APPLICATION OF R-FUNCTIONS TO THE DETERMINATION OF ELECTRIC FIELD IN PIECEWISE HOMOGENEOUS MEDIUM

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Summary

The paper concerns the approximate determination of the electric field in piecewise homogeneous planar regions. The bounding curve of the planar region as well as the curves separating the media of different permittivity are deemed to be approximated by curves given in analytical form. The potential values of the electrodes and the permittivities of the dielectrics are taken to be given.

The method presented determines the potential function with the aid of variational calculus. The satisfaction of the conditions imposed on the potential function is obtained by constructing a function satisfying the Dirichlet boundary conditions on the boundary of the planar region with the aid of R-functions, and by subsequently supplementing this with a term ensuring the satisfaction of the interface conditions on the boundary of the layers.

The application of the method is illustrated by an example.

Introduction

In a previous paper [6] the determination of the electric field in homogeneous medium has been dealt with. The bounding curve of the planar region examined has been deemed to be approximated by curves given in analytical form. The potential values of the electrodes have been taken to be given. The potential function satisfying the Laplace equation approximately and the boundary conditions exactly have been determined by variational method. The satisfaction of the boundary conditions was ensured by the employment of R-functions.

The above procedure will be extended in the present paper to the case of piecewise homogeneous medium. Assuming the curves separating the media of different permittivity to be also given in analytical form and supplementing the potential function of [6] by a suitably selected term, a potential function exactly satisfying the interface conditions along the boundaries of the layers is obtained. The paper concerns the determination of this supplementing term.

Satisfaction of the interface conditions

The permittivities in the planar regions shown in Fig. 1 surrounded by electrodes are ε_1 and ε_2, respectively. The potentials Φ_1 and Φ_2 of electrodes are given.

Let Γ_1 denote the curve with potential Φ_1, Γ_2 the one with potential Φ_2, Ω_1 and Ω_2 the subregions of permittivities ε_1 and ε_2, Γ_s the curve separating the subregions and $\mathbf{v}$ the normal unit vector pointing towards subregion Ω_1.

The potential function $\Phi(x, y)$ satisfying the Laplace equation

$$\Delta\Phi(x, y)=0 \tag{1}$$

in the planar region surrounded by the electrodes is to be determined at the boundary conditions

$$\text{(a)} \qquad \Phi(x, y)\,|_{\Gamma_1}=\Phi_1, \quad \Phi(x, y)\,|_{\Gamma_2}=\Phi_2 \tag{2}$$

and the interface conditions [1]

(b) $\Phi(x, y)$ is continuous on Γ_s

and (c) $$\varepsilon_1 \mathbf{v}\,\mathbf{grad}\,\Phi(x, y)\,|_{\Gamma_s \in \Omega_1}=\varepsilon_2 \mathbf{v}\,\mathbf{grad}\,\Phi(x, y)\,|_{\Gamma_s \in \Omega_2} \tag{3}$$

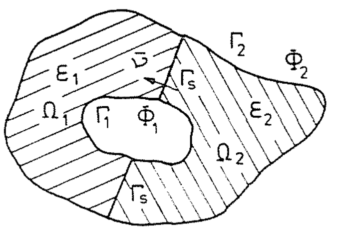

Fig. 1

Applying the variational method, the solution of (1) in case of n_r layers ($n_r=2$) is the potential function $\Phi(x, y)$ extremizing the functional

$$W(\Phi)=\sum_{i=1}^{n_r}\int_{\Omega_i}\frac{\varepsilon_i}{2}\,\mathbf{grad}^2\,\Phi(x, y)\,d\Omega_i \tag{4}$$

at the conditions (a), (b) and (c) [2]. $\Phi(x, y)$ is approximated by a function $\Phi_n(x, y)$ according to Ritz's method. The approximating function satisfies the boundary and interface conditions if it is constructed as follows:

At first, a function $\Omega_D(x, y)$ satisfying the Dirichlet boundary condition (a) is formed as presented in [6]. Thereafter, it is supplemented by a further term to ensure the satisfaction of the interface conditions (b) and (c):

$$\Phi_n(x, y) = \Phi_D(x, y) + \tau(x, y) \cdot \Phi_s(x, y) . \tag{5}$$

In case the function $\tau(x, y)$ fulfills the condition

$$\tau(x, y) \mid_{\Gamma_1, \Gamma_2, \Gamma_s} = 0 \tag{6}$$

the potential function (5) satisfies the conditions (a) and (b) at any choice of $\Phi_s(x, y)$. If $\tau(x, y)$ further satisfies the condition

$$\mathbf{v}\ \mathbf{grad}\ \tau(x, y) \mid_{\Gamma_s} = \begin{cases} 1 & \text{if} \quad \Gamma_s \in \Omega_1, \\ -1 & \text{if} \quad \Gamma_s \in \Omega_2 \end{cases} \tag{7}$$

$\Phi_s(x, y)$ can be determined from the condition (c) imposed on the potential function (5). Hence, the potential function satisfying the conditions (a), (b) and (c) is as follows:

$$\Phi_n(x, y) = \Phi_D(x, y) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \tau(x, y)\ \mathbf{v}\ \mathbf{grad}\ \Phi_D(x, y) \tag{8}$$

$\Phi_D(x, y)$ is sought in accordance with [6] as the sum of a known function $\Phi_\delta(x, y)$ satisfying the Dirichlet boundary condition (a) and of an unknown function $\Phi_\alpha(x, y)$ vanishing on Γ_1 and Γ_2:

$$\Phi_D(x, y) = \Phi_\delta(x, y) + \Phi_\alpha(x, y) . \tag{9}$$

The unknown function $\Phi_\alpha(x, y)$ is approximated by the linear combination of the first n elements of an entire function set:

$$\Phi_\alpha(x, y) = \sum_{k=1}^{n} a_k f_k(x, y) . \tag{10}$$

The unknown coefficients a_k $(k = 1, 2, \ldots, n)$ can be obtained as the solution of the linear set of equations derived by equating the partial derivatives of the functional (4) with respect to the coefficients with zero [2], [5].

Application of the method

Let us determine the potential function in the cylindrical capacitor shown in Fig. 2 by the application of the above method. The terms deemed to be known in (8) are given with the aid of R-functions [3], [4]. The R-functions described by Rvachev are known to express the logical relationships of conjunction, disjunction and negation with the aid of analytical functions. Hence, the function $\tau(x, y)$ satisfying the conditions (6), (7) may be the following:

$$\tau(x, y) = |w_s(x, y)| \frac{w_D^2(x, y)}{w_D^2(x, y) + w_s^2(x, y)}, \tag{11}$$

where

$$w_D(x, y) = (r_2^2 - r^2) \wedge (r^2 - r_1^2) \tag{12}$$

and

$$w_s(x, y) = a - x\,. \tag{13}$$

The term $\Phi_\delta(x, y)$ in (9) is selected as follows:

$$\Phi_\delta(x, y) = \frac{\bar{\Phi}_1 (r_2^2 - r^2) + \bar{\Phi}_2 (r^2 - r_1^2)}{r_2^2 - r_1^2}\,. \tag{14}$$

The elements $f_k(x, y)$ $(k = 1, 2, \ldots, n)$ of the approximating function set have been selected as Chebishev polinomials with the symmetry observed:

$$f_k(x, y) = T_i(x) \cdot T_{2j}(y) \cdot w_D(x, y)\,, \quad i,j = 0, 1, 2, \ldots, \tag{15}$$

where $T_i(x)$ is the i-th order Chebishev polinomial of x, $T_{2j}(y)$ is the 2j-th order Chebishev polinomial of y and $w_D(x, y)$ is the R-function given in (12).

The potential function with $n=6$, at the relative values $r_1/r_2 = 0.4$; $a/r_2 = 0.2$; $\varepsilon_1/\varepsilon_2 = 1/10$; $\bar{\Phi}_1 = 1.0$; $\bar{\Phi}_2 = 0$ has been determined. The capacitance of the capacitor as calculated from $\bar{\Phi}_n(x, y)$ was $C_n = 0.291659$ nF/m. The capacitance obtained by elementary approximations is $C_{\text{elementary}} = 0.273\,345$ nF/m. The relative difference of the two values is 6.7%.

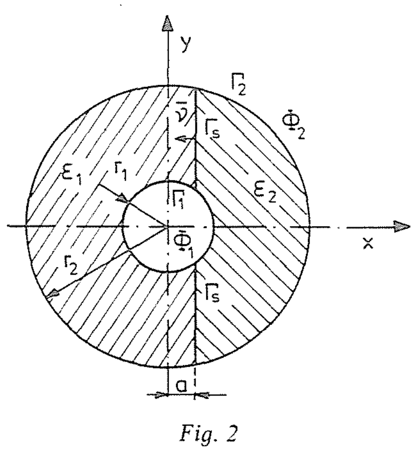

Fig. 2

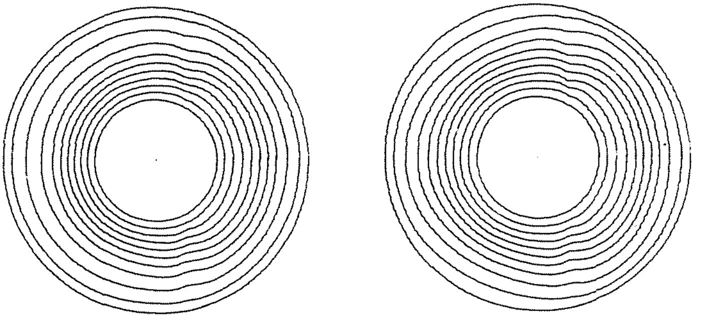

Fig. 3

The potential function has been determined at two different rations $\varepsilon_1/\varepsilon_2$ ($\varepsilon_1/\varepsilon_2 = 1/10$ and $\varepsilon_1/\varepsilon_2 = 1/80$). Their equipotential lines in the two cases with $n = 6$ and $d\Phi/\Phi_1 = 0.1$ are shown in Fig. 3.

The calculations have been carried out on a desk calculator EMG 666. The equipotential curves have been drawn on a plotter NE-2000 connected to the desk calculator EMG 666.

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