IS A TIME-INVARIANT LINEAR OPERATOR ALWAYS CONTINUOUS?

(REMARK TO THE MATHEMATICAL DESCRIPTION OF TIME-INVARIANT LINEAR SYSTEMS)

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Summary

A very important problem in linear system theory is the following: Is the operator of a BIBO-system a continuous operator of the corresponding normed space? The results of functional analysis can be applied in case of an affirmative answer only. It is proved for a time-invariant causal linear system that the operator of a BIBO-system is always continuous under general conditions (Proposition 3).

1. A linear system is called bounded input bounded output (BIBO) system if for any admissible bounded input, the response is bounded and it is called bounded energy system (or energy BIBO) if for any admissible input $x=x(t)$, the condition $\int_{-\infty}^{+\infty} x(t)^2 dt<\infty$ implies $\int_{-\infty}^{+\infty} y(t)^2 dt<\infty$, for the response $y=y(t)$.

More generally, a linear system is called *bounded in the norm* $\|.\|$ *(or BIBO)* if for *every* input $x=x(t)$ with $\|x\|<\infty$ the response is such that $\|y\|<\infty$.

Linear operator is the concept in mathematics corresponding to linear input-output system. The operator corresponding to a linear system is the rule which describes the relation between input and output signals. If y is the (unique) response for the input x, then the operator T of the system is

$$Tx=y \quad \text{(for every admissible } x\text{)}$$

and the system is BIBO if T maps the Banach space X, generated from the input signals x with $\|x\|<\infty$, into itself.

Remark

If $Tx \in X$, then by definition $\|Tx\|<\infty$ and hence the system corresponding to T is BIBO.

Shortly, a linear system is BIBO means exactly that the operator T corresponding to the system is an *(everywhere defined)* linear operator of a *Banach space* X.

A system is called time-invariant if for any admissible input $x=x(t)$, $[U_\tau x](t):=x(t-\tau)$ is also admissible and the response for $U_\tau x$, the shifted x, is the shifted y namely $U_\tau y$. In mathematical expressions, a system is *time-invariant* if

a. $U_\tau x \in X$ for every $x \in X$ and $U_\tau y \in Y$ for every $y \in Y$ (where τ is any positive number):

b. $U_\tau T = T U_\tau$

in this case both X, Y and T are called *translation-*, or *shift-invariant.*

The mathematical description of time-invariant linear system is the theory of convolution operators where T is considered as *continuous* shift-invariant linear operator. Recall, that a linear operator T of a Banach space X is continuous if there exists a positive number $\|T\|$ such that

$$\|Tx\| \leqq \|T\| \, \|x\| \qquad x \in X$$

the geometrical meaning of which is that the copy of the unit sphere $\{x: \|x\| \leqq 1\}$ belongs to the sphere with center Θ and radii $\|T\|$: $\{Tx: \|Tx\| \leqq \leqq \|T\|\}$. For this reason, continuous linear operators are often called *bounded* linear operator.

Obviously, if T is continuous, then the corresponding system is BIBO *but the converse is not necessarily true.* It may happen that $\|Tx\| < \infty$ if $\|x\| \leqq 1$, but walking all over the unit sphere with x, the $\|Tx\|$ should be arbitrary large. Thus, there is a serious gap between the physical and mathematical description of BIBO systems. In the mathematical model of time-invariant systems the theory of continuous linear operators is applied, however, the operator T corresponding to a BIBO system is not necessarily continuous.

The purpose of this paper is to fill the gap. It will be shown that under certain additional conditions, which a "real" system always satisfies, a *shift-invariant* linear T is continuous.

2. There is a more deeper motivation of the automatic continuity problems (e.g. which was raised in the previous section). The subject of the (linear) functional analysis is the bounded operators and the closed operators. If an operator is neither bounded nor closed, then functional analysis is of little use. Recall, that an operator T in a Banach space X is closed if from $x_n \in \mathscr{D}_T$ (where $\mathscr{D}_T$ is the domain of T), $x_n \to x$ and $Tx_n \to y$ it follows, that $x \in \mathscr{D}_T$ and $Tx = y$.

Obviously, every bounded operator T of X is closed, but there are important unbounded closed operators:

I. The differential operator $[Tx](t) := \frac{\mathrm{d}}{\mathrm{d}t} x(t)$ in $X = C[a, b]$ (with the uniform norm) is a typical example for for a closed unbounded operator.

II. Every self-adjoint differential operator is closed. In general, every self-adjoint operator in a Hilbert-space is closed.

However, the Closed Graph Theorem tells us that an *everywhere defined* closed operator T of a Banach space X is a bounded operator.

In the light of the Closed Graph Theorem, if the operator of a BIBO system is not bounded, then the results of functional analysis cannot be applied and it is enough to prove that the operator of a BIBO system is closed.

3. *The main theorem.* Let, X and Y be Banach spaces of functions with support on the half-line $\mathscr{P}=[0, \infty)$. In this case

$$[U_\tau x](t) = \begin{cases} x(t-\tau) & \text{if } t-\tau \in \mathscr{P} \\ 0 & \text{elsewhere.} \end{cases}$$

Moreover, T be a linear operator from X into Y and the following conditions are satisfied for X, Y and T:

a. X and Y are translation-invariant; i.e. if $x \in X$, then $U_\tau x \in X$ and if $y \in Y$, then $U_\tau y \in Y$ for every $\tau \in \mathscr{P}$.

b. T is translation-invariant; i.e. $U_\tau T = TU_\tau$ for every $\tau \in \mathscr{P}$.

c. U_τ for every $\tau \in \mathscr{P}$ is an *isometry*; i.e. $\|U_\tau x\| = \|x\|$ and $\|U_\tau y\| = \|y\|$ $x \in X$, $y \in Y$.

d. For the *truncation operator* P_α

$$\|P_\alpha x\| \leqq \|x\| \quad \text{and} \quad \|P_\alpha y\| \leqq \|y\| \quad x \in X, \quad y \in Y.$$

where $\alpha \in \mathscr{P}$ and

$$[P_\alpha x](t) = \begin{cases} x(t) & \text{if } \alpha - t \in \mathscr{P} \\ 0 & \text{elsewhere.} \end{cases}$$

Theorem

Under the above conditions the operator $P_\alpha T$ is continuous for any $\alpha \in \mathscr{P}$.

Remarks

I It is convenient that the input and output space is considered very different, however, for a "real" system $X = Y$.

II The norm-condition for the shift operator U_τ means that the property expressed by the norm is also time-invariant.

III The norm-condition for P_α means, that the norm (e.g. amplitude, energy, power) of the truncated signal is not greater than the original one.

IV It seems that the properties, expressed by the norm conditions for U_τ and P_α $(\tau, \alpha \in \mathscr{P})$, are satisfied by any "reasonable" system.

In proving the "Main Theorem" we need the following *commutation relation*

$$(*) \qquad P_\beta U_\tau = \begin{cases} U_\tau P_{\beta-\tau} & \text{if } \beta - \tau \in \mathscr{P} \\ 0 & \text{elsewhere.} \end{cases}$$

The commutation relation (*) is obvious, since truncating at β the function shifted by τ is the same as truncating at $\beta-\tau$ and *afterwards* shifted by τ. Moreover, if $\tau \geqq \beta$ then $P_\beta U_\tau = 0$.

The proof of the theorem

It will be proved, that *if $P_\alpha T$ is an unbounded operator for any $\alpha \in \mathcal{P}$, then T is not BIBO.*

If $P_\alpha T$ is unbounded for any $\alpha \in \mathcal{P}$, then a sequence $\{x_n; \|x_n\| < 2^{-n}\}$ can be constructed in an inductive manner as follows:

$$\|P_\alpha T x_1\| > 1$$

$$\|P_\alpha T x_2\| > 2 + \|T x_1\|$$

$$\|P_\alpha T x_n\| > n + \sum_{i=1}^{n-1} \|T x_i\|$$

Let N be an *arbitrary* positive number and

$$x_0 = \sum_{k=1}^{\infty} U_{n(k)} x_k$$

where $n(k) \in \mathcal{P}$ and $n(k+1) - n(k) > \alpha$ for $k = 1, 2, \ldots$ then $x_0 \in X$ and it will be shown that $\|T x_0\| > N$.

If $\beta = n(N) + \alpha$, then

$$n(N) < \beta < n(N+1)$$

and

$$\|T x_0\| \geqq \|P_\beta T x_0\| = \|P_\beta T\Big(\sum_{k=1}^{N-1} U_{n(k)} x_k\Big) + P_\beta T U_{n(N)} x_N +$$

$$+ P_\beta T\Big(\sum_{k=N+1}^{\infty} U_{n(k)} x_k\Big);$$

for the first member in the right side

$$\left\|P_\beta T\Big(\sum_{k=1}^{N-1} U_{n(k)} x_x\Big)\right\| = \left\|P_\beta \sum_{k=1}^{N-1} U_{n(k)} T x_k\right\| \leqq \sum_{k=1}^{N-1} \|T x_k\|;$$

for the third member in the right side

$$P_\beta T\left(\sum_{k=N+1}^{\infty} U_{n(k)}x_k\right)=P_\beta TU_{n(N+1)}x_{N+1}+$$

$$+P_\beta TU_{n(N+1)}\sum_{k=N+2}^{\infty} U_{n(k)-n(N+1)}x_k=$$

$$=P_\beta U_{n(N+1)}Tx_{N+1}+P_\beta U_{n(N+1)}T\left(\sum_{k=N+2}^{\infty}\dots\right)=0$$

since $P_\beta U_{n(N+1)}=0$, by the commutation relation $(*)$. Thus, it is obtained

$$(**)\qquad \|Tx_0\|\geqq\left\|P_\beta T\left(\sum_{k=1}^{N-1} U_{n(k)}x_k\right)+P_\beta TU_{n(N)}x_N\right\|\geqq$$

$$\geqq\|P_\beta TU_{n(N)}x_N\|-\sum_{k=1}^{N-1}\|Tx_k\|.$$

And now we arrived to the decisive part of the proof:

$$\|P_\beta TU_{n(N)}x_N\|=\|P_\beta U_{n(N)}Tx_N\|=\|U_{n(N)}P_{\beta-n(N)}Tx_N\|=$$

$$=\|U_{n(N)}P_\alpha Tx_N\|=\|P_\alpha Tx_N\|$$

hence, it follows from $(**)$ and the inductive hypothesis for $\{x_n\}$

$$\|Tx_0\|\geqq\|P_\alpha Tx_N\|-\sum_{i=1}^{N-1}\|Tx_i\|\geqq\left(N+\sum_{i=1}^{N-1}\|Tx_i\|\right)-\sum_{i=1}^{N-1}\|Tx_i\|=N$$

3. *Conclusions and further problems.* It follows from the foregoing theorem that if the operator T is composed with any truncation P_α then the result, $P_\alpha T$, will be a bounded operator for any reasonable T.

At least in two cases, the boundedness of $P_\alpha T$ for every $\alpha\in\mathscr{P}$ implies that T is bounded:

Proposition 1

If $\lim_{\alpha\to\infty} P_\alpha Tx=Tx$ for every $x\in X$, then T is bounded.

Proof

It follows from the Uniform Boundedness Theorem ([1] II. 1. 11.) or even more, from the Banach-Steinhaus Theorem ([1] II. 1. 18.) that T is bounded since $P_\alpha T$ is a bounded operator for every $\alpha\in\mathscr{P}$ and $\|P_\alpha Tx\|\leqq\|Tx\|$ for every $x\in X$.

Proposition 2

If T maps truncated function into truncated function, then T is bounded on the subspace of function, with compact support.

Proof

If Tx is truncated by $\beta \in \mathscr{P}$, then

$$P_\alpha Tx = Tx$$

for $\alpha > \beta$ and hence, the condition of Proposition 1 is satisfied *for any x with compact support*. (Every x with compact support can be considered as a truncated $x \in X$ and vice versa). If X_0 is the closure of functions in X with compact support, then $P_\alpha Tx \to Tx$ and $\|P_\alpha Tx\| \leqq \|Tx\|$ on a dense subspace of X_0 and the Banach-Steinhaus Theorem can be applied (see [1] II. 1. 18).

Now let X and Y be any translation invariant Banach space of time-functions. That is, *there is no restriction to the support* of the functions in X resp. Y. In this case it can be concluded from the Main Theorem, that the *causal* translation-invariant linear operators are bounded operators. More precisely:

Let X and Y be translation-invariant Banach space of functions on the line $(-\infty, +\infty)$ and T be a linear operator from X into Y.

Proposition 3. If

a. the functions with support bounded below are dense in X and T is causal (i.e. if $\text{supp}\, x \subseteq [t, \infty)$ then $\text{supp}\, Tx \subseteq [t, \infty)$);

b. the conditions b., c. and d. of the Main Theorem are satisfied; then the operator $P_\alpha T$ is bounded for every $\alpha > 0$.

Proof

If $P_\alpha T$ is unbounded on the dense subspace

$$X_b := \{x : x \in X;\ \text{supp}\, x \subset [t_0, \infty);\ t_0 > -\infty\}$$

then, as in the proof of the Main Theorem, a sequence $\{x_n; x_n \in X_b, \|x_n\| < 2^{-n}\}$ can be constructed in an inductive manner such that

$$\|P_\alpha Tx_n\| > n + \sum_{i=1}^{n-1} \|Tx_i\|.$$

Let N be an *arbitrary* positive number and now

$$x_0 = \sum_{k=1}^{\infty} U_{n(k)+t_k} x_k$$

where $t_k \geqq 0$ such that $\text{supp}\, x_k \subseteq [-t_k, \infty)$ and $n(k+1) + t_{k+1} - [n(k) + t_k] > \alpha$ for $k = 1, 2, \ldots$.

It follows that $x_0 \in X$ since U_τ is an isometry and $\|x_k\| < 2^{-k}$. It will be shown that $\|Tx_0\| > N$.

Similarly as in the proof of the Main Theorem, if $\beta = n(N) + t_N + \alpha$, then

$$n(N) + t_N < \beta < n(N+1) + t_{N+1}$$

and for the same reason

$$\left\| P_\beta T\left(\sum_{k=1}^{N-1} U_{n(k)+t_k} x_k\right)\right\| \leqq \sum_{k=1}^{N-1} \|Tx_k\|$$

$$P_\beta T\left(\sum_{k=N+1}^{\infty} U_{n(k)+t_k} x_k\right) = 0;$$

hence, again

$$(**) \qquad \|Tx_0\| \geqq \|P_\beta T U_{n(N)+t_N} x_N\| - \sum_{i=1}^{N-1} \|Tx_i\|$$

For the first member of the right side

$$\|P_\beta T U_{n(N)+t_N} x_N\| = \|U_{n(N)+t_N} P_{\beta-[n(N)+t_N]} Tx_N\| = \|P_\alpha Tx_N\|$$

hence, it follows from $(**)$ and the inductive hypothesis for $\{x_n\}$

$$\|Tx_0\| \geqq \|P_\alpha Tx_N\| - \sum_{i=1}^{N-1} \|Tx_i\| \geqq \left(N + \sum_{i=1}^{N-1} \|Tx_i\|\right) - \sum_{i=1}^{N-1} \|Tx_i\| = N.$$

Thus it is proved: (by the principle of indirect proof) if T is causal and BIBO, then $P_\alpha T$ is bounded on X_b.

As Proposition 1 or Proposition 2, it can be proved

Corollary

If $\lim_{\alpha\to\infty} P_\alpha Tx = Tx$ for every $x \in X_b$ then T is bounded on X_b. (And hence there is a unique bounded extension of T onto X.)

The linear space $\{x: x \in X \text{ supp } x \subset [t_0, \infty);\ t_0 > -\infty\}$ can be considered as the *inductive limit of the linear spaces* $X_t := \{x \in X: \text{supp } x \subseteq [t, \infty)\}$ with the norm inherited from the Banach-space X. Recall, a sequence $\{x_n\}$ is convergent in the inductive limit of the Banach spaces X_t if there is a fixed t such that $x_n \in X_t$ $(n = 1, 2, \ldots)$ and $\{x_n\}$ is convergent in the Banach space X_t.

Since T is causal if and only if $TX_t \subseteq X_t$ for every t and T is continuous in the inductive limit of the Banach spaces X_t if it is continuous in every X_t, it is more convenient to consider T as the operator of the inductive limit of Banach spaces $X_t := \{x \in X, \text{supp } x \subseteq [t, \infty)\}$ then to consider X as a Banach space itself.

In a forthcoming paper we shall deal with "automatic continuity" of causal shift-invariant linear operators in this more natural setting.

References

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