ON THE NUMERICAL SOLUTION OF RETARDED ORDINARY DIFFERENTIAL EQUATIONS BY A CLASS OF FORMULAE BASED ON SPLINE FUNCTION APPROXIMATION

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Summary

This paper presents a class of spline approximation method for the numerical solution of retarded ordinary differential equations (RODE). The proposed method is explicit, for convergence we need several restrictions and assumptions on the right side of the RODE. For other application of spline functions see [1], [2], [3]. A scheme for constructing methods of arbitrary ordei and numerical examples is given.

Introduction

In recent years there has been a growing interest for the numerical treatment of delay differential equations. This is due to the versatility of such equations in the mathematical modeling of processes in various application fields, where they provide the best and sometimes the only realistic simulation of observed phenomena. In difference-differential equations (DDF), or more generally, in functional differential equations (FDE) the past exerts its influence in a significant manner upon the future. For general treatment of FDE and for their history we could refer to the book of Hale [4]. Nevertheless the numerical methods which deal with this kind of equation were not developed so far. Cryer and Tavernini [5] studied the Cauchy problem for Volterra functional differential equations:

$$\begin{aligned} y'(t) &= F(y, t), \qquad t \in [a, b] \\ y(t) &= g(t), \qquad t \in [\alpha, a] \end{aligned} \tag{1.1}$$

Here, $F: C([\alpha, b] \to E^n)\, X[a, b] \to E^n$ is a Volterra functional, that is, $F(y, t)$ depends on t and on $y(s)$ for $s > t$; and the function $g \in C([\alpha, a] \to E^n)$ is specified initial function.

The problem (1.1) includes as special case the initial value problems for ordinary differential equations (ODE), retarded ordinary differential equations (RODE), and Volterra integro-differential equations (VIDE).

Numerical method for solving special cases of (1.1) has been considered by several authors, Feldstein and Goodman [6], Bleyer [7], Bleyer and Preuss [8], Tavernini [9], Kemper [10], Oberle and Pecsh [11].

Description of the method

We consider a RODE in the form

$$y'(x)=f(x, y(x), y(\alpha(x))), \qquad x\in[a, b] \tag{2.1}$$

with

$$y(a)=y_a \quad \text{and} \quad \alpha(a)=a \tag{2.2}$$

where $\alpha(x)\leqq x$, $y(x)$ is an unknown function. The function $\alpha(x)$ is usually called the "retardation" or "lag" function. Generalizations to multiple lags are carried on in the obvious way:

Let us denote $d^j y/dx^j$ by $y^{(j)}$; if $y^{(j)}$ is continuous at $x_i (x_i\in[a, b])$, we denote it by $y_i^{(j)}$. The notation $y_i^{(j)}$ will be for the left-hand limit of $y^{(j)}$ at x_i, when $y^{(j)}$ is discontinuous at x_i and sometimes, for convenience, even when $y^{(j)}$ is continuous there. The solution of the problem (2.1) with (2.2) is denoted by $v(x)$. $f^{(j)}$ stands for the higher derivatives of $y'(x)$.

Choosing $h^*>0$, define $a_0^*=a+h^*=x_0$, then we define the algorithm on $[a, a_0^*]$ by using Taylor's expansion as follows:

Let us start the algorithm by

$$y_a^{(0)}=y(a)=y_a \tag{2.3}$$

$$y_a^{(j)}=f^{(j-1)}(a, y_a^{(0)}, y_{\alpha a}^{(0)}), \quad 1\leqq j\leqq m \tag{2.4}$$

where m is fixed integer.

For $x\in(a, a_0^*]$, $y(x)$ is given by formula

$$y(x) = \sum_{k=0}^{m} \frac{1}{k!} y_a^{(k)} [x-a]^k \tag{2.5}$$

we define for $0\leqq j\leqq p$

$$y_0^{(j)} = \sum_{k=j}^{m} \frac{h^{*k-j}}{(k-j)!} y_a^{(k)} \tag{2.6}$$

where $0\leqq p\leqq m-1$, and for $p+1\leqq j\leqq m$

$$y_0^{(j)}=f^{(j-1)}(x_0, y_0^{(0)}, y_{\alpha_0}^{(0)}) \tag{2.7}$$

where $y_{\alpha_0}^{(0)}$ can be calculated from (2.5)

$$y(\alpha(x_0)) = \sum_{k=0}^{m} \frac{1}{k!} y_a^{(k)} [\alpha(x_0)-a]^k \tag{2.8}$$

and the higher derivatives

$$y^{(j)}(\alpha(x_0)) = \sum_{k=j}^{m} \frac{1}{(k-j)!} y_a^{(k)} [\alpha(x_0)-a]^{k-j} \tag{2.9}$$

we continue the algorithm in term of "mesh points" x_n and "non mesh points" $\alpha(x_n)$ defined as follows.

Let us Pick $h>0$ and integer N so that $Nh=b-a_0^*$, denote $x_n=a_0^*+nh$ for $n=0, 1, \ldots, N$. Let us define

$$a_i^* = \max\{xj/xj>a_{i-1}^*;\, \alpha(x_k)\leqq a_{i-1}^* \text{ for } k\leqq j\leqq N\} \tag{2.10}$$

It may happen that the procedure defining the numbers a_i^*, which are always mesh points, terminates before $a_i^*=x_N=b$. In this case let the last defined a_i^* be denoted by x_j and by definition let $a_{i+1}^*=x_{j+1}$. Since $\alpha(x_{j+1})\geqq a_i^*$ we could say that in the interval (x_j, x_{j+1}) we have no delay and $\alpha(x)\simeq x$ can be assumed. By assumption $\alpha(x)\leqq x$ on $[a, b]$, the sequence a_i^* is finite, the possible number of a_i^*-s is at most N.

The algorithm now will be defined first on $(a_0^*, a_1^*]$, then on $(a_1^*, a_2^*]$ and so on. If $x_n\in(a_0^*, a_1^*]$ then by (2.10) $x_n^*=\alpha(x_n)\leqq a_0^*$. Let $x_i\in(a_0^*, a_1^*]$ and define for $0\leqq j\leqq p$

$$y_i^{(j)} = \sum_{k=j}^{m} \frac{h^{k-j}}{(k-j)!} y_{(i-1)}^{(k)} \tag{2.11}$$

where $0\leqq p\leqq m-1$ is fixed. If $p+1\leqq j\leqq m$ then by (2.5) and (2.7)

$$y_i^{(j)}=f^{(j-1)}(x_i, y_i^{(0)}, y_{\alpha^i}^{(0)}) \tag{2.12}$$

is well — defined. For $x\in(a_0^*, a_1^*]$, $y(x)$ is given by formula

$$y(x) = \sum_{k=0}^{m} \frac{1}{k!} y_{(i-1)}^{(k)} [x-x_{i-1}]^k \tag{2.13}$$

where $x_{i-1}<x\leqq x_i$.

In the next step we move to the interval $(a_1^*, a_2^*]$, $a_2^*\leqq b$. If $x_l\in(a_1^*, a_2^*]$ then $\alpha(x_l)\in[a, a_1^*]$, if $\alpha(x_l)\in(a_0^*, a_1^*]$, then we need the approximation of $v(\alpha(x_l))$. $y_{\alpha^l}^{(0)}$ will be counted from (2.13)

$$y(\alpha(x_l))=y_{\alpha_l}^{(0)} = \sum_{k=0}^{m} \frac{1}{k!} y_{(i-1)}^{(k)} [\alpha(x_l)-x_{i-1}]^k \tag{2.14}$$

whenever $x_{i-1}<\alpha(x_l)\leqq x_i$. In the case $\alpha(x_l)\in(a, a_0^*]$ then $y_{\alpha_l}^{(0)}$ is given by (1.5) at $x=\alpha(x_l)$. Higher derivatives can be get similarly from j-th derivative of (2.13) at $x=\alpha(x_l)$ or respectively from the derivatives of (1.5) evaluated at $x=\alpha(x_l)$.

$$y^{(j)}(\alpha(x_l))=y_{\alpha_l}^{(j)} = \sum_{k=0}^{m} \frac{1}{(k-j)!} y_a^{(k)}[\alpha(x_l)-a]^{k-j} \tag{2.15}$$

and

$$y^{(j)}(\alpha(x_l)) = y_{\alpha_l}^{(j)} = \sum_{k=}^{m} \frac{1}{(k-j)!} y_{(i-1)}^{(j)} [\alpha(x_l) - x_{i-1}]^{k-j} \tag{2.16}$$

for $x_{i-1} < \alpha(x_l) \leqq x_i$ and for $0 \leqq j \leqq m$. Hence similarly to (2.11) and (2.12) for $0 \leqq j \leqq p$

$$y_l^{(j)} = \sum_{k=j}^{m} \frac{h^{k-j}}{(k-j)!} y_{l-1}^{(k)} \tag{2.17}$$

where for the smallest l, $x_{l-1} = a_1^*$,

$$y^{(j)} = f^{(j-1)}(x_l, y_l^{(0)}, y_{\alpha_l}^{(0)}) \tag{2.18}$$

when $p+1 \leqq j \leqq m$. By the assumption on $\alpha(x)$ the algorithm terminates in at most N steps, if we repeat the above method for $(a_2^*, a_3^*]$, ... Obviously (2.11) is the j-th derivative of (2.13) evaluated at x_i. Therefore $y(x) \in C^p[a, b]$ while the $(p+1)$-st through m-th derivatives of $y(x)$ generally have finite jumps at the mesh points, the values given by (2.12) (or for the next interval $(a_1^*, a_2^*]$, by (2.18)) being left-hand limits of these derivatives of $y(x)$.

Convergence

We begin this section with five lemmas which will prove useful in the following convergence (for to proves see [12]).

Lemma 1

If j, p and r are non-negative integers and h and α_l are real numbers, then

$$\sum_{k=j}^{p} \sum_{l=k}^{p} \frac{(rh)^{k-j}}{(k-j)!} \frac{h^{l-k}}{(1-k)!} \alpha_l = \sum_{k=j}^{p} \frac{((r+1)h)^{k-j}}{(k-j)!} \alpha_k .$$

Lemma 2

Let $\varepsilon_{j,i}$, $\eta_{j,i}$ and $\beta_{j,i}$ be real numbers which satisfy

$$\varepsilon_{j,i} = \sum_{k=j}^{p} \frac{h^{k-j}}{(k-j)!} \varepsilon_{k,i-1} + \sum_{k=p+1}^{m} \frac{h^{k-j}}{(k-j)!} \eta_{k,i-1} - \frac{h^{m-j+1}}{(m-j+1)!} \beta_{j,i-1}$$

for $0 \leqq j \leqq p$, $1 \leqq i \leqq N$. Then for any integer q such that $1 \leqq q \leqq i$

$$\varepsilon_{j,i} = \sum_{k=j}^{p} \frac{(qh)^{k-j}}{(k-j)!} \varepsilon_{k,i-q} + \sum_{r=1}^{q} \left[\sum_{k=j}^{p} \sum_{1=p+1}^{m} \frac{((r-1)h)^{k-j}}{(k-j)!} \frac{h^{1-k}}{(1-k)!} \eta_{1,i-r} \right] -$$

$$- \sum_{r=1}^{q} \left[\sum_{k=j}^{p} \frac{((r-1)h)^{k-j}}{(k-j)!} \frac{h^{m-k+1}}{(m-k+1)!} \beta_{k,i-r} \right]$$

for $0 \leqq j \leqq p$, $1 \leqq i \leqq n$

Lemma 3

Let j, p, l and r be non-negative integers, and let $h \geqq 0$ be real. If $l \leqq p \leqq j$ and $r \geqq 1$, then

$$\sum_{k=j}^{p} \frac{((r-1)h)^{k-j}}{(k-j)!} \frac{h^{l-k}}{(l-k)!} \leqq \frac{(2h)^{l-j}}{(l-j)!} r^{p-j}$$

Lemma 4

Let $q \geqq 0$ and $r \geqq 1$ be integers. Then

$$\sum_{r=1}^{i} r^q \leqq i^{q+1}$$

Lemma 5

Let α_i be real numbers and C and D be non-negative real numbers satisfying the recurrent inequality

$$|\alpha_i| \leqq D \sum_{k=0}^{i-1} |\alpha_k| + c$$

Then the following bound holds

$$|\alpha_i| \leqq C\, e^{iD}, \qquad i \geqq 0$$

Let us consider the errors on the interval $(a, a_0^*]$ $(a_0^* = x_0)$.

$$\varepsilon_{k,0} = y_0^{(k)} - v_0^{(k)}, \qquad 0 \leqq k \leqq p \tag{3.1}$$

$$\eta_{k,0} = y_0^{(k)} - v_0^{(k)}, \qquad p+1 \leqq k \leqq m \tag{3.2}$$

Between (a, a_0^*) we obtain by Taylor expansion

$$v_0^{(j)} = \sum_{k=j}^{m} \frac{h^{*k-j}}{(k-j)!} v_a^{(k)} + \frac{h^{*m-j+1}}{(m-j+1)!} v^{(m+1)}(\xi_{j,a}) \tag{3.3}$$

where $a < \xi_{j,a} < a_0$.
From (2.6) and (3.1) it follows that

$$\varepsilon_{j,0} = \sum_{k=j}^{p} \frac{h^{*k-j}}{(k-j)!} \varepsilon_{k,a} + \sum_{k=p+1}^{m} \frac{k^{k-j}}{(k-j)!} \eta_{k,a} \frac{h^{*m-j+1}}{(m-j+1)!} v^{(m+1)}(\xi_{j,a}) \tag{3.4}$$

for $0 \leqq j \leqq p$.
Since $\varepsilon_{k,a} = 0$ and $\eta_{k,a} = 0$, we get

$$\varepsilon_{j,0} = -\frac{h^{*m-j+1}}{(m-j+1)!} v^{(m+1)}(\xi_{j,a}) \tag{3.5}$$

Assuming

$$\sup_{[a,b]} |v^{(m+1)}| \leqq V_{m+1} \tag{3.6}$$

then

$$|\varepsilon_{j,0}| \leqq \frac{h^{*m-j+1}}{(m-j+1)!} V_{m+1} \tag{3.7}$$

For the continuous error bounds, by Taylor expansion of $y(x)$ and $v(x)$ on the right-side end of $(a, a_0^*]$ we can obtain for $0 \leqq j \leqq m$

$$\sup_{[a,a_0^*]} |y_{(x)}^{(j)} - v_{(x)}^{(j)}| \leqq \frac{h^{*m-j+1}}{(m-j+1)!} V_{m+1} \tag{3.8}$$

Assuming $0 \leqq j \leqq p$ and set

$$\delta_j(h_0) = \frac{h_0^{p-j+1}}{(m-j+1)!} V_{m+1} \tag{3.9}$$

$$\sup_{[a,a_0^*]} |y_{(x)}^{(j)} - v_{(x)}^{(j)}| \leqq \delta_j(h_0) h^{*m-p} \tag{3.10}$$

by taking $h^* \leqq h_0$.

For the case $p+1 \leqq j \leqq m$, we have

$$\delta_j(h_0) = \frac{V_{m+1}}{(m-j+1)!}.$$

Then for $p+1 \leqq j \leqq m$

$$\sup_{[a,a_0^*]} |y_{(x)}^{(j)} - v_{(x)}^{(j)}| \leqq \delta_j(h_0) h^{*m-j+1} \tag{3.11}$$

Now we have given the error bound for $\eta_{j,0}$ as follows

$$\eta_{j,0} = f^{(j-1)}(x_0, v_0 + \varepsilon_{0,0}, v_{z_0} + \varepsilon_{z,0}\delta_0(h_0)h^{*m-p}) - f^{(j-1)}(x_0, v_0, v_{z_0})$$

for $p+1 \leqq j \leqq m$, where $0 \leqq |\varepsilon_{z,0}| \leqq 1$.

Therefore by the virtue of the mean value theorem:

$$\eta_{j,0} = \varepsilon_{0,0} \frac{\partial}{\partial y} f^{(j-1)}(x_0, \beta_{j,0} v_0 + (1-\beta_{j,0}) y_0^{(0)}, y_{z_0}^{(0)} +$$

$$+ \varepsilon_{z_0,0} \delta_0(h_0) h^{*m-p} \frac{\delta}{\delta z} f^{(j-1)}(x_0, v_0, \gamma_{j,0} v_{z_0} + (1-\gamma_{j,0}) y_{z_0}^{(0)})$$

where $0 < \beta_{j,0}$ and $\gamma_{j,0} < 1$.

Using the following notation:

$$F_{j,0}^{\langle 1\rangle}=\frac{\partial}{\partial y}f^{(j-1)}(x_0,\beta_{j,0}v_0+(1-\beta_{j,0})y_0^{(0)},y_{\alpha_0}^{(0)})$$

$$F_{j,0}^{\langle 2\rangle}=\frac{\partial}{\partial z}f^{(j-1)}(x_0,v_0,\gamma_{j,0}v_{\alpha_0}+(1-\gamma_{j,0})y_{\alpha_0}) \tag{3.12}$$

we can write

$$\eta_{j,0}=\varepsilon_{0,0}F_{j,0}^{\langle 1\rangle}+\varepsilon_{\alpha_0}\delta_0(h_0)h^{*m-p}F_{j,0}^{\langle 2\rangle} \tag{3.13}$$

Assuming

$$\sup_D\left|\frac{\partial}{\partial y}f^j\right|\leqq M_j^{\langle 1\rangle},\ \sup_D\left|\frac{\partial}{\partial z}f^j\right|\leqq M_j^{\langle 2\rangle} \tag{3.14}$$

then by (3.13)

$$|\eta_{j,0}|\leqq|\varepsilon_{0,0}|M_{j-1}^{\langle 1\rangle}+\delta_0(h_0)h^{*m-p}M_{j-1}^{\langle 2\rangle}, \tag{3.15}$$

set $M_j=\max\{M_j^{\langle 1\rangle},M_j^{\langle 2\rangle}\}$.
Then for $p+1\leqq j\leqq m$

$$|\eta_{j,0}|\leqq[|\varepsilon_{0,0}|+\delta_0(h_0)h^{*m-p}]M_{j-1} \tag{3.16}$$

From (3.7) and (3.9) we have

$$|\varepsilon_{j,0}|=0(h^{*m-p})\qquad 0\leqq j\leqq p \tag{3.17}$$

and

$$|\eta_{j,0}|=0(h^{*m-p})\qquad p+1\leqq j\leqq m \tag{3.18}$$

Now let us consider the errors on the mesh points of $(a_0^*,a_1^*]$

$$\varepsilon_{k,i}=y_i^{(k)}-v_i^{(k)},\qquad 0\leqq k\leqq p \tag{3.19}$$

$$\eta_{k,i}=y_i^{(k)}-v_i^{(k)},\qquad p+1\leqq k\leqq m \tag{3.20}$$

Between mesh points (x_{i-1},x_i) we obtain by Taylor expansion

$$v_i^{(j)}=\sum_{k=j}^{m}\frac{h^{k-j}}{(k-j)!}v_{i-1}^{(k)}+\frac{h^{m-j+1}}{(m-j+1)!}v^{m+1}(\xi_{j,i}) \tag{3.21}$$

where $x_{i-1}<\xi_{j,i-1}<x_i$.
From (2.11) and (3.19) it follows that

$$\varepsilon_{j,i}=\sum_{k=j}^{p}\frac{h^{k-j}}{(k-j)!}\varepsilon_{k,i-1}+\sum_{k=p+1}^{m}\frac{h^{k-j}}{(k-j)!}\eta_{k,i-1}-\frac{h^{m-j+1}}{(m-j+1)!}v^{m+1}(\xi_{j,i-1}) \tag{3.22}$$

By applying *Lemma 2* and by using (3.5) one can obtain

$$\varepsilon_{j,i} = -\sum_{k=j}^{p} \frac{(ih)^{k-j}}{(k-j)!} \frac{h^{m-k+1}}{(m-k+1)!} v^{(m+1)}(\xi_{k,a}) + \\ + \sum_{r=1}^{i} \left[\sum_{k=j}^{p} \sum_{l=p+1}^{m} \frac{((r-1)h)^{k-j}}{(k-j)!} \frac{h^{l-k}}{(l-k)!} \xi_{l,i-r} \right] - \\ - \sum_{r=1}^{i} \left[\sum_{k=j}^{p} \frac{((r-1)h)^{k-j}}{(k-j)!} \frac{h^{m-k+1}}{(m-k+1)!} v^{(m+1)}(\xi_{k,i-r}) \right] \tag{3.23}$$

for $0 \leqq j \leqq p$, $1 \leqq i \leqq N$.
To obtain some estimation for $\varepsilon_{j,i}$ we must have some bounds for $\eta_{j,i}$

$$\eta_{j,i} = f^{(j-1)}(x_i, v_i + \varepsilon_{0,i}, v_{\alpha_i} + \varepsilon_{\alpha_i} \delta_0(h_0) h^{m-p}) - \\ - f^{j-1}(x_i, v_i, v_{\alpha_i}) = \varepsilon_{0,i} \frac{\partial}{\partial y} f^{(j-1)}(x_i, \beta_{j,i} v_i + (1-\beta_{j,i}) y_i^{(0)}, y_{\alpha_i}^{(0)}) + \\ + \varepsilon_{\alpha_i} \delta_0(h_0) h^{m-p} \frac{\partial}{\partial z} f^{(j-1)}(x_i, v_i, \gamma_{j,i} v_{\alpha_i} + (1-\gamma_{j,i}) y_{\alpha_i}^{(0)})$$

where $0 < \beta_{j,i}, \gamma_{j,i} < 1$, $0 \leqq |\varepsilon_{\alpha_i}| \leqq 1$ with notation similar to (3.12) and (3.14), and by applying *Lemma 3*, we obtain for $0 \leqq j \leqq p$ and

$$\{i/1\ i\ N, x_i \in (a_0^*, a_1^*]\}$$

$$|\varepsilon_{j,i}| \leqq V_{m+1} \frac{(2h^*)^{m-j+1}}{(m-j+1)!} (i+1)^{p-j} + V_{m+1} \sum_{r=1}^{i} \frac{(2h)^{m-j+1}}{(m-j+1)!} r^{p-j} + \\ + \sum_{r=1}^{i} \left[\sum_{l=p+1}^{m} (|\varepsilon_{0,i-r}| M_{l-1}^{\langle 1 \rangle} + \delta_0(h_0) h^{*m-p} M_{l-1}^{\langle 2 \rangle}) \frac{(2h)^{l-j}}{(l-j)!} r^{p-j} \right] \tag{3.25}$$

Denoting $I_1 = \{i/1 \leqq i \leqq N, x_i \in (a_0^*, a_1^*]\}$ then by applying *Lemma 4* we get

$$|\varepsilon_{j,i}| \leqq V_{m+1} \frac{(2h)^{m-j+1}}{(m-j+1)!} (i+1)^{p-j} + V_{m+1} \frac{(2h)^{m-j+1}}{(m-j+1)!} i^{p-j+1} + \\ + \max_{I_1} |\varepsilon_{0,i}| i^{p-j+1} \sum_{l=p+1}^{m} \frac{(2h)^{l-j}}{(l-j)!} M_{l-1}^{\langle 1 \rangle} + \\ + \delta_0(h_0) h^{*m-p} i^{p-j+1} \sum_{l=p+1}^{m} \frac{(2h)^{l-j}}{(l-j)!} M_{l-j}^{\langle 2 \rangle}. \tag{3.26}$$

From (3.25) taking $j=0$, since $r^p \leqq i^p$, we obtain

$$|\varepsilon_{0,i}| \leqq \sum_{r=1}^{i} \varepsilon_{0,i-r} i^p \sum_{l=p+1}^{m} M_{l-1}^{\langle 1 \rangle} \frac{(2h)^l}{l!} +$$

$$+ V_{m+1} \frac{(2h)^{m+1}}{(m+1)!} 2^p i^p + V_{m+1} i^{p+1} \frac{(2h)^{m+1}}{(m+1)!} +$$

$$+ \delta_0(h_0) h^{*m-p} i^{p+1} \left[\sum_{l=p+1}^{m} M_{l-1}^{\langle 2 \rangle} \frac{(2h)^l}{l!} \right]. \tag{3.27}$$

In order to write (3.26) and (3.27) in a denser form we introduce the notation

$$A_j = (b-a)^{p-j} \sum_{l=p+1}^{m} M_{l-1}^{\langle 1 \rangle} \frac{2^{l-j} h_0^{l-p-1}}{(l-j)!} \tag{3.28}$$

$$B_j = (b-a)^{p-j+1} V_{m+1} \frac{2^{m-j+1}}{m-j+1} + (b-a)^{p-j} V_{m+1} 2^p \frac{2^{m-j+1}}{(m-j+1)!} +$$

$$+ (b-a)^{p-j+1} \delta_0(h_0) \sum_{l=p+1}^{m} M_{l-1}^{\langle 2 \rangle} \frac{2^{l-j}}{(l-j)!} h_0^{l-p-1} \tag{3.29}$$

Now taking $h \leqq h_0$ and $\bar{h} = \max\{h^*, h\}$,

$$i \leqq N = \frac{b-a_0}{h}$$

$$|\varepsilon_{0,i}| \leqq A_0 \bar{h} \sum_{r=0}^{i-1} |\varepsilon_{0,r}| + B_0 \bar{h}^{m-p} \tag{3.30}$$

by applying *Lemma 5*, (3.30) implies

$$|\varepsilon_{0,i}| \leqq B_0 \bar{h}^{m-p} e^{i\bar{h}A_0} \leqq B_0 e^{(b-a_0^*)A_0} \bar{h}^{m-p} \tag{3.31}$$

therefore

$$|\varepsilon_{j,i}| \leqq [B_0 e^{(b-a_0^* A_0} A_j (b-a) + B_j] \bar{h}^{m-p} = 0(\bar{h}^{m-p}) \tag{3.32}$$

for $0 \leqq j \leqq p, i \in I_1$, and

$$|\eta_{j,i}| \leqq |\varepsilon_{0,i}| M_{j-1}^{\langle 1 \rangle} + \delta_0(h_0) \bar{h}^{m-p} M_{j-1}^{\langle 2 \rangle}.$$

$$\text{set } M_j = \max\{M_j^{\langle 1 \rangle}, M_j^{\langle 2 \rangle}\},$$

then

$$|\eta_{j,i}| \leqq M_{j-1} [B_0 e^{(b-a_0^*)A_0} + \delta_0(h_0) \bar{h}^{m-p} = 0(h^{m-p}) \tag{3.33}$$

Now we give the continuous error bounds on $(a_0, a_1^*]$. By Taylor expansion of $y(x)$ and $v(x)$ on the right end of each subinterval $(x_i, x_{i+1}]$ of $(a_0^*, a_1^*]$ we can obtain for $0 \leqq j \leqq m$

$$\max\{\sup\{|y_{(x)}^{(j)}-v_{(x)}^{(j)}|:[x_i,x_{i+1}]\}:1\leqq i\leqq I_1-1\}\leqq$$
$$\leqq\sum_{k=j}^{p}\frac{\bar{h}^{k-j}}{(k-j)!}\max_{i\in I_1}|\varepsilon_{k,i}|+\sum_{k=p+1}^{m}\frac{\bar{h}^{k-j}}{(k-j)!}\max_{i\in I_1}|\eta_{k,i}|+\frac{\bar{h}^{m-j+1}}{(m-j+1)!}V_{m+1} \tag{3.34}$$

Assuming that $0\leqq j\leqq p$ and set

$$M_j(h_0)=\sum_{k=j}^{p}\frac{h_0^{k-j}}{(k-j)!}[B_0r^{(b-a_0^*)A_0}A_k(b-a)+B_k]+$$
$$+\sum_{k=p+1}^{m}\frac{h_0^{k-j}}{(x-j)!}M_{k-1}[B_0e^{(b-a_0^*)A_0}+\delta_0(h_0)]+$$
$$+\frac{h_0^{m-j+1}}{(m-j+1)}V_{m+1} \tag{3.35}$$

one gets for $0\leqq j\leqq p$

$$\sup_{[a_0^*,a_0^*]}|y^{(j)}(x)-v^{(j)}(x)|\leqq M_j(h_0)\bar{h}^{m-p} \tag{3.36}$$

For $p+1\leqq j\leqq m$, $M_j(h_0)$ is different from (3.35) since in this case the last term of (3.34) gives the essential part to determine the error bounds:

$$(M_j(h_0)=\sum_{k=j}^{m}\frac{h_0^{k-p+1}}{(k-j)!}M_{k-1}[B_0e^{(b-a)A_0}+\delta_0(h_0)+\frac{V_{m+1}}{(m-j+1)!} \tag{3.37}$$

Then for $p+1\leqq j\leqq m$

$$\max\{\sup\{|y^{(j)}(x)-v^{(j)}(x)|:[x_i,x_{i+1}]\}:1\ i\ I_1-1\}\leqq M_j(h_0)\bar{h}^{m-j+1} \tag{3.38}$$

Now we are able to give error bounds whenever our procedure is continued to next steps $(a_1,a_2),\ldots$ and it is obvious that the same error bounds could be obtained as (3.32), (3.33) (3.36) and (3.38).

For given $\in_1,\in_2>0, h_0$ can be chosen such that $\delta_0(h_0)h_0^{m-p}$ and $M_0(h_0)h_0^{m-p}<\in(\in=\max\in_1,\in_2)$. This implies that $y(x)\to v(x)$ in Ghebyshev norm on $[a,b]$. Since A_j and B_j are independent of $\bar{h}$, $\delta_0(h_0)$ and $M_j(h_0)$ are bounded. Hence $\delta_0(h_0)\bar{h}_{0\to 0}^{m-p}$ and $M_0(h_0)h_0^{m-p}\to 0$ as $h_0\to 0$.

We can summarize our results.

Theorem

Let the function $f(x,y,z)$ be k-times continuously differentiable with respect to x,y,z in same domain D of (x,y,z)-space $(k\geqq 1)$. Let us assume that $\alpha(x)\in C^k[a,b]$. The restriction of D into (x,y) — space will be denoted by D_1. Let us assume that D_1 contains the exact solution $v(x)$ of the RODE on $[a,b]$, so that $v(x)\in C^{k+1}[a,b]$. Let $1\leqq m\leqq k$ and $0\leqq p\leqq m-1$. Then there is $h_0>0$

such that the algorithm producing the spline function $y(x) \in PC^{m,p}[a, b]$, satisfies the discrete error bounds

$$\max_{0 \leqq i \leqq N} |y^{(j)}(x_i) - v^{(j)}| \leqq L_j \bar{h}^{m-p} \qquad 0 \leqq j \leqq m \tag{3.39}$$

Moreover, $y(x)$ also satisfies the Chebyshev error bounds

$$\sup_{X \in [a, a_0^*]} |y^{(j)}(x) - v^{(j)}(x)| \leqq L_j(h_0) \bar{h}^{m-p}$$

$$\sup_{[a_0^*, b]} |y^{(j)}(x) - v^{(j)}(x)| \leqq M_j(h_0) \bar{h}^{m-p}, \quad 0 \leqq j \leqq p \tag{3.40}$$

$$\sup_{(a, a_0^*]} |y^{(j)}(x) - v^{(j)}(x)| \leqq L_j(h_0) \bar{h}^{m-j+1}$$

$$\max_{1 \leqq i \leqq N} \sup_{(X_{i-1}, X_i)} |y^{(j)}(x) - v^{(j)}(x)| \leqq M_j(h_0) \bar{h}^{m-j+1}, \quad p+1 \leqq j \leqq m \tag{3.41}$$

where L_j, $\delta_j(h_0)$, $M_j(h_0)$ are given constants.

Numerical Examples

To illustrate convergence, we applied the method to the following examples.

Example 1

$$y'(x) = 2y(\sqrt{x}) \quad y(1) = 1 \quad 1 \leqq x \leqq 2$$

which has the exact solution

$$y = x^2 .$$

Example 2

$$y'(x) = \frac{1}{1 - y^2(\sin x)} \, y(0) = 0 \qquad 0 \leqq x \leqq \frac{\pi}{2}$$

which has the exact solution

$$y = \arcsin x .$$

The errors for $h = 0.05$, $m = 3$, $p = 2$ are tabulated in Table 1, for example 1 and in Table 2, for example 2.

Table 1

h=0.05 $m=3$ $p=2$

X	exact solution				approximate solution			
	y	y'	y''	y'''	y	y'	y''	y'''
1.1	1.21	2.2	2.0	0.0	1.21	2.2	2.0	−0.000 03
1.2	1.44	2.4	2.0	0.0	1.44	2.4	2.0	−0.000 02
1.3	1.69	2.6	2.0	0.0	1.69	2.6	2.0	0.0
1.4	1.96	2.8	2.0	0.0	1.96	2.8	2.0	0.000 01
1.5	2.25	3.0	2.0	0.0	2.25	3.0	2.0	0.0

Table 2

h=0.05 m=3 p=2

X	appr. sol.	error			
	y	y	y'	y''	y'''
0.1	0.100 168 00	0.000 000 58	−0.000 424 33	−0.006 833 70	−0.063 485 75
0.3	0.304 790 35	−0.000 097 00	−0.001 198 56	−0.009 351 49	−0.170 047 52
0.5	0.524 253 01	−0.000 654 23	−0.005 179 53	−0.032 431 05	−0.674 015 91
0.7	0.778 204 89	−0.002 873 87	−0.017 692 41	−0.042 983 30	−0.941 774 84
0.9	1.119 769 55	0.005 680 89	0.290 335 50	7.039 982 04	159.625 309

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