpose.

In § 2 a background necessary material is given. The main results are given as *Theorem 1*, *Theorem 2*, and *Algorithm 1* in § 3. These results will be used in § 4 to construct a new and more efficient algorithm for the diagonal decoupling of systems having unobservable-unstable modes.

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2. Preliminaries

Definition 1. Triangular decoupling through l.s.v.f. A $p \times m$, proper, right-invertible system $\mathbf{T}(s)$, expressed in its minimal controllable form,

$$\mathbf{T}(s) = \mathbf{R}(s)\, \mathbf{P}^{-1}(s) \tag{2.1}$$

where $\mathbf{R}(s)$ and $\mathbf{P}(s)$ are RRP; and $\mathbf{P}(s)$ column proper, is said to be triangularly decoupled by l.s.v.f., if there exists a l.s.v.f. control law,

$$\mathbf{U}(s) = \mathbf{F}(s)\mathbf{X}(s) + \mathbf{G}\mathbf{V}(s) \tag{2.2}$$

such that the C.L.S.,

$$\hat{\mathbf{T}}(s) = \mathbf{R}(s)\, [\mathbf{P}(s) - \mathbf{F}(s)]^{-1}\mathbf{G} \tag{2.3}$$

is in lower-left triangular form with

$$\partial c_j[\mathbf{P}(s)] = \partial c_j[\mathbf{P}(s) - \mathbf{F}(s)]\,, \tag{2.4a}$$

and

$$\mathbf{G}^{-1}\Gamma_c[\mathbf{P}(s)] = \Gamma_c[\mathbf{G}^{-1}(\mathbf{P}(s) - \mathbf{F}(s))] \tag{2.4b}$$

Let us consider the invertible case ($p = m$). By making use of Wang's theorem [6], there is a pair $\{\mathbf{F}(s), \mathbf{G}\}$ which satisfies equation (2.3), i.e.,

$$\hat{\mathbf{T}}(s) = \begin{bmatrix} \dfrac{\bar{r}_{11}(s)}{\bar{p}_{11}(s)} & 0 & \dots & 0 \\ \dfrac{\bar{r}_{21}(s)}{\bar{p}_{21}(s)} & \dfrac{\bar{r}_{22}(s)}{\bar{p}_{22}(s)} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \dfrac{\bar{r}_{m1}(s)}{\bar{p}_{m1}(s)} & \dfrac{\bar{r}_{m2}(s)}{\bar{p}_{m2}(s)} & \dots & \dfrac{\bar{r}_{mm}(s)}{\bar{p}_{mm}(s)} \end{bmatrix} \tag{2.5}$$

It is always possible to put equation (2.5) in the form,

$$\hat{\mathbf{T}}(s) = \hat{\mathbf{R}}(s)\, \hat{\mathbf{P}}^{-1}(s) \tag{2.6a}$$

where $\hat{\mathbf{R}}(s)$ and $\hat{\mathbf{P}}(s)$ are RRP and both are triangular, i.e.,

$$\hat{\mathbf{R}}(s) = \begin{bmatrix} \hat{r}_{11}(s) & 0 & \dots & 0 \\ \hat{r}_{21}(s) & \hat{r}_{22}(s) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \hat{r}_{m1}(s) & \hat{r}_{m2}(s) & \dots & \hat{r}_{mm}(s) \end{bmatrix} \tag{2.6b}$$

$$\hat{\mathbf{P}}(s)=\begin{bmatrix} \hat{p}_{11}(s) & 0 & \dots & 0 \\ \hat{p}_{21}(s) & \hat{p}_{22}(s) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \hat{p}_{m1}(s) & \hat{p}_{m2}(s) & \dots & \hat{p}_{mm}(s) \end{bmatrix} \tag{2.6c}$$

The representation (2.6) is not unique and our study will include the following questions:

(i) What is the relation between $\hat{\mathbf{R}}(s)$ and $\mathbf{R}(s)$, and between $\hat{\mathbf{P}}(s)$ and $\mathbf{P}(s)$?
(ii) Are there other common (invariant) properties than those given by equation (2.4)?
(iii) Is it possible to have $\hat{\mathbf{P}}(s)$ in column or row proper form?

The following lemmas are needed to resolve questions:

Lemma 1

"If $\mathbf{P}(s)$ is an $m\times m$ column (row) proper invertible polynomial matrix, with column (row) indices ∂c_j, $j=1,2,\dots,m\,(\partial\bar{r}_i, i=1,2,\dots,m)$, then its adjugate matrix $\mathbf{B}(s)$ is row (column) proper polynomial matrix with row (column) indices $\partial r_i(\partial\bar{c}_j)$ given by

$$\partial r_i=n-\partial c_i,\ \forall_i \qquad (\partial\bar{c}_j=n-\partial\bar{r}_j,\ \forall_j),$$
$$n=\sum_{j=1}^{m}\partial c_j \qquad \left(n=\sum_{i=1}^{m}\partial\bar{r}_i\right).$$

Lemma 2

"If $\mathbf{T}(s)=\mathbf{R}(s)\,\mathbf{P}^{-1}(s)$ is an $p\times m$ transfer function matrix with $\mathbf{P}(s)$ $m\times m$ invertible, column proper polynomial matrix and $\partial c_j[\mathbf{R}(s)]\leqq\partial c_j[\mathbf{P}(s)],\forall j$, then $\mathbf{T}(s)$ is proper."

Lemma 3 (Wolovich and Falb [8])

"For any $p\times m$ proper rational transfer function matrix $\mathbf{T}(s)$, there is a unique, nonsingular $p\times p$ lower-left triangular matrix $\xi_{\mathbf{T}(s)}$ of the form,

$$\xi_{\mathbf{T}}(s)=\mathbf{H}_{\mathbf{T}}(s)\langle s^{f_1},\dots,s^{f_p}\rangle$$

where,

$$\mathbf{H}_{\mathbf{T}}(s)=\begin{bmatrix} 1 & 0 & \dots & 0 \\ h_{21}(s) & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{p1}(s) & h_{p2}(s) & & 1 \end{bmatrix}$$

and $h_{ij}(s)$ is divisible by s (or zero) such that:

$\lim_{s\to\infty} \xi_{\mathbf{T}}(s)\,\mathbf{T}(s)=\mathbf{K}_{\mathbf{T}}$, with $\mathbf{K}_{\mathbf{T}}$ of full rank p."

Definition 2. The interactor $\xi_{\mathbf{T}}(s)$ and $\{f_i\}$ are called the interactor and the interactor indices respectively.

Lemma 4

"Let $\bar{\mathbf{R}}(s)$ and $\bar{\mathbf{P}}(s)$ be two $m\times m$, invertible polynomial matrices and $\lim_{s\to\infty} \bar{\mathbf{R}}(s)\,\bar{\mathbf{P}}^{-1}=\bar{\mathbf{K}}$ with $\bar{\mathbf{K}}$ nonsingular. Then $\bar{\mathbf{R}}(s)$ is column proper and has the same column indices as $\bar{\mathbf{P}}(s)$ if and only if $\bar{\mathbf{P}}(s)$ is column proper."

Proof

To prove the "if" statement, assume that $\mathbf{P}(s)$ is column proper and ∂c_j is the j-*th* column index. The inverse of $\mathbf{P}(s)$ is:

$$\bar{\mathbf{P}}^{-1}(s)=\mathbf{B}(s)\div\Delta_{\bar{p}}s \tag{2.7}$$

where $\Delta_{\bar{p}}(s)=|\bar{\mathbf{P}}(s)|$ and of degree $n=\sum_{j=1}^{m}\partial c_j$, $\mathbf{B}(s)$ is the adjugate of $\bar{\mathbf{P}}(s)$. By using *Lemma 1*, $\mathbf{B}(s)$ can be written in the form:

$$\mathbf{B}(s)=\tilde{\mathbf{S}}_h(s)\mathbf{B}_h+\tilde{\mathbf{S}}_{h-1}(s)\mathbf{B}_{h-1}+\ldots+\mathbf{B}_0 \tag{2.8}$$

where $h=\max\{\partial r_i\}$, and (2.9a)

$$\tilde{\mathbf{S}}_{h-i}(s)=\langle s^{\partial r_1-i},\ldots,s^{\partial r_m-i}\rangle \tag{2.9b}$$

and $\mathbf{B}_h$ is a nonsingular matrix, and the elements in the $\mathbf{B}_i$ corresponding to the negative power of s are zero.

Now, express $\bar{\mathbf{R}}(s)$ according to its column degrees $\partial\bar{c}_j$,

$$\bar{\mathbf{R}}(s)=\bar{\mathbf{R}}_l\bar{\mathbf{S}}_l(s)+\ldots+\bar{\mathbf{R}}_0 \tag{2.10}$$

where $\bar{\mathbf{S}}_{l-i}(s)=\langle s^{\partial\bar{c}_1-i},\ldots,s^{\partial\bar{c}_m-i}\rangle$

$$\text{Since, }\lim_{s\to\infty}\bar{\mathbf{R}}(s)\bar{\mathbf{P}}^{-1}(s)=\bar{\mathbf{K}}\ (\bar{\mathbf{K}}\text{ nonsingular}), \tag{2.11}$$

$$\text{then, }\left(\lim_{s\to\infty}\bar{\mathbf{R}}(s)\bar{\mathbf{P}}^{-1}(s)\right)\mathbf{L}=\bar{\mathbf{K}}\mathbf{L}=\lim_{s\to\infty}(\bar{\mathbf{R}}(s)\bar{\mathbf{P}}^{-1}(s)\mathbf{L}), \tag{2.12}$$

for any nonsingular scalar matrix $\mathbf{L}$. By postmultiplying equation (2.11) by $\mathbf{B}_h^{-1}$ we have,

$$\lim_{s\to\infty}\frac{1}{\Delta\bar{p}(s)}\{\bar{\mathbf{R}}_l\bar{\mathbf{S}}_l(s)\tilde{\mathbf{S}}_h(s)+(\bar{\mathbf{R}}_{l-1}(s)\bar{\mathbf{S}}_{l-1}(s)\mathbf{B}_h+$$
$$+\bar{\mathbf{R}}_l\bar{\mathbf{S}}_h(s)\tilde{\mathbf{S}}_{h-1}(s)\mathbf{B}_{h-1})\mathbf{B}_h^{-1}+\ldots\}=\bar{\mathbf{K}}\mathbf{B}_h^{-1} \tag{2.13}$$

Let,

$$\mathbf{S}_{h+l-i}(s)=\langle s^{\partial\bar{c}_1+n-\partial c_1-i},\ldots,s^{\partial\bar{c}_m+n-\partial c_m-i}\rangle \tag{2.14}$$

Since the limit is finite, then

$$\lim_{s\to\infty}\left(\frac{1}{s^n}\bar{\mathbf{R}}_l\mathbf{S}_{h+l}(s)\right)=\bar{\mathbf{K}}\mathbf{B}_h^{-1} \tag{2.15}$$

From Sylvester's inequality, $\bar{\mathbf{R}}_l$ and $\dfrac{\mathbf{S}_{h+l}(s)}{s^n}$ are of rank m, i.e., $\bar{\mathbf{R}}_l$ is nonsingular and $\partial\bar{c}_j=\partial c_j$; $\forall_j$, and this means that $\bar{\mathbf{R}}(s)$ is column proper and has the same column indices as $\bar{\mathbf{P}}(s)$.

To prove the "only if" statement we can use the same procedure for the assumption that $\bar{\mathbf{R}}(s)$ is column proper to prove that $\bar{\mathbf{P}}(s)$ will also be column proper and has the same column indices as $\bar{\mathbf{R}}(s)$.

Q.E.D.

3. Triangularization: theorem and algorithm

In this section we will give the theoretical base for triangularization as *Theorem 2*, and an algorithm to construct a l.s.v.f. compensator for such a purpose. The following theorem will also be used.

Theorem 1

"For every right invertible plant transfer function matrix, the interactor is invariant under l.s.v.f. control law."

Proof

Consider a $p\times m$ O.L.S. as given by equation (2.1). The C.L.S. under l.s.v.f. is given by:

$$\mathbf{T}_{\mathrm{F,G}}(s)=\mathbf{R}(s)[\mathbf{P}(s)-\mathbf{F}(s)]^{-1}\mathbf{G} \tag{3.1}$$

with the same conditions given by equation (2.4). If $\xi(s)$ and $\bar{\xi}(s)$ are the interactors of $\mathbf{T}(s)$ and $\mathbf{T}_{\mathrm{F,G}}(s)$ respectively we want to prove that $\xi(s)=\bar{\xi}(s)$.

We begin first with the special case $p=m$. Let,

$$\xi(s)=\mathbf{H}(s)\langle s^{f_i}\rangle \tag{3.2a}$$

$$\bar{\xi}(s)=\bar{\mathbf{H}}(s)\,\langle s^{\bar{f}_i}\rangle \tag{3.2b}$$

From *Lemma 3* and *Lemma 4*, $\xi(s)\mathbf{R}(s)$ and $\bar{\xi}(s)\mathbf{R}(s)$ are column proper with,

$$\partial c_j[\xi(s)\mathbf{R}(s)]=\partial c_j[\mathbf{P}(s)] \tag{3.3a}$$

$$\partial c_j[\bar{\xi}(s)\mathbf{R}(s)]=\partial c_j[\mathbf{P}(s)-\mathbf{F}(s)]=\partial c_j[\mathbf{P}(s)] \tag{3.3b}$$

From *Lemmas* (4) and (2), it follows that

$$[\xi(s)\mathbf{R}(s)]\,[\bar{\xi}(s)\mathbf{R}(s)]^{-1}=\mathbf{H}(s)\,\langle s^{f_i-\bar{f}_i}\rangle\bar{\mathbf{H}}^{-1}(s), \tag{3.4a}$$
$$[\bar{\xi}(s)\mathbf{R}(s)]\,[\xi(s)\mathbf{R}(s)]^{-1}=\bar{\mathbf{H}}(s)\,\langle s^{\bar{f}_i-f_i}\rangle\mathbf{H}^{-1}(s), \tag{3.4b}$$

are both proper. It is easy to prove, by using equation (3.4) and the special form of $\mathbf{H}(s)$ and $\bar{\mathbf{H}}(s)$, that,

$$f_i=\bar{f}_i,\ \forall_i \tag{3.5}$$

Now, both $\mathbf{H}(s)$ and $\bar{\mathbf{H}}(s)$ are unimodular, lower-left, triangular matrices with diagonal entries "1". Hence $\mathbf{H}(s)=\bar{\mathbf{H}}(s)=\mathbf{U}(s)$ is unimodular, proper and satisfies $\lim_{s\to\infty}\mathbf{U}(s)=\mathbf{L}$, with $\mathbf{L}$ nonsingular. Since each $h_{ij}(s)$ and $\bar{h}_{ij}(s)$ are divisible by s (or zero), therefore $\mathbf{U}(s)=\mathbf{I}$ and

$$\mathbf{H}(s)=\bar{\mathbf{H}}(s) \tag{3.6}$$

It is evident from equations (3.5) and (3.6) that:

$$\xi(s)=\bar{\xi}(s) \tag{3.7}$$

If $p\neq m$, but $\mathbf{T}(s)$ is right invertible, then $\mathbf{R}(s)$ is of full rank p and there are "$m-p$" row vectors, $r_{p+1}(s), \ldots, r_m(s)$ (with polynomial entries) such that the extended numerator polynomial,

$$\mathbf{R}_e(s)=\begin{bmatrix}\mathbf{R}(s)\\ \cdots\\ r_{p+1}(s)\\ \vdots\\ r_m(s)\end{bmatrix} \tag{3.8}$$

is nonsingular and $\mathbf{T}_e(s)=\mathbf{R}_e(s)\mathbf{P}^{-1}(s)$ is proper. From *Lemma 3* there is a unique interactor $\xi_e(s)$ having the form,

$$\xi_e(s)=\begin{bmatrix}\xi(s) & \vdots & \mathbf{0}_{p,m-p}\\ \cdots\cdots & \vdots & \cdots\cdots\cdots\\ \mathbf{X}_1(s) & \vdots & \mathbf{X}_2(s)\\ m-p,p & \vdots & m-p,m-p\end{bmatrix} \tag{3.9}$$

such that $\lim_{s\to\infty}\xi_e(s)\mathbf{T}_e(s)=\mathbf{K}_e$. From the previous proof of the special case $p=m$, $\xi_e(s)$ is invariant under l.s.v.f. and hence $\xi(s)$ is also invariant.

Q.E.D.

Hint. This proof is an almost word by word repetition of the one used by Wolovich and Falb [8], to establish the uniqueness and existence of $\xi(s)$ for $p \neq m$.

Theorem 2

"Let, $\mathbf{T}(s) = \mathbf{R}(s)\mathbf{P}^{-1}(s)$ be a $p \times m$ right invertible plant transfer function matrix, $\mathbf{R}(s)$ and $\mathbf{P}(s)$ RRP, and $\hat{\mathbf{T}}(s)$ be a C.L.S., triangularized by the l.s.v.f. pair $\{\mathbf{F}(s), \mathbf{G}\}$. For all the MFRD possibilities of $\hat{\mathbf{T}}(s)$ as a triangular pair $\{\hat{\mathbf{R}}(s), \hat{\mathbf{P}}(s)\}$, the degrees of the diagonal entries of $\{\hat{\mathbf{R}}(s), \hat{\mathbf{P}}(s)\}$ denoted by $\{\hat{k}_i, \hat{d}_i\}$ are fixed and given by the relations:

$$\hat{k}_i = k_i,$$

$$\hat{d}_i = k_i + f_i, \qquad i = 1, \ldots, p$$

where, $\{k_i\} \equiv$ degrees of the diagonal entries of any lower-left, triangular form of $\mathbf{R}(s)$, and $\{f_i\}$ are the interactor indices of $\mathbf{T}(s)$."

Proof

We assume that $\mathbf{R}(s)$ is in lower-left, triangular form. It is well-known, Wolovich [7], that $\mathbf{R}(s)$ (in equation (2.3)) may be varied through l.s.v.f. within only a unimodular factor, i.e., $\partial|\mathbf{R}(s)|$ and the roots of $|\mathbf{R}(s)|$ are invariant under l.s.v.f. Hence the effect of a l.s.v.f. on $\mathbf{R}(s)$ can be described by some unimodular matrix $\mathbf{U}_j(s)$,

$$\hat{\mathbf{T}}(s) = \mathbf{R}(s)\,[\hat{\mathbf{P}}(s)\mathbf{U}_j^{-1}(s)]^{-1} \tag{3.10a}$$

$$= \mathbf{R}(s)\mathbf{U}_j(s)\hat{\mathbf{P}}^{-1}(s) \tag{3.10b}$$

$$= \hat{\mathbf{R}}(s)\hat{\mathbf{P}}^{-1}(s) \tag{3.10c}$$

Let $\hat{\mathbf{T}}(s) = \hat{\mathbf{R}}(s)\hat{\mathbf{P}}^{-1}(s)$ be a MFRD of $\hat{\mathbf{T}}(s)$ as a triangular pair $\{\hat{\mathbf{R}}(s), \hat{\mathbf{P}}(s)\}$, then

$$\hat{k}_i \triangleq \partial\{\hat{r}_{ii}(s)\} \tag{3.11a}$$

$$\hat{d}_i \triangleq \partial\{p_{ii}(s)\}. \tag{3.11b}$$

To show that $\{\hat{k}_i\}$ are fixed, suppose that $\{\tilde{\mathbf{R}}(s), \tilde{\mathbf{P}}(s)\}$ is another triangular MFRD of $\hat{\mathbf{T}}(s)$. This new pair can be obtained from $\{\hat{\mathbf{R}}(s), \hat{\mathbf{P}}(s)\}$ through $\hat{\mathbf{U}}(s)$. Since $\tilde{\mathbf{R}}(s)$ and $\hat{\mathbf{R}}(s)$ are triangular and $\tilde{\mathbf{U}}(s)$ is unimodular, then the diagonal elements of $\tilde{\mathbf{U}}(s)$ in its upper left $p \times p$ block are nonzero constants, and so neither the degrees of the diagonal elements $\{k_i\}$ of $\hat{\mathbf{R}}(s)$ nor those of $\hat{\mathbf{P}}(s)$ are changed. For the same reason but by using $\mathbf{U}_j(s)$ and equations (3.10b) and (3.10c) we get,

$$\hat{k}_i = k_i, \qquad i = 1, \ldots, p \tag{3.12}$$

On the other hand, by using *Theorem 1*, $\xi_{\mathbf{T}}(s)=\xi_{\hat{\mathbf{T}}}(s)$

$$\lim_{s\to\infty} \xi_{\mathbf{T}}(s)\hat{\mathbf{T}}(s)=\mathbf{K}_{\hat{\mathbf{T}}} \tag{3.13}$$

where $\mathbf{K}_{\hat{\mathbf{T}}}$ is of full rank p and has a lower-left triangular form. It follows from the form of $\hat{\mathbf{T}}(s)$ (equations (2.5) and (2.6)), and $\xi_{\mathbf{T}}(s)$ that:

$$\hat{d}_i=\hat{k}_i+f_i=k_i+f_i, \qquad i=1, \ldots, p \tag{3.14}$$

and since $\hat{p}_{ii}(s)=\alpha_i\tilde{p}_{ii}(s)$ (where α_i is some constant), then

$$\hat{d}_i\triangleq\partial\{\hat{p}_{ii}(s)\}=k_i+f_i, \qquad i=1, \ldots, p \tag{3.15}$$

Q.E.D.

Lemma 5

"Let $\bar{\mathbf{R}}(s)$ and $\bar{\mathbf{P}}(s)$ be two $m\times m$, lower-left, triangular invertible polynomial matrices, and $\lim\limits_{s\to\infty} \bar{\mathbf{R}}(s)\bar{\mathbf{P}}^{-1}(s)=\bar{\mathbf{K}}$; with $\bar{\mathbf{K}}$ nonsingular. Then $\bar{\mathbf{P}}(s)$ will be row proper if and only if $\bar{\mathbf{R}}(s)$ is row proper, and in this case the two polynomial matrices will have the same row indices."

Proof

We shall prove only the "if" statement, and the "only if" one may be proved in a similar way. Assume that $\bar{\mathbf{R}}(s)$ is row proper with row indices $\partial\bar{r}_i$. By expressing $\bar{\mathbf{R}}(s)$ and the adjugate of $\bar{\mathbf{P}}(s)$ according to their row degrees and column degrees respectively and by using the same procedure given before in the proof of *Lemma 4*, we have

$$\lim_{s\to\infty} \frac{\bar{\mathbf{S}}_l(s)\bar{\mathbf{R}}\mathbf{B}\tilde{\mathbf{S}}_h(s)}{s^n}=\bar{\mathbf{K}} \tag{3.16}$$

where

$$\bar{\mathbf{S}}_l(s)=\langle s^{\partial\bar{r}_i}\rangle \tag{3.17a}$$

$$\tilde{\mathbf{S}}_l(s)=\langle s^{\partial c_i}\rangle \tag{3.17b}$$

$$\bar{\mathbf{R}}_l=[\bar{r}_{ij}^{(l)}] \quad i,j=1,2, \ldots, m \tag{3.17c}$$

and

$$\mathbf{B}_h=[b_{ij}^{(h)}], \quad i,j=1,2, \ldots, m \tag{3.17d}$$

with $\mathbf{R}_l$ nonsingular.
Let

$$\mathbf{C}\triangleq\bar{\mathbf{R}}_l\mathbf{B}_h \tag{3.18}$$

then,

$$\lim_{s\to\infty}\frac{1}{s^n}\begin{bmatrix} c_{11}s^{\partial\bar{r}_1+\partial c_1} & 0 & \dots & 0 \\ c_{21}s^{\partial\bar{r}_2+\partial c_1} & c_{22}s^{\partial\bar{r}_2+\partial c_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ c_{m1}s^{\partial\bar{r}_m+\partial c_1} & c_{m2}s^{\partial\bar{r}_m+\partial c_2} & \dots & c_{mm}s^{\partial\bar{r}_m+\partial c_m} \end{bmatrix}=\bar{\mathbf{K}} \tag{3.19}$$

Since the limit is finite and $\bar{\mathbf{K}}$ is nonsingular, then

$$\partial c_j = n - \partial\bar{r}_j, \forall_j, \quad \text{and} \tag{3.20}$$

$$c_{jj} \neq 0, \forall_j \tag{3.21}$$

Since $c_{jj}=\bar{r}_{jj}^{(l)}b_{jj}^{(h)}$ and $\bar{r}_{jj}^{(l)}\neq 0$, $\forall_j$ (from nonsingularity of $\bar{\mathbf{R}}_l$), then

$$b_{jj}^{(h)} \neq 0, \forall_j \tag{3.22}$$

Hence $\mathbf{B}_h$ is nonsingular and the column indices of the adjugate of $\bar{\mathbf{P}}(s)$ are given by equation (3.20), i.e., $\mathbf{B}(s)$ is row proper with row indices equal to $\partial\bar{r}_i$, $\forall_i$ *(Lemma 1)*

Q.E.D.

Corollary 1

"Let $\mathbf{T}(s)=\mathbf{R}(s)\mathbf{P}^{-1}(s)$ be an invertible plant transfer function matrix, $\mathbf{R}(s)$ and $\mathbf{P}(s)$ are RRP, and suppose that it is triangularized through l.s.v.f. alone and the C.L.S. $\hat{\mathbf{T}}(s)$ is described by the triangular pair $\{\hat{\mathbf{R}}(s)\ \hat{\mathbf{P}}(s)\}$. Then the degree of the i-*th* row of $\xi_{\mathbf{T}}(s)\ \hat{\mathbf{R}}(s)$ is equal to the degree of the i-*th* diagonal element in $\hat{\mathbf{P}}(s)$."

In light of the previous analysis, we can construct now the following algorithm to achieve the triangular form of an invertible system, through l.s.v.f. alone.

Algorithm 1

Step 1 Find the interactor $\xi_{\mathbf{T}}(s)$ and the interactor indices $\{f_i\}$, of the uncompensated system $\mathbf{T}(s)$.

Step 2 Find the unimodulator matrix $\mathbf{U}(s)$ which reduces $\mathbf{R}(s)$ into a row proper, lower-left, triangular form. Denote $\hat{\mathbf{R}}(s)=\mathbf{R}(s)\mathbf{U}(s)$.

Find $\quad \hat{k}_i \triangleq \partial[\hat{r}_{ii}(s)], \quad i=1,2,\dots,m.$

Step 3 The degrees of the diagonal entries of the desired triangular denominator polynomial matrix, $\hat{\mathbf{P}}(s)$ are given by: $d_i \triangleq \partial[\hat{p}_{ii}(s)] = = f_i+\hat{k}_i$. Choose any arbitrary polynomial (representing the characteristic equation of the compensated system), with those degrees.

Step 4 Put arbitrary polynomials on the lower, off-diagonal entries of $\hat{\mathbf{P}}(s)$, with degrees constrained by the row indices of $\bar{\mathbf{R}}(s) \triangleq \xi_{\mathbf{T}}(s)\hat{\mathbf{R}}(s)$ in the sense of *Lemma 5* and *Corollary 1*. The coefficients of those polynomials are chosen such that,

$$\partial c_j[\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)] = \partial c_j[\mathbf{P}(s)], \qquad j=1, \ldots, m \tag{3.23}$$

and

$$|\mathbf{\Gamma}_c[\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)| \neq 0 \tag{3.24}$$

Step 5 Find $\mathbf{G}$ and $\mathbf{F}(s)$ from the relations:

$$\mathbf{G\Gamma}_c[\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)] = \Gamma_c[\mathbf{P}(s)] \qquad \text{and}$$

$$\mathbf{F}(s) = \mathbf{P}(s) - \mathbf{G}\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s). \qquad \text{STOP.}$$

Example 1

Consider the following unstable system

$$\mathbf{T}(s) = \begin{bmatrix} 1 & \dfrac{-3}{s^3+2s+3} \\ \dfrac{1}{s^2+1} & \dfrac{2s^2-s-1}{(s^2+1)(s^3+2s+3)} \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} s^2+1 & s \\ 1 & 2 \end{bmatrix}}_{\mathbf{R}(s)} \underbrace{\begin{bmatrix} s^2+1 & s+3 \\ 0 & s^3+2s+3 \end{bmatrix}^{-1}}_{\mathbf{P}^{-1}(s)}$$

Step 1

$$\xi_{\mathbf{T}}(s) = \begin{bmatrix} 1 & 0 \\ -s & s^3 \end{bmatrix}, \quad f_1 = 0 \quad \text{and} \quad f_2 = 3$$

Step 2

$$\mathbf{U}(s) = \begin{bmatrix} 1 & -s \\ -s & s^2+1 \end{bmatrix}, \qquad \hat{\mathbf{R}}(s) = \begin{bmatrix} 1 & 0 \\ 1-2s & 2s^2-s+2 \end{bmatrix},$$

$$\hat{k}_1 = 0 \quad \text{and} \quad \hat{k}_2 = 2$$

Step 3

$$d_1 = 0 \quad \text{and} \quad d_2 = 5$$

$$\hat{\mathbf{P}}(s) = \begin{bmatrix} 1 & 0 \\ \hat{p}_{21}(s) & s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 \end{bmatrix}$$

It is obvious that all the five poles (closed loop poles) can be arbitrarily chosen, e.g., at: -1, $-1 \pm j$, -2 and -2, i.e.,

$$\hat{p}_{22}(s) = s^5 + 7s^4 + 20s^3 + 30s^2 + 24s + 8$$

Step 4 $\hat{p}_{21}(s)=x_0s^5+x_1s^4+x_2s^3+x_3s^2+x_4s+x_5$

The values of x_i satisfying equations (3.23) and (3.24) are: $x_0=0, x_1=-1$, $x_2=-7$, $x_3=-19$, $x_4=-23$ and x_5 arbitrary. For simplicity set $x_5=0$.

$$\hat{\mathbf{P}}_{\mathbf{F},\mathbf{G}}(s)=\mathbf{G}^{-1}[\mathbf{P}(s)-\mathbf{F}(s)]=\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)=$$

$$=\begin{bmatrix} s^2+1 & s \\ 5s^2-15s & s^3+7s^2+24s+8 \end{bmatrix}$$

Step 5

$$\mathbf{G}=\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{F}(s)=\begin{bmatrix} 0 & 3 \\ 15s+5 & -7s^2-17s-5 \end{bmatrix}$$

and under this l.s.v.f. pair, the c.l.s. will be

$$\hat{\mathbf{T}}(s)=\hat{\mathbf{R}}(s)\hat{\mathbf{P}}^{-1}(s)=\begin{bmatrix} 1 & 0 \\ \dfrac{s^3-3s^2+54s+8}{(s+1)(s^2+2s+2)(s+2)^2} & \dfrac{2s^2-s+2}{(s+1)(s^2+2s+2)(s+2)^2} \end{bmatrix}$$

Remarks:

(i) The system given in the previous example has weak inherent coupling, so a precompensator; in addition to the l.s.v.f. is needed for diagonal (exact) decoupling. Here, we achieved triangular decoupling through l.s.v.f. alone.

(ii) Even the diagonal decoupling condition is achieved through a precompensator, the invertible decoupling algorithm (Wolovich [7]), will yield unstable-unobservable modes corresponding to the roots of $|\mathbf{R}(s)|=2s^2-s+2$, and this problem disappears in the triangular decoupling case.

We can extend the previous results to the more general case $p\neq m$, by using one of the following two methods:

(a) Make an extension to $\mathbf{R}(s)$ by adding $(m-p)$, linearly independent rows, $r_{p+1}(s), \ldots, r_m(s)$ to form $\mathbf{R}_e(s)$. Those rows are chosen such that:

(i) $\partial c_j[\mathbf{R}_e(s)]\leq \partial c_j[\mathbf{P}(s)]; \forall_j$,

(ii) $[\mathbf{R}_e(s)\mathbf{U}_\mathbf{R}(s)]$ is lower-left, triangular, where $\mathbf{U}_\mathbf{R}(s)$ is an $m\times m$ unimodular matrix, which reduces $\mathbf{R}(s)$ to the lower-left triangular form,

(iii) the product of each added "$p+i$"-*th* row of $\mathbf{R}_e(s)$ and the corresponding "$p+i$"-*th* column of $\mathbf{U}_\mathbf{R}(s)$ is any desired stable polynomial of largest possible degree, consistent with the other two conditions (Wolovich [7]). Then we use Algorithm 1.

(b) We can use the same analysis described before, since *Theorem 1* has a general form ($p\neq m$). The differences are:

(i) The last "$m-p$" columns of $\hat{\mathbf{R}}(s)$ will be zero,

(ii) $d_i=\hat{k}_i+f_i$, $i=1;2;\ldots,p_m$, and the other "$m-p$" indices, $d_{p+1},\ldots,d_m$ are chosen such that $\sum_{i=1}^{m} d_i=n$,

(iii) there is no guarantee of row properness of $\hat{\mathbf{P}}(s)$ (*Lemma 5* cannot be applied in *Step 4* of Algorithm 1), and hence trial and error may be used to estimate the maximum degrees of the lower, off-diagonal polynomials of $\hat{\mathbf{P}}(s)$.

Example 2

$$\mathbf{T}(s)=\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} & \frac{1}{s+3} \\ 0 & \frac{1}{s+3} & 1 \end{bmatrix}=$$

$$=\underbrace{\begin{bmatrix} 1 & s+3 & 1 \\ 0 & s+2 & s+3 \end{bmatrix}}_{\mathbf{R}(s)}\underbrace{\begin{bmatrix} s+1 & 0 & 0 \\ 0 & (s+2)(s+3) & 0 \\ 0 & 0 & (s+3) \end{bmatrix}^{-1}}_{\mathbf{P}^{-1}(s)},$$

$$n=\sum_{j=1}^{3}\partial c_j=4$$

Step 1

$$\xi_{\mathbf{T}}(s)=\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix},\quad f_1=1 \quad \text{and} \quad f_2=0$$

Step 2

$$\mathbf{U}(s)=\begin{bmatrix} 1 & s+2 & -s^2-5s-7 \\ 0 & -1 & s+3 \\ 0 & 1 & -(s+2) \end{bmatrix},\quad \hat{\mathbf{R}}(s)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{d}_1=1,\quad \hat{d}_2=0 \quad \text{and} \quad \hat{d}_3=3$$

Step 3

$$\hat{\mathbf{P}}(s)=\begin{bmatrix} s+a_1 & 0 & 0 \\ \hat{p}_{21}(s) & 1 & 0 \\ \hat{p}_{31}(s) & p_{32}(s) & s^3+b_1s^2+b_2s+b_3 \end{bmatrix}$$

If it is desired to assign the c.l.s. poles at -1; then $a_1=1$. The three disappeared modes can also be chosen arbitrarily, e.g., at -2, -3, and -3, i.e., $b_1=8$, $b_2=21$ and $b_3=18$.

Step 4

$$\mathbf{U}^{-1}(s)=\begin{bmatrix} 1 & s+3 & -1 \\ 0 & s+2 & s+3 \\ 0 & 1 & 1 \end{bmatrix},$$

$$\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)=\begin{bmatrix} s+1 & (s+1)(s+3) & -(s+1) \\ \hat{p}_{21}(s) & \hat{p}_{21}(s)(s+3)+(s+2) & -\hat{p}_{21}(s)+(s+3) \\ \hat{p}_{31}(s) & \hat{p}_{31}(s)(s+3)+p_{32}(s)(s+2)+\hat{p}_{33}(s) & -\hat{p}_{31}(s)+\hat{p}_{32}(s)(s+3)+\hat{p}_{33}(s) \end{bmatrix}$$

Si..ce $\partial c_1=1$, $\partial c_2=2$ and $\partial c_3=1$, then:

$$\partial(\hat{p}_{21}(s))\leqq 1,\ \partial(\hat{p}_{31}(s))\leqq 1 \text{ and } \partial(\hat{p}_{32}(s))=2.$$

We have a big freedom of choice, e.g., $\hat{p}_{21}(s)=s$, $\hat{p}_{31}(s)=0$, $\hat{p}_{32}(s)=-s^2-5s$ satisfying equations (3.23) and (3.24).

Step 5

$$\mathbf{G}=\begin{bmatrix} -6 & 7 & -1 \\ 6 & 6 & 1 \\ -1 & -1 & 0 \end{bmatrix} \text{and} \quad \mathbf{F}(s)=\begin{bmatrix} 7 & 22+7s & -9 \\ -6 & -18-6s & 6 \\ 1 & 1 & -1 \end{bmatrix}$$

and under this l.s.v.f. pair the C.L.S. will be:

$$\hat{\mathbf{T}}(s)=\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ -\frac{s}{s+1} & 1 & 0 \end{bmatrix}$$

4. The use of triangular decoupling for diagonalization of systems having unobservable-unstable modes

It is known (Wolovich [7]) that, if the numerator polynomial matrix $\mathbf{R}(s)$ of the O.L.S, as given by equation (2.1) is factorized as:

$$\mathbf{R}(s)=\mathbf{R}_{\mathbf{d}}(s)\mathbf{R}_0(s) \tag{4.1}$$

where $\mathbf{R}_{\mathbf{d}}(s)$ is a nonsingular diagonal polynomial matrix, with each diagonal element $r_{d_i}(s)$ equal to the gcd of the corresponding i-*th* row of $\mathbf{R}(s)$, then the roots of $|\mathbf{R}_0(s)|$ represent the only fixed poles of an invertible system decoupled by combined l.s.v.f. and input dynamics, and appear as cancellation pole-zero terms, or unobservable modes. Wolovich [7], suggested the use of multi-stages compensator to overcome this difficulty. In this section we will use the triangular decoupling concept to give a new strategy for attacking this problem, but first we outline Wolovich's suggestion in the following finite steps.

Algorithm 2

Step 1 Design an input dynamic $\mathbf{P}_{c1}(s)$ (as shown in *Fig. 1.a*) such that

$$\mathbf{P}_1(s)=\mathbf{P}_{c_1}(s)\mathbf{P}(s) \tag{4.2}$$

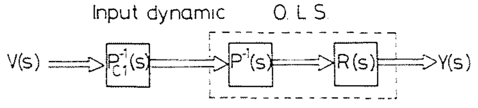

Fig. 1.a. The first-stage compensator

is a column proper polynomial matrix with equally chosen column degrees μ. These conditions may be satisfied through two substeps:
1.1) Compute a gain matrix $\mathbf{G}_1$ as

$$\mathbf{G}_1^{-1}=\mathbf{\Gamma}_c[\mathbf{P}(s)] \tag{4.3a}$$

1.2) Construct a diagonal, stable compensator $\bar{\mathbf{P}}_{c_1}^{-1}(s)$, where $\bar{\mathbf{P}}_{c_1}(s)$ consists of arbitrarily monic diagonal polynomials $\bar{p}_{c_1}^{(i)}$ of degree,

$$\partial\bar{c}_i=\mu-\partial c_i\,, \tag{4.3b}$$

$$\mu=\max\{\partial c_i\}\,, \tag{4.3c}$$

$$\mathbf{P}_{c_1}(s)\triangleq\bar{\mathbf{P}}_{c_1}(s)\mathbf{G}_1\,. \tag{4.4}$$

Step 2 Use the l.s.v.f. technique to design the system feedback compensator (see *Fig. 1.b*) such that the new denominator polynomial matrix $\mathbf{P}_2(s)$ will have the form,

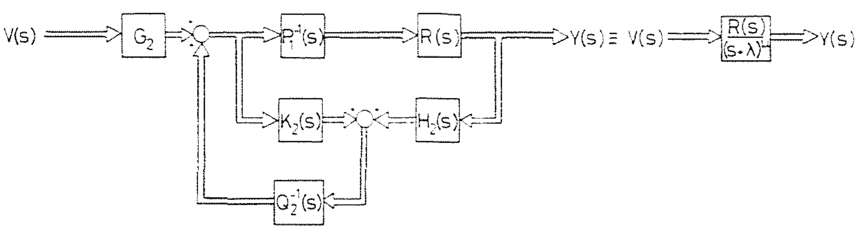

Fig. 1.b. The second-stage compensator

$$\mathbf{P}_2(s)=\langle(s+\lambda)^{\mu}\rangle \tag{4.5}$$

where λ represents the desired, equally chosen C.L.S poles.

Step 3 Use a second series precompensator $\mathbf{T}_c(s)$ of the form:

$$\mathbf{T}_c(s)=\bar{\mathbf{R}}^{+}(s)\mathbf{D}^{-1}(s) \tag{4.6}$$

where $\mathbf{R}^{+}(s)$ is an $m\times p$ polynomial matrix chosen such that,

$$\mathbf{R}(s)\mathbf{R}^{+}(s)=\Delta(s)\mathbf{I}_p, \tag{4.7}$$

$\mathbf{D}(s)$ is an $p \times p$ diagonal polynomial matrix, with stable arbitrary entries having degrees just enough to ensure that $\mathbf{R}^+(s)\mathbf{D}^{-1}(s)$ is proper (see *Fig. 1.c*).

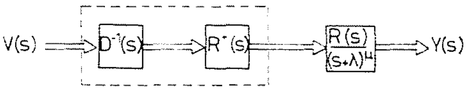

Fig. 1.c. The third-stage compensator

If $m=p$ (the invertible case), it is obvious that,

$$\mathbf{R}^+(s)=\text{adj } \mathbf{R}(s)\,, \tag{4.8a}$$

$$\Delta(s)=|\mathbf{R}(s)| \tag{4.8b}$$

For the general case ($p \neq m$), $\mathbf{R}^+(s)$ may be obtained in the following way:

3–1) Find $\mathbf{U_R}(s)$ the unimodulator matrix which reduces $\mathbf{R}(s)$ into the lower-left, triangular form,

$$\mathbf{R_R}(s)=\mathbf{R}(s)\,\mathbf{U_R}(s)=[\tilde{\mathbf{P}}_1(s)\;0_{p\times(m-p)}] \tag{4.9}$$

3–2) Let,

$$\tilde{\mathbf{R}}^+(s)=\text{adj } \tilde{\mathbf{R}}(s) \tag{4.10a}$$

$$\mathbf{R}^+(s)=\mathbf{U_R}(s)\begin{bmatrix}\mathbf{R}^+(s)\\ \cdots\\ 0_{(m-p),\,p}\end{bmatrix} \tag{4.10b}$$

and hence,

$$\Delta(s)=|\tilde{\mathbf{R}}(s)| \tag{4.10c}$$

STOP.

In our opinon better results can be obtained by using the triangularization technique as follows:

Algorithm 3

Step 1 Use Algorithm 1 to obtain the stabilized and triangular form $\hat{\mathbf{T}}(s)$.

Step 2 Omit the last $(m-p)$-zero columns of $\hat{\mathbf{T}}(s)$ and factorize the remainder, $p \times p$ submatrix $\hat{\mathbf{T}}_0(s)$ as:

$$\hat{\mathbf{T}}_0(s)=\mathbf{Q}^{-1}(s)\mathbf{L}(s) \tag{4.11}$$

where $\mathbf{Q}(s)$ is a $p \times p$, nonsingular, diagonal, polynomial matrix whose i-*th* entry $q_{ii}(s)$ represents the l.c.m. (*monic*) of the denominator polynomials in the i-*th* row of $\hat{\mathbf{T}}_0(s)$.

Factorize $\mathbf{L}(s)$ as:

$$\mathbf{L}(s)=\mathbf{L_d}(s)\hat{\mathbf{L}}(s) \tag{4.12}$$

where $\mathbf{L_d}(s)$ is the diagonal left divisor of $\mathbf{L}(s)$ of maximum row degree.

Step 3 Design a precompensator $\mathbf{T}_c(s)$ as follows. Let $\mathbf{B}(s) \triangleq \operatorname{adj} \hat{\mathbf{L}}(s)$. The precompensator description $\mathbf{T_c}(s)$ will be given by:

$$\mathbf{T}_c(s) = [\mathbf{B}(s)\mathbf{D}_c^{-1}(s) \;\; \mathbf{0}_{p \times (m-p)}] \tag{4.13}$$

where $\mathbf{D_c}(s)$ is a diagonal polynomial matrix of arbitrarily chosen nonzero entries such that $\mathbf{T_c}(s)$ is proper. A sufficient condition for this purpose is $\partial c_j[\mathbf{B}(s)] = \partial c_j[\mathbf{D_c}(s)]$, $j = 1, 2, \ldots, p$. The C.L.S. (diagonal form) will be:

$$\mathbf{T_d}(s) = \hat{\mathbf{T}}(s)\mathbf{T_c}(c) = |\hat{\mathbf{L}}(s)|\,\mathbf{L_d}(s)\,(\mathbf{D}_c(s)\mathbf{Q}(s))^{-1} \tag{4.14}$$

STOP

Example 3

$$\mathbf{T}(s) = \begin{bmatrix} \dfrac{s^2+1}{s^3} & \dfrac{s^3+s^2+1}{s^3} \\[2ex] \dfrac{s+3}{s^2} & \dfrac{2s^2+s+3}{s^2} \end{bmatrix} = \underbrace{\begin{bmatrix} s^2+1 & 1 \\ s^2+3s & 2 \end{bmatrix}}_{\mathbf{R}(s)} \underbrace{\begin{bmatrix} s^3 & -1 \\ 0 & 1 \end{bmatrix}^{-1}}_{\mathbf{P}^{-1}(s)}$$

It is evident that the use of Wolovich's invertible decoupling algorithm will yield unobservable-unstable modes at $s = 1$ and at $s = 2$.

Step 1

$$\mu = 3$$

$$\mathbf{P}_{c_1}(s) = \begin{bmatrix} 1 & 1 \\ 0 & s^3 + a_1^{(1)}s^2 + a_2^{(1)}s + a_3^{(1)} \end{bmatrix}$$

where $a_i^{(1)}$ are arbitrarily chosen,

$$\mathbf{P}_1(s) = \begin{bmatrix} s^3 & 0 \\ 0 & s^3 + a_1^{(1)}s^2 + a_2^{(1)}s + a_3^{(1)} \end{bmatrix}$$

Step 2

$$\mathbf{G} = \mathbf{I}_2,$$

$$\mathbf{F}(s) = \begin{bmatrix} -(3\lambda s^2 + 3\lambda^2 s + \lambda^3 & 0 \\ 0 & (a_1^{(1)} - 3\lambda)s^2 + (a_2^{(1)} - 3\lambda^2)s + (a_3 - \lambda^3) \end{bmatrix}$$

$$\mathbf{R}(s)\mathbf{P}_2^{-1}(s) = \frac{\mathbf{R}(s)}{(s+\lambda)^3}$$

Step 3

$$\mathbf{R}^{+}(s)=\begin{bmatrix} 2 & -1 \\ -(s^2+3s) & s^2+1 \end{bmatrix}, \quad \mathbf{D}(s)=\begin{bmatrix} s^2+d_1^{(1)}s+d_2^{(1)} & 0 \\ 0 & s^2+d_1^{(2)}s+d_2^{(2)} \end{bmatrix}$$

where $d_i^{(1)}$ and $d_i^{(2)}$ are arbitrarily chosen.

The diagonal system will be:

$$\mathbf{T_d}(s)=\begin{bmatrix} \dfrac{(s-1)(s-2)}{(s+\lambda)^3(s^2+d_1^{(1)}s+d_2^{(1)})} & 0 \\ 0 & \dfrac{(s-1)(s-2)}{(s+\lambda)^3(s^2+d_1^{(2)}s+d_2^{(2)})} \end{bmatrix}$$

By using Algorithm 3

Step 1

$$\xi_{\mathbf{T}}(s)=\begin{bmatrix} 1 & 0 \\ -2s & s \end{bmatrix}, \quad f_1=0, \quad \text{and} \quad f_2=1$$

$$\mathbf{U}(s)=\begin{bmatrix} 0 & 1 \\ 1 & -1-s^2 \end{bmatrix}, \quad \mathbf{U}^{-1}(s)=\begin{bmatrix} 1+s^2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{\mathbf{R}}(s)=\mathbf{R}(s)\,\mathbf{U}(s)=\begin{bmatrix} 1 & 0 \\ 2 & -2+3s-s^2 \end{bmatrix}$$

$$\hat{k}_1=0, \quad \hat{k}_2=2, \quad \hat{d}_1=0, \quad \text{and} \quad \hat{d}_2=3$$

$$\hat{\mathbf{P}}(s)=\begin{bmatrix} 1 & 0 \\ \hat{p}_{2,1}(s) & s^3+a_1s^2+a_2s+a_3 \end{bmatrix}, \ \Delta(s)\triangleq s^3+a_1s^2+a_2s+a_3$$

$$\hat{\mathbf{P}}(s)\,\mathbf{U}^{-1}(s)=\begin{bmatrix} 1 & 0 \\ \hat{p}_{21}(s) & \Delta(s) \end{bmatrix} \begin{bmatrix} 1+s^2 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $\partial c_2[\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)]=0$, then $\partial\{\hat{p}_{21}(s)\}=0$. Let $\hat{p}_{21}(s)=\alpha$, $\alpha\neq 0$ (from equations (3.23) and (3.24)), then

$$\hat{\mathbf{P}}(s)\mathbf{U}^{-1}(s)=\begin{bmatrix} 1+s^2 & 1 \\ \alpha(1+s^2)+\Delta(s) & \alpha \end{bmatrix},$$

$$\hat{\mathbf{T}}(s)=\hat{\mathbf{R}}(s)\hat{\mathbf{P}}^{-1}(s)=\begin{bmatrix} 1 & 0 \\ \dfrac{2\Delta(s)+\alpha(2-3s+s^2)}{\Delta(s)} & \dfrac{-2+3s-s^2}{\Delta(s)} \end{bmatrix}$$

For $\alpha=1$ (e.g.)

$$\mathbf{G}=\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix},$$

$$\mathbf{F}(s)=\begin{bmatrix}-(a_3-1)-a_2s-(a_1-1)s^2 & 0\\ -(1+s^2) & 0\end{bmatrix}$$

Step 2

$$\hat{\mathbf{T}}_0(s)=\hat{\mathbf{T}}(s),$$

$$\mathbf{Q}(s)=\begin{bmatrix}1 & 0\\ 0 & \Delta(s)\end{bmatrix},\quad \hat{\mathbf{L}}(s)=\mathbf{L}(s)=\begin{bmatrix}1 & 0\\ 2\Delta(s)+(2-3s+s^2) & -2+3s-s^2\end{bmatrix}$$

Step 3

$$\mathbf{B}(s)=\begin{bmatrix}-2+3s-s^2 & 0\\ -2\Delta(s)-(2-3s+s^2) & 1\end{bmatrix},$$

$$\mathbf{D}_c(s)=\begin{bmatrix}d_3+d_2s+d_1s^2+s^3 & 0\\ 0 & 1\end{bmatrix}$$

$$\mathbf{T}_c(s)=\begin{bmatrix}\dfrac{-(2-3s+s^2)}{d_3+d_2s+d_1s^2+s^3} & 0\\ \dfrac{-2\Delta(s)-(2-3s+s^2)}{d_3+d_2s+d_1s^2+s^3} & 1\end{bmatrix},$$

where d_i are arbitrarily chosen.

$$\mathbf{T}_d(s)=\begin{bmatrix}\dfrac{-2+3s-s^2}{d_3+d_2s+d_1s^2+s^3} & 0\\ 0 & \dfrac{-2+3s-s^2}{a_3+a_2s+a_1s^2+s^3}\end{bmatrix}$$

For this particular example, Algorithm 3 yields one precompensator of order “3” and l.s.v.f. of order “2”. The use of Algorithm 2, yields two precompensators of orders “3” and “4” respectively and l.s.v.f. of order “4”.

5. Conclusion

This paper deals with the triangular decoupling problem through l.s.v.f. by using the frequency-domain approach. We actually use the MFRD to prove our results, given as *Theorem 2*, *Lemma 3*, and *Corollary 1*. Those results enable us to learn the degrees of the diagonal entries of the compensated system, and accordingly to construct a concrete algorithm (Algorithm 1) for triangularization of any right-invertible system by using l.s.v.f. alone.

The triangular form is less simple than the diagonal one, but it has the following two advantages:

i — It is a less expensive technique, especially for systems having weak inherent coupling since we do not need an input dynamic in addition to the l.s.v.f.
ii — The problem of unstable — unobservable modes does not appear.

We make use of the latter advantage to develop a new technique, given as Algorithm 3, for the exact decoupling of systems having unstable-unobservable modes. Our algorithm has the following advantages over the only known suggestion dealing with this problem:

i — We use only one precompensator, and in turn a limited increase in the C.L.S. order.
ii — The l.s.v.f. order is less than or equal to that obtained by the old method.
iii — All the C.L.S. poles are arbitrarily chosen.

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