A PRECOMPENSATOR DESIGN TO ACHIEVE DECOUPLING IN THE FREQUENCY DOMAIN

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Summary

This paper deals with the problem of designing a precompensator, by using the frequency domain approach, for systems which have weak inherent coupling. A necessary and sufficient new condition for decoupling is given which is the basis of the first algorithm suggested to achieve dynamic decoupling. Another algorithm is presented, for the same purpose, by making use of the interactor matrix idea. For each method all the poles of the precompensator can be assigned arbitrarily and the construction does not depend upon the control law used.

1. Introduction

The problem of decoupling was introduced by Morgan [6] about two decades ago. Falb and Wolovich [4] were the first to give the necessary and sufficient condition for decoupling by using linear state variable feedback (l.s.v.f) alone. Gilbert [5] made a broad extension to their results, while Morse and Wonham [7] discussed the problem by using the geometric approach, and they introduced new classes of decoupling such as triangular and block decoupling.

Algebraically a proper, right invertible system can be dynamically decoupled by using l.s.v.f. alone if $\mathbf{B}^*$ (for definitions see § 2) is of full rank. Sometimes this condition is not satisfied and even the system is right invertible. In this case Gilbert [5] said the system has weak inherent coupling. The problem of designing a precompensator for systems having weak inherent coupling was studied by Cremer [3], Panda [8] and Sinha [10] by using the time domain approach, and in the freqency domain by Wolovich [12].

Wolovich invertible decoupling algorithm (Wolovich, [12]) yields a minimal order precompensator (or input dynamics), whose dynamics are governed by both the open loop system (o.l.s) transfer function matrix, and the

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desired closed loop system (c.l.s). Due to this constraint some practical problems of compensator design may arise, such as undesirable valued modes or even unstable ones. Moreover, the algorithm is derived by using l.s.v.f, and if an output feedback control law is desired for decoupling (see, e.g., Wolovich [13] and Bayoumi et al. [1]) other techniques are needed.

In § 3 and § 4 of the paper we present two methods for precompensator design such that the system after precompensation does not have weak inherent coupling ($\mathbf{B}^*$ is of full rank). The first algorithm is based on a necessary and sufficient new condition for decoupling in *Theorem 1*. The second algorithm makes use of the interactor idea. An introductory scientific material is given in § 2.

2. Preliminary definitions and basis

In this section the definitions and the theoretical basis needed in § 3 and § 4 are summarized. The symbol $\langle \cdot \rangle$ is used to denote diagonal matrices.

Definition 1 Gilbert matrix and indices.[+]

For any $p \times m$ proper matrix $\mathbf{T}(s)$ which has no zero rows we define the Gilbert index, d_{ii} of the ith row of $\mathbf{T}(s)$ as the least nonnegative integer for which $\lim_{s \to \infty} s^{d_{ii}}[t_{ij}(s)]_{1 \times m}$ is finite and nonzero. We shall call the diagonal collection of such s-powers, $\mathbf{D}(s) \triangleq \langle s^{d_{ii}} \rangle$; $i = 1, \ldots, p$, the Gilbert matrix. We use the notation $\mathbf{B}^* \triangleq \lim_{s \to \infty} \mathbf{D}(s)\mathbf{T}(s)$.

Definition 2 Column (row) proper polynomial matrix with respect to (w.r.t.) some indices.

We say that a $p \times m$ polynomial matrix $\mathbf{R}(s)$ is column (row) proper w.r.t. some set consisting of $m(p)$ index elements, if the scalar matrix whose columns (rows) consist of the coefficients of the s-powers corresponding to those indices is of full rank.

Definition 3 Column (row) proper polynomial matrix $\mathbf{R}(s)$ is called column (row) proper if the index set in *Definition 2* consists of the column (row) degrees. The scalar matrix of the coefficients is denoted by $\mathbf{\Gamma}_c[\mathbf{R}(s)]$ ($\mathbf{\Gamma}_r[\mathbf{R}(s)]$).

Example 1

$$\mathbf{R}(s) = \begin{bmatrix} s^2+1 & s \\ s+2 & 3s^2+1 \end{bmatrix}$$

is column proper w.r.t. the following set of indices: (0, 0), (0, 1), (0, 2), (1, 1), (2, 0) and (2, 2). However it is not column proper w.r.t. the following set of indices:

[+] $\mathbf{B}^*$ was defined first by Falb and Wolovich in 1967. Gilbert gave the frequency domain definition and interpretation in 1969.

(1, 0), (1, 2) and (2, 1). It must be noted that $\mathbf{R}(s)$ may be column proper w.r.t. some indices but not row proper for the same set, e.g., $\mathbf{R}(s)$ is column proper w.r.t. (0, 1) while it is not row proper for the same set.

Definition 4 Linear state variable feedback (l.s.v.f).

For a proper system, whose input-output behaviour is described by the $p \times m$ transfer function matrix, $\mathbf{T}(s) = \mathbf{R}(s)\mathbf{P}^{-1}(s)$ with $\mathbf{R}(s)$ and $\mathbf{P}(s)$ relatively right prime (*rrp*) and $\mathbf{P}(s)$ column proper, the l.s.v.f. is defined as the control law

$$\mathbf{U}(s) = \mathbf{F}(s)\mathbf{Z}(s) + \mathbf{G}\mathbf{V}(s),$$

i.e., the pair $\{\mathbf{F}(s), \mathbf{G}\}$ for which the closed loop system $\mathbf{T}_{\mathbf{F},\mathbf{G}}(s) = \mathbf{R}(s)[\mathbf{P}(s) - \mathbf{F}(s)]^{-1}\mathbf{G}$ satisfies,

$$\partial c_j[\mathbf{P}(s)] = \partial c_j[\mathbf{P}(s) - \mathbf{F}(s)] \text{ and}$$

$$\mathbf{\Gamma}_c[\mathbf{P}(s)] = \mathbf{\Gamma}_c[\mathbf{P}(s) - \mathbf{F}(s)]$$

where $\mathbf{Z}(s)$ is an m-vector representing the partial states, $\mathbf{U}(s)$ is an m-vector representing the control signal, and $\mathbf{V}(s)$ is an m-vector representing the input signal.

Definition 5 Dynamic decoupling

A linear multivariable system is said to be dynamically decoupled if its transfer function matrix is diagonal and nonsingular. If the o.l.s. input-output behaviour is given by a $p \times m$ proper transfer function matrix $\mathbf{T}(s)$, then we say that the system is decouplable through l.s.v.f. if there exists a pair $\{\mathbf{F}(s), \mathbf{G}\}$ of dimensions $m \times m$ and $m \times p$ respectively; and $\mathbf{G}$ of full rank p, such that the c.l.s. $\mathbf{T}_{\mathbf{F},\mathbf{G}}(s)$ will be a $p \times p$ nonsingular diagonal transfer function matrix.

We shall also make use of the following known theorems,

W1. Wolovich's first theorem (Wolovich [12], p: 288) "*A linear multivariable system characterized by a $p \times m$ proper transfer function matrix* $\mathbf{T}(s)$ *can be decoupled by input dynamics (precompensator) in combination with l.s.v.f. if and only if it is right invertible.*"

W2. Wolovich's second theorem (Wolovich [12], p: 296) "*A system with a proper, right invertible $p \times m$ transfer function matrix* $\mathbf{T}(s)$ *can be decoupled via l.s.v.f. alone if and only if there exists some constant $m \times p$ matrix* $\mathbf{G}$ *such that* $\mathbf{B}^*(\mathbf{T}(s)\mathbf{G})$ *is nonsingular.*"

In light of *W1* we assume that we have a right invertible system $\mathbf{T}(s)$, and our aim in § 3 and in § 4 is to design a precompensator $\mathbf{T}_c(s)$ whose poles may be arbitrarily chosen such that $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s))$ is of full rank.

3. A new necessary and sufficient condition for decoupling and its application for the precompensator design

In this section we give a new necessary and sufficient condition for decoupling a right invertible system expressed in the matrix fraction right description (MFRD) form. Then we construct an algorithm to achieve decoupling by making use of this theorem. The following lemmas are needed to establish *Theorem 1*, and they can be easily proved.

Lemma 1. "*If* $\mathbf{T}(s)=\mathbf{R}(s)\mathbf{P}^{-1}(s)$ *is a proper rational function matrix, then* $\partial c_j[\mathbf{R}(s)] \leqq \partial c_j[\mathbf{P}(s)]$, $\forall_j$."

Lemma 2. "*If* $\mathbf{T}(s)=\mathbf{R}(s)\mathbf{P}^{-1}(s)$ *is a proper rational function matrix, and* $\mathbf{D}(s)$ *is its Gilbert matrix, then* $\partial c_j[\mathbf{D}(s)\mathbf{R}(s)] \leqq \partial c_j[\mathbf{P}(s)]$, $\forall_j$."

Theorem 1

"*A system described by a* $p \times m$, *right-invertible, proper transfer function matrix* $\mathbf{T}(s)=\mathbf{R}(s)\mathbf{P}^{-1}(s)$, *with* $\mathbf{R}(s)$ *and* $\mathbf{P}(s)$ *rrp and* $\mathbf{P}(s)$ *column proper, can be decoupled by l.s.v.f. alone, if and only if* $\bar{\mathbf{R}}(s) \triangleq \mathbf{D}(s)\mathbf{R}(s)$ *is column proper w.r.t. the column indices of* $\mathbf{P}(s)$, *where* $\mathbf{D}(s)$ *is the Gilbert matrix of* $\mathbf{T}(s)$."

Proof

Since $\mathbf{P}(s)$ is column proper, its adjugate matrix $\mathbf{B}(s)$ must be row proper (*Lemma 1*, [15]). Let $\mathbf{R}(s)$ and $\mathbf{P}(s)$ be the $p \times m$ and $m \times m$ irreducible matrix fraction description as we assumed by *rrp* constraint, then the system degree "n" is equal to the determinental degree of $\mathbf{P}(s)$, i.e., $n=\sum_{j=1}^{m} \partial c_j$, where ∂c_j is the j*th* column degree (or index) of $\mathbf{P}(s)$. The i-*th* row index of $\mathbf{B}(s)$, ∂r_i, is equal to $\sum_{\substack{j=1\\ j\neq i}}^{m} \partial c_j$. $\mathbf{B}(s)$ can be expressed in the form:

$$\mathbf{B}(s)=\bar{\mathbf{S}}_l(s)\mathbf{B}_0+\bar{\mathbf{S}}_{l-1}(s)\mathbf{B}_1+\ldots\mathbf{B}_l \tag{1}$$

where $\mathbf{B}_0$ is an $m \times m$ nonsingular matrix, which represents the coefficients corresponding to the row degree,

$$l \triangleq \max\{\partial r_i\}, \qquad i=1,2,\ldots,m, \tag{2}$$

$\bar{\mathbf{S}}_{l-k}(s)$ is an $m \times m$ diagonal, polynomial matrix having the form:

$$\bar{\mathbf{S}}_{l-k}(s) \triangleq \langle s^{\partial r_1-k}, s^{\partial r_2-k}, \ldots s^{\partial r_m-k}\rangle, k=0,\ldots,l-1 \tag{3}$$

and the elements of $\mathbf{B}_i$ corresponding to negative s-powers are zeros.

To prove necessity, we assume that the system can be decoupled through l.s.v.f. alone, i.e., there exists a pair $\{\mathbf{F}(s), \mathbf{G}\}$ of dimensions $m \times m$ and $m \times p$

respectively, with $\partial c_j[\mathbf{F}(s)] < \partial c_j[\mathbf{P}(s)], \forall_j$, such that the c.l.s. transfer function matrix is given by:

$$\mathbf{T}^{(s)}_{\mathbf{F},\mathbf{G}} = \mathbf{R}(s)\,[\mathbf{P}(s) - \mathbf{F}(s)]^{-1}\mathbf{G} \qquad (4)$$
$$= \mathbf{T}_d(s)$$

where $\mathbf{T}_d(s)$ is the desired $p \times p$ nonsingular, diagonal transfer function matrix. Premultiplication of both sides of equation (4) by the Gilbert matrix $\mathbf{D}(s)$ of $\mathbf{T}(s)$ yields:

$$\mathbf{D}(s)\mathbf{R}(s)\,[\mathbf{P}(s) - \mathbf{F}(s)]^{-1}\mathbf{G} = \mathbf{D}(s)\mathbf{T}_d(s) \qquad (5)$$

Now, let:

$$\bar{\mathbf{R}}(s) \triangleq \mathbf{D}(s)\mathbf{R}(s) \qquad (6)$$

By writing equation (6) in the column proper form we have:

$$\bar{\mathbf{R}}(s) = \bar{\mathbf{R}}_0\hat{\mathbf{S}}_h(s) + \bar{\mathbf{R}}_1\hat{\mathbf{S}}_{h-1}(s) + \ldots + \bar{\mathbf{R}}_h \qquad (7)$$

where

$$\hat{\mathbf{S}}_{h-k}(s) = \langle s^{\partial\bar{c}_1 - k}, s^{\partial\bar{c}_2 - k}, \ldots, s^{\partial\bar{c}_m - k}\rangle, k = 0, 1, \ldots, h \qquad (8a)$$

$$h \triangleq \max\,\{\partial\bar{c}_j\}, j = 1, 2, \ldots, m \qquad (8b)$$

and $\partial\bar{c}_j$ is the jth column degree of $\bar{\mathbf{R}}(s)$. It follows from *Lemma 2* that $\partial\bar{c}_j \leqq \partial c_j, \forall_j$. By taking the limits of both sides of equation (5) and using *Theorem W2*, *Definition 1* and *Definition 4*, the R.H.S. gives:

$$\lim_{s\to\infty} \mathbf{D}(s)\mathbf{T}_d(s) = \mathbf{B}^*(\mathbf{T}(s)\mathbf{G}) = \mathbf{B}^*(\mathbf{T}(s))\mathbf{G} \qquad (9)$$

with $\mathbf{B}^*$ of rank p from the assumption ($p \leqq m$ follows from the right invertibility of $\mathbf{T}(s)$).

By making use of the constraint $\partial c_j[\mathbf{F}(s)] < \partial c_j[\mathbf{P}(s)], \forall_j$ and the form of equation (6), the L.H.S. gives,

$$\lim_{s\to\infty}\{\mathbf{D}(s)\mathbf{R}(s)\,[\mathbf{P}(s) - \mathbf{F}(s)]^{-1}\mathbf{G}\}\ \lim_{s\to\infty}\left\{\frac{\bar{\mathbf{R}}_0\hat{\mathbf{S}}_h(s)\bar{\mathbf{S}}_l(s)\mathbf{B}_0\mathbf{G}}{s^n}\right\} \qquad (10)$$

Now let:

$$\mathbf{S}_{h+l}(s) \triangleq \hat{\mathbf{S}}_h(s)\hat{\mathbf{S}}_l(s) = \langle s^{\partial\bar{c}_1 + n - \partial c_1}, \ldots, s^{\partial\bar{c}_m + n - \partial c_m}\rangle \qquad (11)$$

From *Lemma 1* it follows $\partial\bar{c}_j + n - \partial c_j \leqq n$. Equation (10) may be rewritten as:

$$\text{L.H.S.} = \lim_{s\to\infty}\left\{\frac{\bar{\mathbf{R}}_0\mathbf{S}_{h+l}(s)\mathbf{B}_0\mathbf{G}}{s^n}\right\}, \qquad (12)$$

and since $\lim_{s\to\infty}\dfrac{\mathbf{S}_{h+l}(s)}{s^n}$ is finite, another form of equation (12) may be:

$$\text{L.H.S.} = \lim_{s\to\infty} \left\{ \bar{\mathbf{R}}_0 \frac{\mathbf{S}_{h+l}(s)}{s^n} \right\} \mathbf{B}_0 \mathbf{G} \tag{13.a}$$

$$= \bar{\mathbf{R}}_0 \lim_{s\to\infty} \left\{ \frac{\mathbf{S}_{h+l}(s)}{s^n} \right\} \mathbf{B}_0 \mathbf{G} \tag{13.b}$$

From equation (9), the rank of the above equation must be equal to p and the Sylvester's inequality gives:

$$q \triangleq \operatorname{rank}\left(\lim_{s\to\infty} \frac{\mathbf{S}_{h+l}(s)}{s^n} \right) \geqq p\,, \quad \text{and} \tag{14}$$

$$\operatorname{rank}(\bar{\mathbf{R}}_0) = p \tag{15}$$

Equation (14) means that there are at least q columns of $\mathbf{S}_{h+l}(s)$ of degrees equal to "n", or equivalently, q-column indices of $\bar{\mathbf{R}}(s)$ are equal to the corresponding ones in $\mathbf{P}(s)$, and by combining it with equation (15), give that $\bar{\mathbf{R}}(s)$ is column proper w.r.t. column indices of $\mathbf{P}(s)$.

To prove sufficiency we assume that $\bar{\mathbf{R}}(s)$ is column proper w.r.t. the column indices of $\mathbf{P}(s)$. It follows from *Lemma 2* that there exist at least p indices $i_1, i_2, \ldots, i_p$ satisfying $\partial \bar{c}_{i_k} = \partial c_{i_k}$, $k = 1, \ldots, p$, and the corresponding columns of $\bar{\mathbf{R}}_0$ are linearly independent. Hence $\bar{\mathbf{R}}_0 \left(\lim_{s\to\infty} \frac{\mathbf{S}_{h+l}(s)}{s^n} \right)$ is of rank p. Since $\mathbf{B}_0$ is of rank m, it follows from the Sylvester's inequality that $\bar{\mathbf{R}}_0 \left(\lim_{s\to\infty} \frac{\mathbf{S}_{h+l}(s)}{s^n} \right) \mathbf{B}_0$ is of rank p. Let $\mathbf{G}$ be the right inverse of $\bar{\mathbf{R}}_0 \left(\lim_{s\to\infty} \frac{\mathbf{S}_{h+l}(s)}{s^n} \right) \mathbf{B}_0$, then $\bar{\mathbf{R}}_0 \left(\lim_{s\to\infty} \frac{\mathbf{S}_{h+l}(s)}{s^n} \right) \mathbf{B}_0 \mathbf{G} = \mathbf{B}^*(\mathbf{T}(s)\mathbf{G})$ is nonsingular and $\mathbf{T}(s)$ can be decoupled via l.s.v.f. alone by *Theorem W2*.

Q.E.D.

Corollary 1

"A proper invertible system ($p = m$) can be decoupled by l.s.v.f. alone if and only if $\bar{\mathbf{R}}(s)$ is column proper and $\partial c_j[\bar{\mathbf{R}}(s)] = \partial c_j[\mathbf{P}(s)]$, $\forall_j$."

Example 2

Consider the following, proper and right invertible transfer function matrix,

$$T(s) = \begin{bmatrix} \dfrac{s+1}{s} & \dfrac{1}{s+1} & \dfrac{1}{s+2} \\[2ex] \dfrac{1}{s} & 0 & 0 \end{bmatrix} =$$

$$
= \underbrace{\begin{bmatrix} s+1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{R}(s)} \underbrace{\begin{bmatrix} s & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}}_{\mathbf{P}^{-1}(s)}
$$

The Gilbert matrix is

$$
\mathbf{D}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \quad \text{and} \quad \mathbf{B}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

and the column indices are: $\partial c_1 = \partial c_2 = \partial c_3 = 1$. By using the suggested theorem,

$$
\bar{\mathbf{R}}(s) = \mathbf{D}(s)\,\mathbf{R}(s) = \begin{bmatrix} s+1 & 1 & 1 \\ s & 0 & 0 \end{bmatrix}
$$

and w.r.t. the indices $\{1, 1, 1\}$, $\Gamma_c^{\{1,1,1\}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, which is equal to $\mathbf{B}^*$ and each one means that an input dynamic is needed to achieve decoupling.

Remark: From the above example, it is evident why we give the new definition of column properness w.r.t. some indices *(Definition 2)*. If we use *Definition 3*, $\Gamma_c = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, which will give an incorrect result.

Since we used the irreducible (rrp) MFRD in *Theorem 1*, an important question arising due to this special form of description is how the system is still controllable and observable after compensation.* A partial answer to this question, and which will be used later, is given as *Theorem 2*. First we shall make use of the following lemma (Rosenbrock, [9], p. 71)

Lemma 3

"*If* $\mathbf{R}(s)$ *and* $\mathbf{P}(s)$ *are two polynomial matrices of dimensions* $p \times m$ *and* $m \times m$ *respectively, with* $\mathbf{P}(s)$ *nonsingular, then* $\{\mathbf{R}(s), \mathbf{P}(s)\}$ are rrp if and only if rank $\begin{bmatrix} \mathbf{P}(s_0) \\ \hline \mathbf{R}(s_0) \end{bmatrix} = m$, *for all zeros* s_0 *of* $|\mathbf{P}(s)|$".

* If $\mathbf{T}(s) = \mathbf{R}(s)\mathbf{P}^{-1}(s)$ is a system transfer function and $\{\mathbf{R}(s), \mathbf{P}(s)\}$ are rrp, then $\mathbf{T}(s)$ describes a controllable and observable realization of the system [14], p. 440).

Theorem 2

"*Let* $\{\mathbf{R}(s), \mathbf{P}(s)\}$ be two invertible, *rrp*, $m \times m$ polynomial matrices. Then $\{\mathbf{R}(s), \mathbf{P_c}(s)\,\mathbf{P}(s)\}$ are also *rrp* (where $\mathbf{P_c}(s)$ is an invertible polynomial matrix) if,

$$\{r_0\} \cap \{s_{c_0}\} = \emptyset \text{ and } \{s_0\} \cap \{s_{c_0}\} = \emptyset, \text{ where}$$

$$\{r_0\} \equiv \{r_0 \colon |\mathbf{R}(r_0)| = 0\},$$

$$\{s_0\} \equiv \{s_0 \colon |\mathbf{P}(s_0)| = 0\},$$

$$\{s_{c_0}\} \equiv \{s_{c_0} \colon |\mathbf{P_c}(s_0)| = 0\}."$$

Proof

Since $|\mathbf{P_c}(s)\,\mathbf{P}(s)| = |\mathbf{P_c}(s)|\,|\mathbf{P}(s)|$, then

$$\{\bar{s}_0\} \equiv \{\bar{s}_0 \colon |\mathbf{P_c}(\bar{s}_0)\,\mathbf{P}(\bar{s}_0)| = 0\}$$

$$= \{s_0\} \cup \{s_{c_0}\} \tag{16}$$

Let $\bar{s}_0 \in \{\bar{s}_0\}$. From equation (16) we have the following two cases:

Case 1: $\bar{s}_0 \in \{s_0\}$ but $\bar{s}_0 \notin \{s_{c_0}\}$.

From the assumptions,

$$rank\,[\mathbf{P_c}(\bar{s}_0)] = m. \tag{17}$$

By Rosenbrock's lemma (*Lemma 3*) and Sylvester inequality,

$$rank\begin{bmatrix}\mathbf{P_c}(\bar{s}_0)\mathbf{P}(\bar{s}_0)\\ \mathbf{R}(\bar{s}_0)\end{bmatrix} = rank\left[\begin{pmatrix}\mathbf{P_c}(\bar{s}_0) & 0\\ 0 & \mathbf{I_p}\end{pmatrix}\begin{pmatrix}\mathbf{P}(\bar{s}_0)\\ \mathbf{R}(\bar{s}_0)\end{pmatrix}\right] = m \tag{18}$$

Case 2: $\bar{s}_0 \in \{s_{c_0}\}$ but $\bar{s}_0 \notin \{r_0\}$

From the assumptions,

$$rank\,[\mathbf{R}(\bar{s}_0)] = m, \tag{19}$$

and hence

$$rank\begin{bmatrix}\mathbf{P_c}(\bar{s}_0)\mathbf{P}(\bar{s}_0)\\ \mathbf{R}(\bar{s}_0)\end{bmatrix} = m \tag{20}$$

So, from equations (18) and (20) $\{\mathbf{R}(s), \mathbf{P_c}(s)\mathbf{P}(s)\}$ are rrp.

Q.E.D.

By using a similar proof to that of *Theorem 2*, we can establish the following theorem for the general case $p \neq m$.

Theorem 3

"*If* $\{\mathbf{R}(s), \mathbf{P}(s)\}$ *are rrp polynomial matrices of dimensions* $p \times m$ *and* $m \times m$ *respectively with* $\mathbf{R}(s)$ *of full rank* p *and* $\mathbf{P}(s)$ *is invertible, then* $\{\mathbf{R}(s), \mathbf{P_c}(s)\mathbf{P}(s)\}$ *is also rrp (where* $\mathbf{P_c}(s)$ *is invertible) if,* $\{s_0\} \cap \{s_{c0}\} = \emptyset$ *and*

$$rank \begin{bmatrix} \mathbf{P_c}(s_{c0}) \\ \hline \mathbf{R}(s_{c0})\mathbf{P}^{-1}(s_{c0}) \end{bmatrix} = m, \forall s_{c0} \in \{s_{c0}\}$$

In spirit of *Theorem 1* and *Theorem 2* we outline now an algorithm to design an $m \times m$ precompensator $\mathbf{T_c}(s)$ for $\mathbf{T}(s)$ such that $\mathbf{B^*}(\mathbf{T}(s)\mathbf{T_c}(s))$ is of full rank and the compensated system will still be controllable and observable. The algorithm is constructed for the invertible case ($p = m$), and a modification will be given for the more general case $p \neq m$.

Algorithm 1

Step 1 Find an $m \times m$ polynomial matrix $\mathbf{X}(s)$ such that $\tilde{\mathbf{R}}(s) \triangleq \mathbf{R}(s)\mathbf{X}(s)$ is column proper with equal column indices and $\Gamma_c[\tilde{\mathbf{R}}(s)] = \mathbf{I_m}$. This can be computed by using the following substeps:

a) Find an $m \times m$ unimodular matrix $\mathbf{X}_1(s)$ such that $\mathbf{R}_1(s) \triangleq \mathbf{R}(s)\mathbf{X}_1(s)$ is column proper (Wolovich 1974, Theorem 2.5.7, p. 27).

b) If the column indices of $\mathbf{R}_1(s)$ are equal set, let $\mathbf{R}_2(s) = \mathbf{R}_1(s)$ and go to (c). Otherwise define $\mathbf{X}_2(s)$ as a diagonal matrix of monomial entries of minimum degree such that $\mathbf{R}_2(s) \triangleq \mathbf{R}_1(s)\mathbf{X}_2(s)$ has equal column indices.

c) Set $\mathbf{X}_3 = \Gamma_c^{-1}(\mathbf{R}_2(s))$ and $\mathbf{X}(s) = \mathbf{X}_1(s)\mathbf{X}_2(s)\mathbf{X}_3$.

Step 2 Find a unimodular matrix $\mathbf{U}(s)$ such that $\tilde{\mathbf{P}}(s) = \mathbf{U}(s)\bar{\mathbf{P}}(s)$ is column proper, where $\bar{\mathbf{P}}(s) \triangleq \mathbf{P}(s)\mathbf{X}(s)$ (Wolovich, [12], Theorem 2.5.11, p. 30 yields $\tilde{\mathbf{P}}(s)$ in Hermite-form).

Step 3 Find $\mathbf{U}^{-1}(s)$ (by any suitable method as that given by Buslowicz [2]), and choose an arbitrary diagonal polynomial matrix $\mathbf{P_c}(s)$ (under only the constraint of *Theorem 2*) of minimal degree such that $\partial c_j[\mathbf{U}^{-1}(s)] \leqq \partial c_j[\mathbf{P_c}(s)], \forall_j$ and $\mathbf{P_c}(s)\tilde{\mathbf{P}}(s)$ is column proper and let $\mathbf{T_c}(s) = (\mathbf{P_c}(s)\mathbf{U}(s))^{-1}$. STOP.★

Remark 1: Similar to the proof of *Theorem 1* we get

$$\tilde{\mathbf{R}}(s) = \mathbf{I_m}s^l + \tilde{\mathbf{R}}_{l-1}s^{l-1} + \ldots + \tilde{\mathbf{R}}_0,$$

$$(\mathbf{P_c}(s)\tilde{\mathbf{P}}(s))^{-1} = \frac{\mathbf{S_h}(s)\bar{\bar{\mathbf{P}}}_h + \ldots}{(s^{\bar{\bar{n}}} + \ldots)},$$

$$\mathbf{S_h}(s) \triangleq \langle s^{\bar{\bar{n}} - \partial\bar{\bar{c}}_1}, s^{\bar{\bar{n}} - \partial\bar{\bar{c}}_2}, \ldots, s^{\bar{\bar{n}} - \partial\bar{\bar{c}}_m} \rangle,$$

★ It is an open subject to study how to choose $\mathbf{U}(s)$ so that $\mathbf{P_c}(s)$ will be of minimum order.

then

$$\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s)) = \lim_{s\to\infty} \mathbf{D}(s)\mathbf{T}(s)\mathbf{T}_c(s) =$$

$$= \lim_{s\to\infty} \frac{\mathbf{D}(s)\mathbf{S}_{h+l}(s)\bar{\bar{\mathbf{P}}}_h}{s^{\bar{\bar{n}}}} = \bar{\bar{\mathbf{P}}}_h = \Gamma_c^{-1}(\mathbf{P}_c(s)\tilde{\mathbf{P}}(s)).$$

So $\mathbf{B}^*$ will be nonsingular and the Gilbert indices are given by $d_{ii} = \partial\bar{\bar{c}}_i - l$.
Remark 2: $\mathbf{U}(s)$ does not affect the controllability of the compensated system since it is a unimodular matrix.

Example 3

Consider the following transfer function matrix describing an O.L.S.,

$$T(s) = \begin{bmatrix} \frac{s+1}{s^2} & \frac{s+2}{s^2+1} \\ \frac{2}{s} & \frac{2s+3}{s^2+1} \end{bmatrix} = \underbrace{\begin{bmatrix} s+1 & s+2 \\ 2s & 2s+3 \end{bmatrix}}_{\mathbf{R}(s)} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s^2+1 \end{bmatrix}^{-1}}_{\mathbf{P}^{-1}(s)}$$

Wolovich's invertible decoupling algorithm yields a precompensator of order "*1*", whose dynamics depend upon the O.L.S. $\mathbf{T}(s)$, and a C.L.S. $\mathbf{T}_d(s)$ of order "*4*". Suppose that the desired form is:

$$\mathbf{T}_d(s) = \begin{bmatrix} \frac{1}{(s^2+0.8s+0.15)} & 0 \\ 0 & \frac{1}{(s^2+3s+2)} \end{bmatrix},$$

then Wolovich's precompensator will be

$$\mathbf{T}_c(s) = \begin{bmatrix} \frac{2s+9}{s-0.6} & -\frac{s+2.8}{s-0.6} \\ -\frac{2s+6}{s-0.6} & \frac{s+1.8}{s-0.6} \end{bmatrix}$$

and hence the precompensator will be unstable.

Now by Algorithm 1,

Step 1

$$\mathbf{X}(s) = \begin{bmatrix} 2s & -s \\ -2s+3 & s-1 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{R}}(s) = \mathbf{R}(s)\mathbf{X}(s) = \begin{bmatrix} s+6 & -2 \\ 9 & s-3 \end{bmatrix}$$

Step 2

$$\bar{\mathbf{P}}(s)=\mathbf{P}(s)\mathbf{X}(s)=\begin{bmatrix}2s^3 & -s^3\\ -2s^2+3s^2-2s+3 & s^3-s^2+s-1\end{bmatrix}.$$

$$\mathbf{U}(s)=\begin{bmatrix}-s & -(s+1)\\ 1 & 1\end{bmatrix},$$

$$\tilde{\mathbf{P}}(s)=\mathbf{U}(s)\bar{\mathbf{P}}(s)=\begin{bmatrix}-s^3-s^2-s-3 & 1\\ 3s^2-2s+3 & -s^2+s-1\end{bmatrix}$$

Step 3

$$\mathbf{U}^{-1}(s)=\begin{bmatrix}1 & s+1\\ -1 & -s\end{bmatrix}$$

We can choose $\mathbf{P}_c(s)$ to be diagonal and having $\partial c_1=0$ and $\partial c_2=1$

$$\mathbf{P}_c(s)=\begin{bmatrix}1 & 0\\ 0 & s+a\end{bmatrix},\quad \mathbf{P}_c(s)\tilde{\mathbf{P}}(s)=\begin{bmatrix}-s^3-s^2-s-3 & 1\\ (s+a)(3s^2-2s+3) & (s+a)(-s^2+s-1)\end{bmatrix}$$

and

$$\mathbf{T}_c(s)=\begin{bmatrix}1 & \dfrac{s+1}{s+a}\\ -1 & \dfrac{-s}{s+a}\end{bmatrix}$$

For check,

$$\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s))=\begin{bmatrix}-1 & 0\\ -3 & -1\end{bmatrix}$$

From the controllability and observability point of view, "a" can be chosen arbitrarily under the following constraints:

$$a\neq 0,\quad a\neq 3\quad \text{and}\quad a\neq \pm j$$

To extend Algorithm 1 for the general case $p\neq m$, *Theorem 3* is used in *Step 3*, instead of *Theorem 2* and the first step in the algorithm can be modified as follows:

Step 1. Find an $m\times m$ polynomial matrix $\mathbf{X}(s)$ such that the $p\times p$ submatrix consisting of the first p-columns of $\tilde{\bar{\mathbf{R}}}(s)$ is column proper with equal column indices and $\Gamma_c[\tilde{\bar{\mathbf{R}}}(s)]=[\mathbf{I}_p\ \ \mathbf{A}]$, where $\mathbf{A}$ is an $p\times(m-p)$ scalar matrix. This can be computed using the following substeps:

i — From $\mathbf{R}(s)$ find $\mathbf{R}_p(s)$, the $p\times p$ minor of maximum determinental degree among the $p\times p$ minors of $\mathbf{R}(s)$ having nonvanishing determinant.

ii — Rearrange the columns of $\mathbf{R}(s)$ by an $m\times m$ scalar matrix $\mathbf{X}_0$ such that the first p-columns will be those of $\mathbf{R}_p(s)$, i.e.,

$$\mathbf{R}_0(s)=[\mathbf{R}_p(s)\ \vdots\ \mathbf{R}_{px(m-p)}(s)\ ]$$

iii — Use the substeps a, b, and c of Algorithm 1 for the $p\times p$ submatrix $\mathbf{X}_{\mathbf{p}}(s)$ such that $\mathbf{R}_{\mathbf{p}}(s)\mathbf{X}_{\mathbf{p}}(s)$ has column proper form of equal column indices and $\Gamma_{\mathbf{c}}[\mathbf{R}_{\mathbf{p}}(s)\mathbf{X}_{\mathbf{p}}(s)]=\mathbf{I}_{\mathbf{p}}$.

Let

$$\mathbf{X}(s)=\mathbf{X}_0\begin{bmatrix}\mathbf{X}_{\mathbf{p}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{\mathbf{m}-\mathbf{p}}\end{bmatrix}$$

Example 4

$$\mathbf{T}(s)=\begin{bmatrix}\dfrac{s+1}{s} & \dfrac{1}{s+1} & \dfrac{1}{s+2}\\ \dfrac{1}{s} & 0 & 0\end{bmatrix}=$$

$$=\underbrace{\begin{bmatrix}s+1 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}}_{\mathbf{R}(s)}\underbrace{\begin{bmatrix}s & 0 & 0\\ 0 & s+1 & 0\\ 0 & 0 & s+2\end{bmatrix}}_{\mathbf{P}^{-1}(s)}{}^{-1}$$

Step 1

$$\mathbf{X}(s)=\begin{bmatrix}0 & 1 & 0\\ 1 & -(s+1) & 0\\ 0 & 0 & 1\end{bmatrix},\quad \tilde{\mathbf{R}}(s)=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

Step 2

$$\bar{\mathbf{P}}(s)=\begin{bmatrix}0 & s & 0\\ s+1 & -(s+1)^2 & 0\\ 0 & 0 & s+2\end{bmatrix},\quad \mathbf{U}(s)=\begin{bmatrix}1 & 0 & 0\\ s & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\tilde{\mathbf{P}}(s)=\begin{bmatrix}0 & s & 0\\ s+1 & -2s-1 & 0\\ 0 & 0 & s+2\end{bmatrix}$$

Step 3

$$\mathbf{U}^{-1}(s)=\begin{bmatrix}1 & 0 & 0\\ -s & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

We can choose $\mathbf{P}_c(s)$ to be diagonal and having $\partial c_1 = 1$ and $\partial c_2 = \partial c_3 = 0$:

$$\mathbf{P}_c(s) = \begin{bmatrix} s+a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P}_c(s)\tilde{\mathbf{P}}(s) = \begin{bmatrix} 0 & s(s+a) & 0 \\ s+1 & -2s-1 & 0 \\ 0 & 0 & s+2 \end{bmatrix},$$

$$\mathbf{T}_c(s) = \begin{bmatrix} \dfrac{1}{s+a} & 0 & 0 \\ \dfrac{-s}{s+a} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence we achieve a minimum order compensator with one pole arbitrarily assigned. By using *Theorem 3* controllability can be ensured if $a \neq 0$, $a \neq 1$ and $a \neq 2$.

For check $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, i.e., of full rank $p=2$.

4. Precompensator design using the interactor idea

Wolovich and Falb introduced the idea of the interactor in 1976. They showed that for an $p \times m$ proper system there exists as associated, unique polynomial matrix of special form, and they called it the interactor of the system. We shall make use of the interactor idea [15] to construct another technique in order to achieve the decoupling condition, i.e., $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s))$ is of full rank.

Theorem 4

"For an invertible system described by an $m \times m$ proper transfer function matrix $\mathbf{T}(s)$, it is always possible to design an $m \times m$ proper precompensator $\mathbf{T}_c(s)$ of order n_c such that $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s))$ is nonsingular, and all the nonzero poles are arbitrarily assigned. The precompensator order is bounded by the inequality

$$n_c \leqq \sum_{i=1}^{m} (\delta - \delta_g) - (f'_1 - \delta_g)$$

where $\delta =$ degree $[\xi_{\mathbf{T}'}(s)]$, $\mathbf{T}'(s)$, *and* $\{f'_i\}_{i=1}^m$ *are the transpose of* $\mathbf{T}(s)$ *and the interactor indices of* $\mathbf{T}'(s)$ *respectively and* δ_g *is g.c.d. of the interactor indices.*"

Proof

Let the transpose of the plant description be $\mathbf{T}'(s)$, then from *Lemma 3* in [15], there is a unique interactor $\xi_{\mathbf{T}'}(s)$ such that

$$\lim_{s\to\infty} \xi_{\mathbf{T}'}(s)\mathbf{T}'(s)=\mathbf{K}_{\mathbf{T}'} \tag{21}$$

with $\mathbf{K}_{\mathbf{T}'}$ nonsingular, and

$$\xi_{\mathbf{T}'}(s)=\begin{bmatrix} s^{f'_1} & 0 & \dots & 0 \\ s^{f'_1}h'_{21}(s) & s^{f'_2} & & 0 \\ \vdots & \vdots & & \\ s^{f'_1}h'_{m-1,1}(s) & s^{f'_2}h'_{m-1,2}(s) & \dots & 0 \\ s^{f'_1}h'_{m1}(s) & s^{f'_2}h'_{m2}(s) & \dots & s^{f'_m} \end{bmatrix} \tag{22}$$

If
$$\delta_g \triangleq \text{g.c.d}\,\{f'_i\}\ \forall_i, \tag{23}$$

then equation (21) may be written in the following form:

$$\lim_{s\to\infty} \mathbf{D}'(s)\mathbf{N}'_{\mathbf{c}}(s)\mathbf{T}'(s)=\mathbf{K}_{\mathbf{T}'} \tag{24a}$$

where

$$\mathbf{D}'(s)=\langle s^{\delta_g}, \dots, s^{\delta_g}\rangle=\mathbf{D}(s)=s^{\delta_g}\,\mathbf{I}_{\mathbf{m}} \tag{24b}$$

$$\mathbf{N}'_{\mathbf{c}}(s)=\begin{bmatrix} s^{f'_1-\delta_g} & 0 & \dots & 0 \\ s^{f'_1-\delta_g}h'_{21}(s) & s^{f'_2-\delta_g} & \dots & 0 \\ & & & \vdots \\ s^{f'_1-\delta_g}h'_{m1}(s) & s^{f'_2-\delta_g}h'_{m2}(s) & \dots & s^{f'_m-\delta_g} \end{bmatrix} \tag{24c}$$

Taking the transpose of equation (24)

$$\lim_{s\to\infty} \mathbf{D}(s)\mathbf{T}(s)\mathbf{N}_{\mathbf{c}}(s)=\mathbf{K}'_{\mathbf{T}'} \tag{25}$$

From the above equation, it is obvious that there is a polynomial matrix $\mathbf{N}_{\mathbf{c}}(s)$ such that $\mathbf{B}^*(\mathbf{T}(s)\mathbf{N}_{\mathbf{c}}(s))$ is nonsingular and all the Gilbert indices of $\mathbf{T}(s)\mathbf{N}_{\mathbf{c}}(s)$ are equal to δ_g.

Let $\mathbf{N}_c(s)$ be the numerator part of a precompensator $\mathbf{T}_c(s)$, then the denominator part $\mathbf{D}_c(s)$ must satisfy the following two conditions:
(i) $\mathbf{N}_c(s)\mathbf{D}_c^{-1}(s)$ is proper (for the practical realization), and
(ii) $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s)$ is nonsingular.

A choice of $\mathbf{D}_c(s)$ which satisfies the above two conditions may be:

$$\mathbf{D}_c(s)=\begin{bmatrix} s^{f'_1-\delta_g}(s^{\delta-f'_1}+d_1^1 s^{\delta-f'_1-1}+\ldots), & 0, & 0 \\ 0, & s^{f'_2-\delta_g}(s^{\delta-f'_2}+d_2^1 s^{\delta-f'_2-1}+\ldots),\ldots, & 0 \\ \vdots & & \\ 0, & 0, & \ldots, s^{f'_m-\delta_g}(s^{\delta-f'_m}+d_m^1 s^{\delta-f'_m-1}+\ldots) \end{bmatrix} \tag{26}$$

where d_i^j are arbitrarily scalars, $i=1,\ldots,m, j=1,2,\ldots,\delta-f_i'$, Equations (24c) and (26) give the precompensator description $\mathbf{T}_c(s)$

$$\mathbf{T}_c(s)=\begin{bmatrix} \frac{1}{(s^{\delta-f'_1}+d_1^1 s^{\delta-f'_1-1}+\ldots)}, & \frac{s^{f'_1-\delta_g}h'_{21}(s)}{s^{f'_2-\delta_g}(s^{\delta-f'_2}+d_1^2 s^{\delta-f'_2-1}+\ldots)}, & \ldots, \frac{s^{f'_1-\delta_g}h'_{m1}(s)}{s^{f'_m-\delta_g}(s^{\delta-f'_m}+d_1^m s^{\delta-f'_m-1}+\ldots)} \\ 0 & \frac{1}{(s^{\delta-f'_2}+d_1^2 s^{\delta-f'_2-1}+\ldots)}, & \ldots, \frac{s^{f'_2-\delta_g}h'_{m2}(s)}{s^{f'_m-\delta_g}(s^{\delta-f'_m}+d_2^m s^{\delta-f'_m-1}+\ldots)} \\ \vdots & & \\ 0 & 0 & \ldots, \frac{1}{(s^{\delta-f'_m}+d_1^m s^{\delta-f'_m-1}+\ldots)} \end{bmatrix} \tag{27}$$

From equation (26) and (24c) it is clear that the sufficient order of $\mathbf{T}_c(s)$ (sufficiency is a result of the special choice of $\mathbf{D}(s)$ of equal index set) is bounded by

$$n_c \leqq \sum_{i=1}^{m} (\delta-\delta_g)-(f'_1-\delta_g) \tag{28}$$

In light of the above theorem the following algorithm is constructed.

Algorithm 2

Step 1 Find the interactor $\xi_{\mathbf{T}'}(s)$ *of* $\mathbf{T}'(s)$.

Step 2 Factorized $(\xi_{\mathbf{T}'}(s))$ *as*,

$$(\xi_{\mathbf{T}'}(s))=\mathbf{N}_{c0}(s)\,\mathbf{N}_d(s)\,,$$

where $\mathbf{N}_d(s)$ is the diagonal right divisor of $(\xi_{\mathbf{T}'}(s))'$ of maximum column degree (i.e., the entries of $\mathbf{N}_d(s)$ will be monomials due to the special form of the interactor).

Step 3 Let $\bar{\mathbf{T}}_0(s)=\mathbf{T}(s)\,\mathbf{N}_{c0}(s)$. If $\mathbf{B}^*(\bar{\mathbf{T}}_0(s))$ is nonsingular then set $\mathbf{N}_{c_{min}}(s)=$ $=\mathbf{N}_{c0}(s)$ and go to (5).

Step 4 a— Compute k, the number of all the possible variety sets among the column indices of $\mathbf{N_d}(s)$.

b— Set $l=1$

c— Multiply each column of $\bar{\mathbf{T}}_0(s)$ by the s-power, equal to the corresponding element in the *lth* set to form $\bar{\mathbf{T}}_l(s)$. Actually this is equivalent to multiplying the columns of $\mathbf{N}_{c0}(s)$ by the same index-powers to form $\mathbf{N}_{cl}(s)$.

d— Find $\mathbf{B}^*(\bar{\mathbf{T}}_l(s))$. If $\mathbf{B}^*$ is singular go to (*e*), otherwise compute the precompensator order, designed according to the l*th* index set, after division by G.C.R.D. of $\mathbf{N}_{cl}(s)$ *and* $\mathbf{P}_{cl}(s)$.

e— Set $l+1$. If $l \leqq k$ go to (*c*) otherwise go to (5).

Step 5 For $n_{c_{min}}$, the minimum value of n_{c_l}, set $s^{\delta g}\mathbf{N_c}(s) \triangleq \mathbf{N}_{c_{min}}$, and compute $\mathbf{P_c}(s)$ in a similar way to that given in Theorem 4 (here f'_i and δg belong to $\mathbf{N}_{c_{min}}$) and STOP.

Remark. 3: Using Algorithm 2, there is no guarantee for the controllability of the precompensated system. The number of nonzero poles of the precompensator is $\bar{n}_{c_l} = \sum_{i=1}^{m} (\delta_i - f'_{i,l})$ and they may be assigned arbitrarily.

Example 5 (Cremer [3], Sinha [10])

$$T(s) = \begin{bmatrix} \dfrac{1}{s} & 0 & \dfrac{s+2}{s(s+1)} \\ \dfrac{s-3}{s(s+2)(s+3)} & \dfrac{1}{s+2} & \dfrac{2(s^2+3s-1)}{s(s+1)(s+2)} \\ 0 & \dfrac{1}{s+2} & \dfrac{2s+5}{(s+1)(s+2)} \end{bmatrix}$$

Step 1

$$\xi_{\mathbf{T}'}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ -s^2-s^3 & -3s^2-2s^3 & s^3 \end{bmatrix}$$

Step 2

$$(\xi_{\mathbf{T}'}(s))' = \underbrace{\begin{bmatrix} 1 & 0 & -1-s \\ 0 & 1 & -3-2s \\ 0 & 0 & s \end{bmatrix}}_{\mathbf{N}_{co}(s)} \underbrace{\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s^2 \end{bmatrix}}_{\mathbf{N_d}(s)}$$

Table 1

$\mathbf{B}^*$ and n_c for the different sets

The set	$\mathbf{B}^*$	n_c	The set	$\mathbf{B}^*$	n_c
(1, 0, 0)	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	–	(1, 1, 0)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	–
(1, 1, 1)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	–	(1, 0, 1)	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$	4
(1, 0, 2)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$	–	(1, 1, 2)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$	5
(0, 1, 0)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	–	(0, 1, 1)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	–
(0, 1, 2)	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$	–	(0, 0, 1)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$	5
(0, 0, 2)	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$	–			

Step 3

$$\bar{\mathbf{T}}_0(s) = \begin{bmatrix} \dfrac{1}{s} & 0 & \dfrac{-1}{s(s+1)} \\ \dfrac{s-3}{s(s+2)(s+3)} & \dfrac{1}{s+2} & \dfrac{s^2-10s+3}{s(s+1)(s+2)(s+3)} \\ 0 & \dfrac{1}{s+2} & \dfrac{-3}{(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{B}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{i.e., singular.}$$

Step 4 $k = 11$ and the sets are:
(1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 0, 1), (1, 0, 2), (1, 1, 2), (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 0, 1), and (0, 0, 2).
Table 1 shows $\mathbf{B}^*$ and n_c for the different sets. There are three sets with nonsingular $\mathbf{B}^*$. Actually the set (0, 0, 1) which gives a precompensator of order "5", is the direct application of *Theorem 4*.

Step 5 The set (1, 0, 1) yields the minimum value, $n_{c\min} = 4$, i.e.

$$\mathbf{N}_c(s) = \begin{bmatrix} s & 0 & -s(s+1) \\ 0 & 1 & -s(2s+3) \\ 0 & 0 & s^2 \end{bmatrix},$$

$$\mathbf{P}_c(s) = \begin{bmatrix} s(s+a) & 0 & 0 \\ 0 & (s+b)(s+c) & 0 \\ 0 & 0 & s^2 \end{bmatrix}$$

$$\mathbf{T}_c(s) = \begin{bmatrix} \dfrac{1}{s+a} & 0 & \dfrac{-(1+s)}{s} \\ 0 & \dfrac{1}{(s+b)(s+c)} & \dfrac{-(3+2s)}{s} \\ 0 & 0 & 1 \end{bmatrix}$$

The precompensator order given by the algorithm is equal 4, while that one, which can be obtained by using Wolovich's invertible algorithm, has order only 3, but here three poles are arbitrarily assigned, and the fourth one is at the origin.

We can extend the previous algorithm to the more general case of $p \neq m$, but full rank replaces nonsingularity.

Example 6

Consider the same system given in *Example 4*.

Step 1

$$\xi_T'(s) = \begin{bmatrix} 1 & 0 & 0 \\ -s & s^2 & 0 \\ -1 & 0 & s \end{bmatrix}$$

Step 2

$$\ddot{\xi}_{\mathbf{T}'}(s)' = \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}}_{\mathbf{N}_{\mathbf{co}}(s)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{N}_{\mathbf{d}}(s)}$$

Step 3

$$\bar{\mathbf{T}}_0(s) = \begin{bmatrix} \frac{s+1}{s} & \frac{-2s-1}{s(s+1)} & \frac{-3s-2}{s(s+2)} \\ \frac{1}{s} & \frac{-1}{s} & \frac{-1}{s} \end{bmatrix}$$

$$\mathbf{B}^*(\mathbf{T}_0(s)) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

It is clear that $\mathbf{B}^*$ is of full rank $p=2$.

$$\mathbf{N}_c(s) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} \text{ and } \mathbf{P}_c(s) = \begin{bmatrix} s+a & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix}$$

Step 5

$$\mathbf{T}_c(s) = \begin{bmatrix} \frac{1}{s+a} & \frac{-1}{s} & \frac{-1}{s} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So we achieve a precompensator of order "1" and equal to the minimum one.

The integrator compensator design

In some cases it is possible to design an integrator precompensator (with all its poles at the origin), which has an order less than that obtained by Algorithm 2. The following simple steps describe the method.

Algorithm 3

Step 1 Find $\mathbf{N}_c(s)$ as in *Theorem 4*

Step 2 Find $\mathbf{B}(s) \triangleq adj\,\mathbf{N}_c(s)$, and factorize it as:

$$\mathbf{B}(s) = s^{\delta_g}\hat{\mathbf{N}}_c(s).$$

where δ_g is g.c.d. of the elements of $\mathbf{B}(s)$.

Step 3 Set $\mathbf{T}_c(s) = \hat{\mathbf{N}}_c(s))^{-1}$. STOP

Example 7

For the same system given by *Example 5* (Cremer [3], Sinha [10])

$$\mathbf{N}_c(s) = \begin{bmatrix} s & 0 & -s-s^2 \\ 0 & s & -3s-2s^2 \\ 0 & 0 & s^2 \end{bmatrix}, \qquad \mathbf{B}(s) = \begin{bmatrix} s^3 & 0 & s^2+s^3 \\ 0 & s^3 & 3s^2+2s^3 \\ 0 & 0 & s^2 \end{bmatrix}$$

$$\hat{\mathbf{N}}_c(s) = \begin{bmatrix} s & 0 & 1+s \\ 0 & s & 3+2s \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and } \mathbf{T}_c(s) = \begin{bmatrix} s & 0 & 1+s \\ 0 & s & 3+2s \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

So, we can achieve decoupling condition using an integrator compensator of order 2 only.

5. Conclusion

Two methods have been presented for designing a precompensator $\mathbf{T}_c(s)$ of any given proper system $\mathbf{T}(s)$, such that $\mathbf{B}^*(\mathbf{T}(s)\mathbf{T}_c(s))$ will be of full rank. The first method is described in Algorithm 1 and based on *Theorem 1*, which gives a new necessary and sufficient condition of decoupling a system described in its minimal, controllable and observable form in the frequency domain. The second method applies the constructive proof of *Theorem 4* for precompensator design, which makes use of the interactor of the transposed system.

The methods presented here ensure arbitrary pole-assignement of the precompensator, but the order is not necessarily the minimum one. The methods have the advantage that they use only frequency domain approach without reference to the state-space description. The suggested methods can also be used for constructing an intermediate stage compensator for the output feedback decoupling purpose, where nonsingularity of $\mathbf{B}^*$ is one of the conditions necessary for the existence of such control law.

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