PHYSICAL INTERPRETATION OF INTERFACE AND BOUNDARY CONDITIONS OF ELECTROMAGNETIC FIELD

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Summary

When solving electromagnetic field problems, it is usual to introduce the electric and magnetic scalar and vector potentials. So, the solution of the Maxwell equations can be derived back to the solution of differential equation relative to the suitable potential. However, the boundary and interface conditions refer to the field intensities. The aim of the paper is to present and summarize the boundary and interface conditions relative to the potentials based on physical meaning, derived them from the conditions referring to the field intensities. Such an interpretation of boundary and interface conditions makes it easier to take them into account by the use of variational methods.

Introduction

Nowadays, numerical methods are widely used in electromagnetic field calculations. The methods use the scalar and vector potentials for the solution. A suitable partial differential equation referring to the potentials is to be solved and the interface and boundary conditions have to be satisfied. The interface and boundary conditions refer to the field intensities.

The aim of this paper is to state and summarize the interface and boundary conditions for the potentials based on a physical interpretation of those for electromagnetic field intensities.

The formulas of magnetic potentials rarely used in literature are also presented.

The physical interpretation of interface and boundary conditions makes it easier to take them into account by using variational methods.

The interface conditions

Let us consider a surface separating two regions of different material characteristics. The material characteristics of the regions are denoted by subscripts 1 and 2, respectively (Fig. 1). The field intensities have to fulfil the following interface conditions:

$$\mathbf{H}_1 \times \mathbf{n} = \mathbf{K}^e + \mathbf{H}_2 \times \mathbf{n} \tag{1}$$

$$\mathbf{E}_1 \times \mathbf{n} = \mathbf{K}^m + \mathbf{E}_2 \times \mathbf{n} \tag{2}$$

$$\mathbf{B}_1 \mathbf{n} = \sigma^m + \mathbf{B}_2 \mathbf{n} \tag{3}$$

$$\mathbf{D}_1 \mathbf{n} = \sigma^e + \mathbf{D}_2 \mathbf{n} \tag{4}$$

where $\mathbf{K}^e$ and $\mathbf{K}^m$ are the electric and magnetic surface current densities, while σ^e and σ^m are the electric and magnetic surface charge densities. The relevant

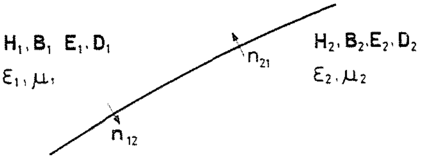

Fig. 1

components of the field intensities change abruptly owing to the surface current and charge densities. In time-varying case, (3) and (4) are the consequence of (1) and (2), therefore (1) and (2) only are independent. In the case of static and stationary electric field, (1) and (3) apply and in the case of static and stationary magnetic field, (2) and (4) have to be used.

If there are neither current nor charge densities on the separating surface, the appropriate components of the field intensities are continuous. This case can be taken as one with unknown surface current densities $\mathbf{K}^e$ and $\mathbf{K}^m$ on the separating surface and the tangential components of the magnetic and electric field intensities in the two regions equal those:

$$\mathbf{H}_1 \times \mathbf{n}_{12} = \mathbf{K}^e \tag{5}$$

$$\mathbf{E}_1 \times \mathbf{n}_{12} = \mathbf{K}^m \tag{6}$$

$$\mathbf{H}_2 \times \mathbf{n}_{21} = \mathbf{K}^e \tag{7}$$

$$\mathbf{E}_2 \times \mathbf{n}_{21} = \mathbf{K}^m \tag{8}$$

In the case of static and stationary electric and magnetic field, the same consideration can be made for the relevant charge densities. The interface conditions for the electric field are:

$$\mathbf{E}_1 \times \mathbf{n}_{12} = \mathbf{K}^m \tag{9}$$

$$\mathbf{D}_1 \mathbf{n}_{12} = \sigma^e \tag{10}$$

$$\mathbf{E}_2 \times \mathbf{n}_{21} = \mathbf{K}^m \tag{11}$$

$$\mathbf{D}_2 \mathbf{n}_{21} = \sigma^e \tag{12}$$

and for the magnetic field they are:

$$\mathbf{H}_1 \times \mathbf{n}_{12} = \mathbf{K}^e \tag{13}$$

$$\mathbf{B}_1 \mathbf{n}_{12} = \sigma^m \tag{14}$$

$$\mathbf{H}_2 \times \mathbf{n}_{21} = \mathbf{K}^e \tag{15}$$

$$\mathbf{B}_2 \mathbf{n}_{21} = \sigma^m \tag{16}$$

Boundary conditions

The solution of Maxwell equations are sought in a region V bounded by a closed surface S (Fig. 2). The field intensities of the internal and external regions are denoted by subscripts 1 and 2, respectively. In order to fulfil the interface conditions on the surface S, appropriate surface current and charge densities are imagined on S, as is known above. In the case when the external field is known and satisfies Maxwell equations, one surface excitation can be given independently on the surface and this is the boundary condition for the internal field.

There are boundary value problems where different kinds of excitation are given on the surface. The sections of the surface with different excitations are denoted by S_1 and S_2 (Fig. 3).

Thus, the boundary conditions are:
in time-varying case:

$$\mathbf{H}_1 \times \mathbf{n} = \mathbf{K}^e \text{ on the surface } S_1 \tag{17}$$

$$\mathbf{E}_1 \times \mathbf{n} = \mathbf{K}^m \text{ on the surface } S_2 \tag{18}$$

in the case of static and stationary electric field:

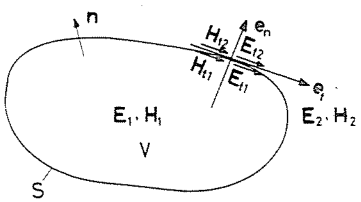

Fig. 2

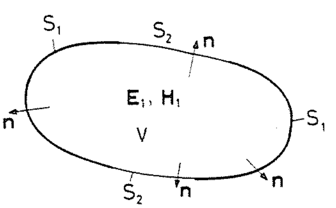

Fig. 3

$$\mathbf{D}_1 \times \mathbf{n} = \sigma^e \text{ on the surface } S_1 \tag{19}$$

$$\mathbf{E}_1 \times \mathbf{n} = \mathbf{K}^m \text{ on the surface } S_2 \tag{20}$$

in the case of static and stationary magnetic field:

$$\mathbf{H}_1 \times \mathbf{n} = \mathbf{K}^e \text{ on the surface } S_1 \tag{21}$$

$$\mathbf{B}_1 \mathbf{n} = \sigma^m \text{ on the surface } S_2 \tag{22}$$

Time-varying electromagnetic field is seen to be generated by a surface electric or magnetic current density. Static and stationary electric field is generated by a surface electric charge density or by a surface magnetic current density. Static and stationary magnetic field is generated by a surface electric current density or by a surface magnetic charge density.

Boundary conditions for the potentials

Generally, the electric or magnetic potentials are employed at the solution of Maxwell equations. Therefore, it is expedient to formulate the boundary conditions for the potentials as well.

Introducing the electric scalar and vector potentials, the field intensities are known to be expressed as:

$$\mathbf{B} = \operatorname{curl} \mathbf{A}^e \tag{23}$$

$$\mathbf{E} = -\operatorname{grad} \varphi^e - \frac{\partial \mathbf{A}^e}{\partial t} \tag{24}$$

where φ^e and $\mathbf{A}^e$ are the electric scalar and vector potentials, respectively.

In the case of time-varying electromagnetic field, the boundary conditions for the electric potentials are:

$$\operatorname{curl} \mathbf{A}^e \times \mathbf{n} = \mu \mathbf{K}^e \text{ on the surface } S_1 \tag{25}$$

$$\left(-\operatorname{grad} \varphi^e - \frac{\partial \mathbf{A}^e}{\partial t}\right) \times \mathbf{n} = \mathbf{K}^m \text{ on the surface } S_2 \tag{26}$$

In the case of static and stationary electric field, the boundary conditions for the electric potentials are:

$$-\varepsilon \operatorname{grad} \varphi^e \times \mathbf{n} = \sigma^e \quad \text{on the surface } S_1 \tag{27}$$

$$-\operatorname{grad} \varphi^e \times \mathbf{n} = \mathbf{K}^m \quad \text{on the surface } S_2 \tag{28}$$

If the potential φ_0^e is known on the surface S_2, there exists a unique function $\mathbf{f}(\varphi_0^e)$ so that $\mathbf{K}^m$ can be expressed:

$$\mathbf{f}(\varphi_0^e)=\mathbf{K}^m \tag{29}$$

This means that instead of (28)

$$\varphi^e=\varphi_0^e \text{ on the surface } S_2 \tag{30}$$

can be used.

In the case of static and stationary magnetic field, the boundary conditions for the electric potentials are:

$$\operatorname{curl} \mathbf{A}^e \times \mathbf{n}=\mu\mathbf{K}^e \quad \text{on the surface } S_1 \tag{31}$$

$$\operatorname{curl} \mathbf{A}^e \times \mathbf{n}=\sigma^m \quad \text{on the surface } S_2 \tag{32}$$

If the tangential components $\mathbf{A}_{t0}^e$ of the vector potential are known on the surface S_2, there exists a unique function $f(\mathbf{A}_{t0}^e)$ so that σ^m can be expressed:

$$f(\mathbf{A}_{t0}^e)=\sigma^m \tag{33}$$

This means that instead of (32)

$$\mathbf{A}_t^e=\mathbf{A}_{t0}^e \text{ on the surface } S_2 \tag{34}$$

can be used.

Introducing the magnetic scalar and vector potentials, the field intensities are known to be expressed as:

$$\mathbf{D}=\operatorname{curl} \mathbf{A}^m \tag{35}$$

$$\mathbf{H}=-\operatorname{grad} \varphi^m+\frac{\partial \mathbf{A}^m}{\partial t} \tag{36}$$

where φ^m and $\mathbf{A}^m$ are the magnetic scalar and vector potentials, respectively.

In the case of time-varying electromagnetic field, the boundary conditions for the magnetic potentials are:

$$\left(-\operatorname{grad} \varphi^m+\frac{\partial \mathbf{A}^m}{\partial t}\right)\times \mathbf{n}=\mathbf{K}^e \quad \text{on the surface } S_1 \tag{37}$$

$$\operatorname{curl} \mathbf{A}^m=\varepsilon\mathbf{K}^m \quad \text{on the surface } S_2 \tag{38}$$

In the case of static and stationary electric field, the boundary conditions for the magnetic potentials are:

Table 1

<table>
<tr><th colspan="2" rowspan="3">Type of potential</th><th colspan="4">Type of excitation</th></tr>
<tr><th colspan="2">electrical excitation</th><th colspan="2">magnetic excitation</th></tr>
<tr><th>σ^e</th><th>$\mathbb{K}^e$</th><th>σ^m</th><th>$\mathbb{K}^m$</th></tr>
<tr><td rowspan="2">Electric potentials</td><td>Time varying case</td><td></td><td rowspan="2">$\operatorname{curl} \mathbf{A}^e \times \mathbf{n} = \mu \mathbb{K}^e$</td><td></td><td>$\left(-\operatorname{grad} \varphi^e - \frac{\partial \mathbf{A}^e}{\partial t}\right) \times \mathbf{n} = \mathbb{K}^m$</td></tr>
<tr><td>Static and stationary case</td><td>$-\varepsilon \operatorname{grad} \varphi^e \mathbf{n} = \sigma^e$</td><td>$f(\mathbf{A}_{t0}^e) = \sigma^m$
$\mathbf{A}_t^e = \mathbf{A}_{t0}^e$</td><td>$\mathbf{f}(\varphi_0^e) = \mathbb{K}^m$
$\varphi^e = \varphi_0^e$</td></tr>
<tr><td rowspan="2">Magnetic potentials</td><td>Time varying case</td><td colspan="2">$\left(-\operatorname{grad} \varphi^m + \frac{\partial \mathbf{A}^m}{\partial t}\right) \times \mathbf{n} = \mathbb{K}^e$</td><td></td><td rowspan="2">$\operatorname{curl} \mathbf{A}^m \times \mathbf{n} = \varepsilon \mathbb{K}^m$</td></tr>
<tr><td>Static and stationary case</td><td>$f(\mathbf{A}_{t0}) = \sigma^e$
$\mathbf{A}_t^m = \mathbf{A}_{t0}^m$</td><td>$\mathbf{f}(\varphi_0^m) = \mathbb{K}^e$
$\varphi^m = \varphi_0^m$</td><td>$\mu \operatorname{grad} \varphi^m \mathbf{n} = \sigma^m$</td></tr>
</table>

$$\text{curl } \mathbf{A}^m \times \mathbf{n} = \sigma^e \qquad \text{on the surface } S_1 \tag{39}$$

$$\text{curl } \mathbf{A}^m \times \mathbf{n} = \varepsilon \mathbf{K}^m \qquad \text{on the surface } S_2 \tag{40}$$

As seen above

$$\mathbf{A}_t^m = \mathbf{A}_{t0}^m \quad \text{on the surface } S_1 \tag{41}$$

can be used instead of (39), if the tangential component $\mathbf{A}_{t0}^m$ of the magnetic vector potential is known on the surface S_1.

In the case of static and stationary magnetic field, the boundary conditions for the magnetic potentials are:

$$\text{grad } \varphi^m \times \mathbf{n} = \mathbf{K}^e \qquad \text{on the surface } S_1 \tag{42}$$

$$\mu \text{ grad } \varphi^m \times \mathbf{n} = \sigma^m \qquad \text{on the surface } S_2 \tag{43}$$

Instead of (42),

$$\varphi^m = \varphi_0^m \quad \text{on the surface } S_1 \tag{44}$$

can be used if the magnetic scalar potential φ_0^m is known on the surface S_1.

The results are summarized in Table 1.

It can be seen that in the case when electric excitations are given and the electric potentials are used or magnetic excitations are given and the magnetic potentials are used, the boundary conditions are expressed by the derivatives of the potentials. This type of boundary condition is called Neumann boundary condition in the literature. In the case when electric excitations are given and the magnetic potentials are used or magnetic excitations are given and electric potentials are used, the boundary conditions can be expressed by the potentials themselves. This type of boundary condition is called Dirichlet boundary condition in the literature.

References

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