ELECTRICAL FIELD IN CYLINDRICAL SYMMETRICAL LAYOUTS BY VARIATIONAL CALCULUS

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To determine the static electrical field in a charge-free ($\rho = 0$) region with a homogeneous medium the potential function satisfying the Laplace equation

$$\Delta \Phi = 0 \tag{1}$$

is to be derived [1], [2]. It is known [3], [4] that one way of solving Laplace's equation by variational calculus is to reduce it to the determination of the extremum of a functional. Accordingly, in a volume v bounded by lines of force and equipotential surfaces the potential function Φ extremizing the functional

$$I(\Phi) = \frac{1}{2} \int_v \operatorname{grad}^2 \Phi \, dv \tag{2}$$

among the functions satisfying the prescribed boundary conditions will also solve Laplace equation (1). Thus, the electrical field problem is equivalent to the determination of the function extremizing the functional (2) and satisfying the prescribed boundary conditions. One way to satisfy the boundary conditions is the application of logical R-functions.

In the following the variational method applied in [7] to the solution of planar problems will be extended to the determination of the electrical field in cylidrical symmetrical layouts. The medium is presumed to be homogeneous. In a plane $\varphi = \text{const.}$ of the cylindrical arrangement the bounding curve of the planar region bounded by lines of force and equipotential curves is assumed to be given or approximated analytically. The functions necessary for satisfying of the Dirichlet boundary conditions are constructed in the form of R-functions known from the literature [5], [6]. The method is illustrated on an example.

The examination of cylindrical symmetrical layouts is advantageously carried out in a cylindrical coordinate system r, φ, z with the axis z coinciding with the axis of the arrangement. Now, since the geometry is independent of the

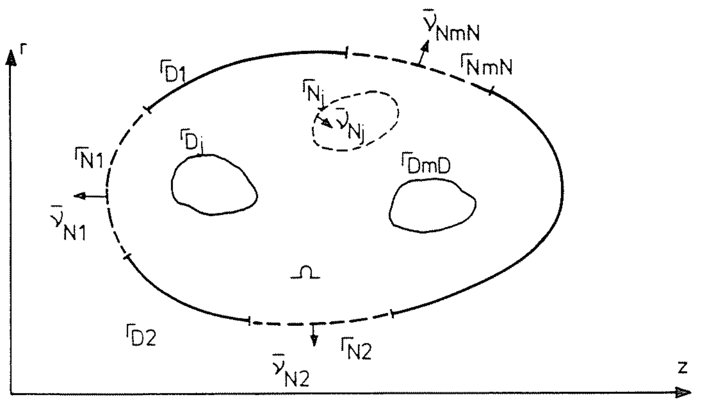

Fig. 1

angle φ, the electrical field has no azimuthal component. This means that the potential function depends only on radius r and coordinate z, i.e.

$$\frac{\partial}{\partial \varphi} = 0 . \tag{3}$$

Accordingly, in order to determine the electrical field in cylindrical symmetrical layouts it suffices to examine a plane $\varphi = \text{const.}$ of the arrangement, a planar region Ω bounded by lines of force and equipotential curves (Fig. 1). The equipotential curves bounding the planar region Ω are $\Gamma_{\mathrm{D}i}$, $i = 1, 2, \ldots, m_D$. The value of the potential is given on these curves:

$$\Phi(r, z)|_{\Gamma_{\mathrm{D}i}} = \Phi_i(r, z) , \qquad i = 1, 2, \ldots, m_D . \tag{4}$$

This is a Dirichlet-type boundary condition. Curves Γ_{Nj}, $j = 1, 2, \ldots, m_N$ are the lines of forces bounding the planar region. On these sections the prescribed boundary condition is of Neumann-type:

$$\bar{\nu} \cdot \operatorname{grad} \Phi(r, z)|_{\Gamma_{Nj}} = 0 , \qquad j = 1, 2, \ldots, m_N , \tag{5}$$

where $\bar{\nu}$ is the normal unit vector pointing away from the region examined. Taking the cylindrical symmetry into account the potential function satisfying two-dimensional Laplace equation

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Phi(r, z)}{\partial r} \right) + \frac{\partial^2 \Phi(r, z)}{\partial z^2} = 0 \tag{6}$$

is to be determined. Since the potential function is independent of the angle φ of the coordinate system, the integration by φ between the bounds 0 and 2π can be carried out, the elementary volume being

$$\mathrm{d}v = r\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}z\,. \tag{7}$$

Hence, the potential function $\Phi(r, z)$ extremizes the functional

$$I(\Phi) = \pi \iint_{\Omega} \left[\left(\frac{\partial \Phi(r, z)}{\partial r} \right)^2 + \left(\frac{\partial \Phi(r, z)}{\partial z} \right)^2 \right] r\,\mathrm{d}\Omega\,, \tag{8}$$

where

$$\mathrm{d}\Omega = \mathrm{d}r\,\mathrm{d}z\,. \tag{9}$$

The potential function zeroing the first variation of functional (8) satisfies the homogeneous Neumann-type boundary condition (5) as natural boundary condition. The Dirichlet-type boundary condition (4) can be satisfied by separating the potential function into two terms [4], [1]:

$$\Phi(r, z) = \Phi_\delta(r, z) + \Phi_\alpha(r, z)\,. \tag{10}$$

$\Phi_\delta(r, z)$ is a suitably chosen function, continuous and at least twice differentialable in the bounded planar region Ω. $\Phi_\delta(r, z)$ satisfies the prescribed Dirichlet-type boundary condition (4) on the bounding curves $\Gamma_{\mathrm{D}i}$, $i = 1, 2, \ldots, m_D$:

$$\Phi_\delta(r, z)|_{\Gamma_{\mathrm{D}i}} = \Phi_i(r, z)\,, \qquad i = 1, 2, \ldots, m_D\,. \tag{11}$$

The unknown term $\Phi_\alpha(r, z)$ of the potential function satisfies a homogeneous Dirichlet-type boundary condition on the bounding curves $\Gamma_{\mathrm{D}i}$, $i = 1, 2, \ldots, m_D$:

$$\Phi_\alpha(r, z)|_{\Gamma_{\mathrm{D}i}} = 0\,, \qquad i = 1, 2, \ldots, m_D\,. \tag{12}$$

According to Ritz's process, $\Phi_\alpha(r, z)$ can be approximated by a linear combination of the first n terms of a function set:

$$\Phi_\alpha(r, z) \cong \sum_{k=1}^{n} a_k f_k(r, z) w_D(r, z)\,, \tag{13}$$

where $f_k(r, z) w_D(r, z)$ is the k-th element of the approximating function set. $w_D(r, z)$ is a positive, continuous and at least twice differentialable function in the studied region Ω. Function $w_D(r, z)$ has to be selected so as to satisfy the boundary condition (12) prescribed for $\Phi_\alpha(r, z)$:

$$w_D(r, z)|_{\Gamma_{Di}} = 0\,, \qquad i = 1, 2, \ldots, m_D\,, \tag{14}$$

and on the sections Γ_{Nj}, $j = 1, 2, \ldots, m_N$ (if any) to be of arbitrary nonzero value. In Eq. (13), $f_k(r, z)$ is the k-th element of an entire function set defined in the studied planar region, while a_k is the k-th coefficient in the approximating function.

In order to determine the potential function, functions $\Phi_\delta(r, z)$ and $w_D(r, z)$ satisfying condition (11) and (14), resp., are to be derived. These functions are constructed as shown in [7], with the aid of the R-functions developed by Rvachev [5], [6] for the description of planar regions.

Taking Eq. (13) into account, the function approximating the potential function (10) satisfying the conditions (4) and (5) is the approximate solution of Laplace equation (6) at parameters a_k, $k = 1, 2, \ldots, n$ zeroing the first derivatives of integral (8). This condition yields a set of linear algebraic equations for coefficients a_k, $k = 1, 2, \ldots, n$, permitting their computation [4], [1].

Application of the method

Let us now determine the electrical field of a cylindrical symmetrical layout by means of the above method. A planar section $\varphi = \text{const.}$ of the arrangement is shown in Fig. 2. The potential function $\Phi(r, z)$ satisfying Laplace equation (6) in the plane r, z has to be determined. The planar region bounded by lines of force and equipotential curves is once connected. Sections Γ_{D1}, Γ_{D2} and Γ_{D3} of the bounding curve are equipotential, section Γ_{N1} is a line of force. The boundary conditions for the potential are the following:

$$\begin{aligned} \Phi(r, z)|_{\Gamma_{D1}} &= \Phi_1 = 0\,, \\ \Phi(r, z)|_{\Gamma_{D2}} &= \Phi_2 = \Phi_0\,, \\ \Phi(r, z)|_{\Gamma_{D3}} &= \Phi_3 = 0\,, \end{aligned} \tag{15}$$

and

$$\bar{\nu} \cdot \text{grad}\, \Phi(r, z)|_{\Gamma_{N1}} = 0\,. \tag{16}$$

The boundary conditions are of Dirichlet-type on curves Γ_{D1}, Γ_{D2} and Γ_{D3} and of Neumann-type on curve Γ_{N1}.

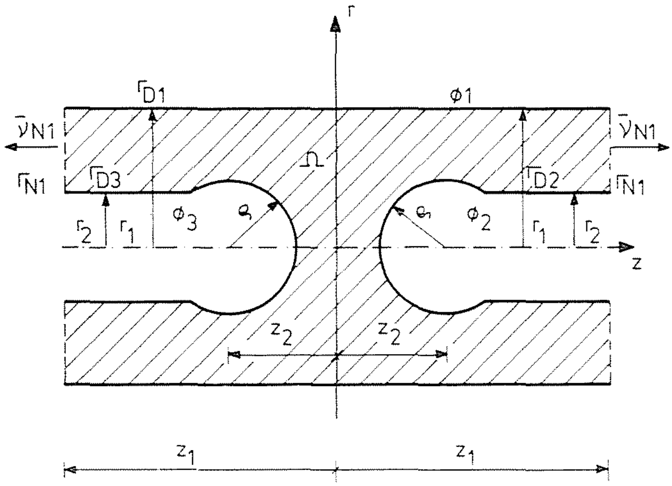

Fig. 2

Using variational principles, the potential function satisfying conditions (15) and (16) and approximately solving the Laplace equation (6) is

$$\Phi_n(r, z) = \Phi_\delta(r, t) + \sum_{k=1}^{n} a_k f_k(r, z) w_D(r, z) \tag{17}$$

with (10) and (13) taken into account. For the construction of functions $\Phi_\delta(r, z)$ and $w_D(r, z)$ the planar region shown in Fig. 2 is obtained from regions Ω_{D1} and Ω_{D2} associated with curves $\Gamma_{D1} \cup \Gamma_{D3}$ and Γ_{D2} (Fig. 3). With

$$\Omega_{11}(r_1^2 - r^2 \geq 0),$$

$$\Omega_{12}(r^2 - r_2^2 \geq 0),$$

$$\Omega_{13}(r^2 + (z + z_2)^2 - \rho^2 \geq 0),$$

$$\Omega_{14}(z + z_2 \geq 0)$$

as elementary regions, region Ω_{D1} (Fig. 3a) is the following:

$$\Omega_{D1} = (\Omega_{12} \cup \Omega_{14}) \cap \Omega_{13} \cap \Omega_{11} \,. \tag{18}$$

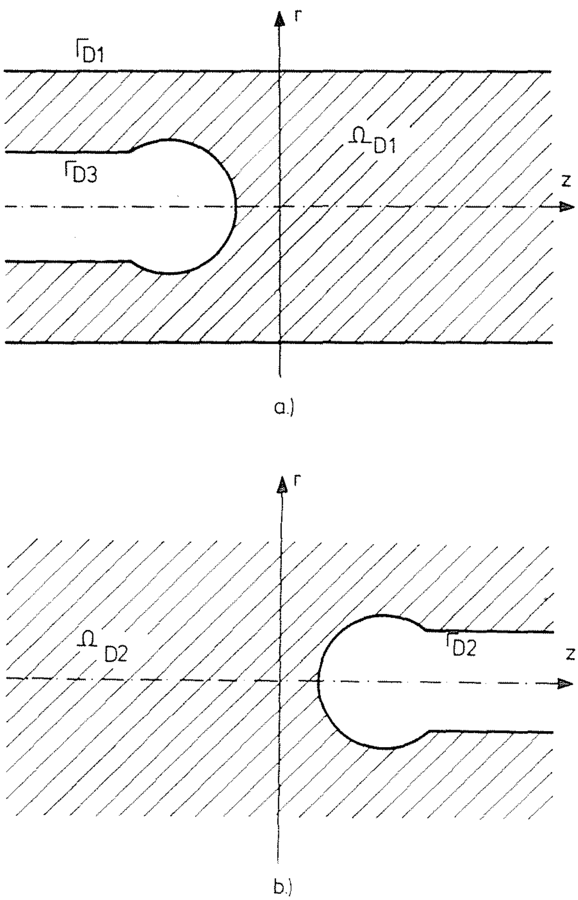

Fig. 3

The R-functions describing the elementary regions of Ω_{D1} are the following [5], [6]:

$$w_{11}(r, z) = r_1^2 - r^2 ,$$

$$w_{12}(r, z) = r^2 - r_2^2 ,$$

$$w_{13}(r, z) = r^2 + (z + z_2)^2 - \rho^2 ,$$

$$w_{14}(r, z) = z + z_2 .$$

In accordance with (18), Ω_{D1} is described by the R-function

$$w_{D1}(r, z) = [w_{12}(r, z) \vee w_{14}(r, z)] \wedge w_{13}(r, z) \wedge w_{11}(r, z) . \tag{19}$$

With

$$\Omega_{21}(r^2 - r_2^2 \geq 0),$$

$$\Omega_{22}(r^2 + (z - z_2)^2 - \rho^2 \geq 0),$$

$$\Omega_{23}(z - z_2 \geq 0)$$

as elementary regions, the region Ω_{D2} is the following:

$$\Omega_{D2} = (\Omega_{21} \cup \Omega_{23}) \cap \Omega_{22}. \tag{20}$$

The elementary regions of Ω_{D2} are described by the R-functions

$$w_{21}(r, z) = r^2 - r_2^2,$$

$$w_{22}(r, z) = r^2 + (z - z_2)^2 - \rho^2,$$

$$w_{23}(r, z) = z - z_2.$$

The *R*-function describing the region Ω_{D2} is

$$w_{D2}(r, z) = [w_{21}(r, z) \vee w_{23}(r, z)] \wedge w_{22}(r, z). \tag{21}$$

For the layout shown in Fig. 2, function $w_D(r, z)$ satisfying condition (14) is

$$w_D(r, z) = w_{D1}(r, z) \wedge w_{D2}(r, z), \tag{22}$$

while function $\Phi_\delta(r, z)$ satisfying condition (15) is

$$\Phi_\delta(r, z) = \Phi_0 \frac{w_{D1}(r, z)}{w_{D1}(r, z) + w_{D2}(r, z)}. \tag{23}$$

The elements of the approximating function have been chosen from the function set consisting of the products of the Chebishev polynomials $T_i(r)$, $i = 0, 1, 2, \ldots, n_1$ and $T_j(z)$, $j = 0, 1, 2, \ldots, n_2$ depending on r and z, respectively. Since due to the cylindrical symmetry, the electrical field intensity has no radial component at $r = 0$ in the arrangement seen in Fig. 2, the elements of the approximative function set have been selected to satisfy this condition.

In order to make the examination complete, a layout has been at first examined with an electrical field having a singularity along the intersection of the cylinder of radius r_2 and the sphere of radius $\rho (r_2 \neq \rho)$. In this case, the

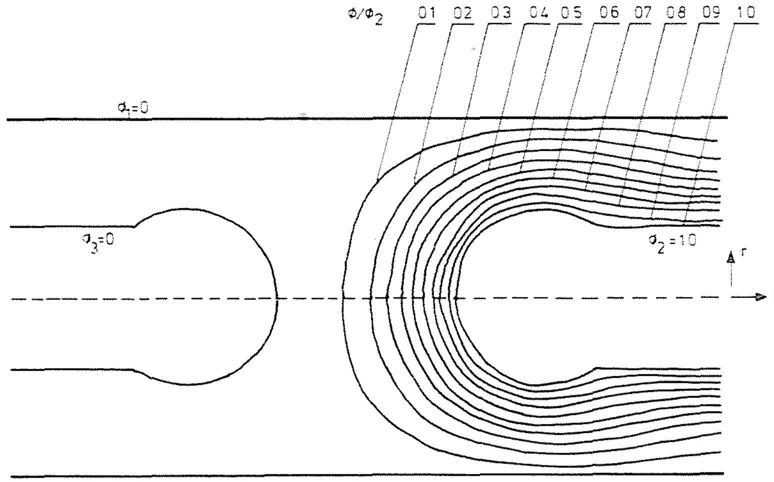

Fig. 4

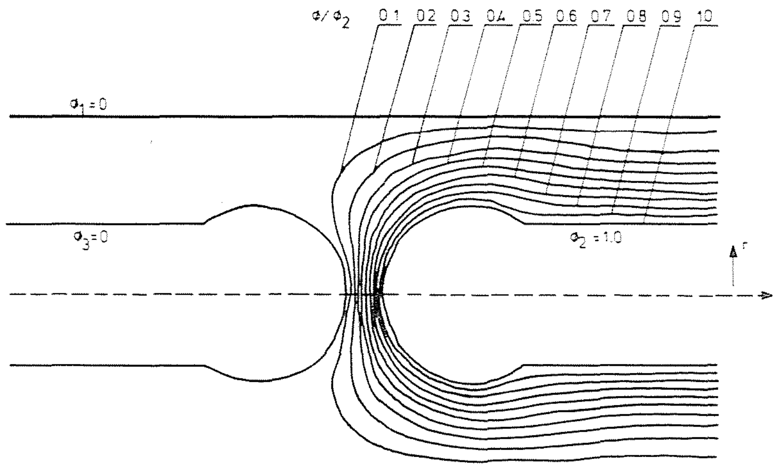

Fig. 5

equipotential lines of the potential function approximated by

$$\sum_{k=1}^{n} a_k f_k(r,z) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij} T_{2i}(r/r_1) T_j(z/z_1) \tag{24}$$

($n=6$, $n_1=1$, $n_2=2$) have been plotted in Figs 4 and 5 for electrode distances $z_2/z_1=0{,}5$ and $z_2/z_1=0.3$, respectively, with the geometrical ratios $r_1/z_1=0.5$, $r_2/z_1=0.20$, $\rho/z_1=0.25$.

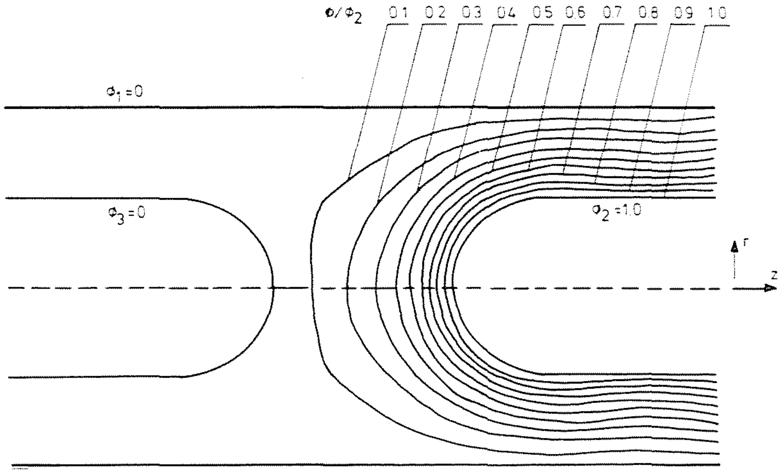

Fig. 6

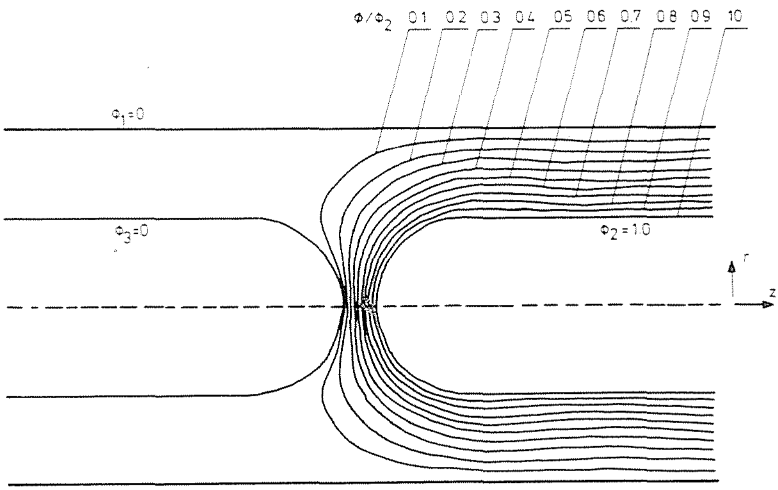

Fig. 7

In the case $r_2 = \rho$ where the electrical field intensity is not of extreme value along the intersection of the cylinder and the sphere, the approximation

$$\sum_{k=1}^{n} a_k f_k(r, z) = \sum_{j=0}^{n_2} T_j(z/z_1)\left[a_j + r^2 \sum_{i=0}^{n_1} a_{ji} T_i(r/r_1)\right] \tag{25}$$

has been used with $n = 20$ ($n_1 = 2$, $n_2 = 4$), and the equipotential lines of the potential function (17) have been plotted in Figs 6 and 7 with unchanged geometrical dimensions and two different electrode distances.

The numerical experience made in the solution of the problem indicates that if the R-functions applied to the construction of the potential function have their derivatives in a finite number of points on the bounding curve of the region defined, irrespective of whether the electrical field intensity is of extreme value in these points, these points should be excluded from the investigation. The potential function is distorted in the neighbourhood of these singular points. By increasing the number of terms in the approximation this distortion can be limited to a narrow vicinity of the singular points, but it can't be avoided.

Summary

The paper deals with the variational solution of static and stationary fields in cylindrical symmetrical layouts of a homogeneous medium. An approximate solution of the two-dimensional Laplace equation exactly satisfying the prescribed Dirichlet-type boundary conditions is constructed in analytical form. The functions needed for satisfying the boundary conditions on curves described at least piecewise analytically are constructed with the aid of R-functions. The application of the method is illustrated on an example.

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