ATTRACTORS OF SYSTEMS CLOSE TO THE LIÉNARD'S EQUATION

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1. Introduction

Liénard's equation is valid in a wide range of applications, first of all in electrical circuitry. In what follows, it will be studied in the form

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u) = F(t, u, \dot{u}), \tag{1}$$

where dot denotes differentiation with respect to t, $\varphi \in C^2[\mathbb{R}, \mathbb{R}]$; $\psi \in C^2[\mathbb{R}, \mathbb{R}]$, $F \in C^1[\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}]$, moreover

(i) φ is an even function such that $\varphi(0) > 0$, and

$$\Phi(u) = \int_0^u \varphi(s)\,\mathrm{d}s$$

is monotonous decreasing and has a single positive zero point $u^* > 0$, with

$$\Phi(u) > 0 \quad \text{for} \quad 0 < u < u^*, \quad \text{and} \quad \Phi(u) < 0 \quad \text{for} \quad u > u^*.$$

(ii) ψ is an odd function such that $\psi(0) = 0$, $\dot{\psi}(0) > 0$, and $\psi(u) > 0$ for $u > 0$.

(iii) $$-\int_0^\infty \varphi(u)\,\mathrm{d}u = \int_0^\infty \psi(u)\,\mathrm{d}u = \infty$$

(iv) there exists an $\eta > 0$ such that

$$|F(t, u, \dot{u})| < \eta, \quad \text{for} \quad (t, u, \dot{u}) \in \mathbb{R}^+ \times \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is an open set containing the origin. Under the above conditions, for the equation

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u) = 0 \tag{2}$$

the origin is known to be an asymptotically stable equilibrium, and its region of attractivity is an open region inside the path of the non-constant periodic solution [3].

The aim of this paper is to apply the estimates of M. FARKAS [2], for giving an explicit upper estimate of the attractor and a lower estimate to its region of attractivity for Eq. (1). For this reason the next items will repeat the results of M. FARKAS for the general case, and apply these results for Eq. (1).

The existence of the attractor is ensured by theorems due to T. YOSHIZAWA [1].

2. The attractor and the region of attractivity of a non-autonomous system close to an autonomous one

Assume $\Omega \subset \mathbb{R}^n$ to be an open set containing the origin, $\mathbb{R}^+ = [0, \infty)$ and

$$f \in C^\circ[\mathbb{R}^+ \times \Omega, \mathbb{R}^n],\ f'_x \in C^\circ[\mathbb{R}^+ \times \Omega, \mathbb{R}^n],\ g \in C^2[\Omega, \mathbb{R}^n],$$

where $x = \mathrm{col}\,(x_1, \ldots, x_n) \in \mathbb{R}^n$ and x^T is the tranpose of x. Consider the systems of differential equations

$$\dot{x} = f(t, x), \qquad t \in \mathbb{R}^+, \tag{3}$$

and

$$\dot{x} = g(x) \tag{4}$$

Assume that there exists an $\eta > 0$, such that

$$|f(t, x) - g(x)| < \eta, \qquad (t, x) \in \mathbb{R}^+ \times \Omega. \tag{5}$$

Let $g(0) = 0$, and assume that the real parts of all the eigenvalues of matrix $g'(0)$ are negative.

Under these conditions, a positive definite quadratic form $w(x) = x^T Wx$, is known to exist, such that the derivative of $w(x)$ with respect to system

$$\dot{y} = g'(0)y \tag{6}$$

is negative definite. Moreover, there exist constants $\rho_1 > 0$, $\rho_2 > 0$ such that

$$U_{\rho_1} = \{x \in \mathbb{R}^n : |x| < \rho_1\} \subset \Omega$$

$$w_{(4)}(x) \leqq -\rho_2 w(x), \quad \text{for} \quad x \in U_{\rho_1}. \tag{7}$$

Let us denote the eigenvalues of the positive definite matrix W by λ_i, $i=1, 2, \ldots, n$, and let $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$. Clearly

$$\lambda_1 |x|^2 \leqq w(x) \leqq \lambda_n |x|^2, \qquad x \in \mathbb{R}^n . \tag{8}$$

Using the notations

$$A_\eta = \{x \in \mathbb{R}^n\colon w(x) \leqq 4\lambda_n^2 \eta^2 | \rho_2^2 \lambda_1\}$$

$$B = \{x \in \mathbb{R}^n\colon w(x) \leqq \rho_1^2 \lambda_1\} \tag{9}$$

the following theorem can be stated:

If $0 < \eta < \lambda_1 \rho_1 \rho_2 | 2\lambda_n$, then set $\mathbb{R}^+ \times A\eta$ is a uniform asymptotically stable invariant set of system (3) and its region of attractivity contains set $\mathbb{R}^+ \times B$.

Using the above results, let us investigate Eq. (1).

3. Linéard's equation under bounded perturbation

Using substitution $(u, \dot{u}) = (x_1, x_2) = x$ in Eq. (2) results in the equivalent two-dimensional system:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\psi(x_1) - \varphi(x_1) x_2 . \tag{10}$$

Its linearized form is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\psi'(0) x_1 - \varphi(0) x_2 . \tag{11}$$

Under conditions $\varphi(0) > 0$, $\psi'(0) > 0$, the origin is an asymptotically stable equilibrium of (10). System (11) is asymptotically stable, and a positive definite quadratic form

$$w(x) = (\varphi^2(0)) + \psi'^2(0) + \psi'(0)) x_1^2 + 2\varphi(0) x_1 x_2 + (\psi'(0) + 1) x_2^2 , \tag{12}$$

is found, with a negative definite derivative with respect to system (11)

$$\dot{w}_{(11)}(x) = -2\varphi(0)\psi'(0) x_1^2 - 2\varphi(0)\psi'(0) x_2^2 , \tag{13}$$

and

$$\dot{w}_{(11)}(x) \leq -\alpha w(x)\,, \tag{14}$$

if

$$0 < \alpha \leq \varphi(0)\left(1 - \sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right).$$

The eigenvalues of $w(x)$ are

$$\lambda_{1,2} = \frac{1}{2}(\varphi^2(0)+(\psi'(0)+1)^2)\left(1 \mp \sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right). \tag{15}$$

The derivate of $w(x)$ with respect to system (10) is

$$\dot{w}_{(10)}(x) = 2(\varphi^2(0)+\psi'^2(0)+\psi'(0))x_1x_2 + 2\varphi(0)x_2^2 +$$

$$+2(\psi'(0)+1)\,(-\varphi(x_1)x_2 - \psi(x_1))x_2 +$$

$$+2\varphi(0)\,(-\varphi(x_1)x_2 - \psi(x_1)x_2\,.$$

Let us determine ρ_1 and ρ_2 such a way, that

$$\dot{w}_{10}(x) \leq -\rho_2 w(x)\,, \quad \text{for} \quad |x| < \rho_1\,. \tag{16}$$

Introducing notation

$$\rho_2 = \varphi(0)\left(1 - \sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right) - \delta \tag{17}$$

Where

$$0 < \delta \leq \varphi(0)\left(1 - \sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right),$$

(16) can be written as

$$-\dot{w}_{(10)}(x) + \delta w(x) + \dot{w}_{(11)}(x) - \left[\dot{w}_{(11)}(x) + \right.$$
$$\left. + \varphi(0)\left(1 - \sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right)w(x)\right] \geq 0. \tag{18}$$

In view of (14) the expression in square brackets is negative definite, so if

$$-\dot{w}_{(10)}(x)+\delta w(x)+\dot{w}_{(11)}(x)\geq 0 \tag{19}$$

then (18) holds.

In case $x_1=0$, (19) becomes

$$\delta(\psi'(0)+1)x_2^2\geq 0$$

If $x_1\neq 0$, then the left-hand side of (19) can be considered as a "quadratic form". It is

$$\left[2\varphi(0)\left(\frac{\psi(x_1)}{x_1}-\psi'(0)\right)+\delta(\varphi^2(0)+\psi'^2(0)+\psi'(0)\right]x_1^2+$$

$$+\left[2(\psi'(0)+1)\left(\frac{\psi(x_1)}{x_1}-\psi'(0)\right)+2\varphi(0)\,(\varphi(x_1)-\varphi(0))+2\varphi(0)\delta\right]x_1x_2+$$

$$+\left[2(\psi'(0)+1)\,(\varphi(x_1)-\varphi(0))+\delta(\psi'(0)+1)\right]x_2^2\,.$$

The sufficient condition of non-negativeness is

$$\left|\frac{\psi(x_1)}{x_1}-\psi'(0)-\frac{\varphi(0)}{\psi'(0)+1}(\varphi(x_1)-\varphi(0))\right|\leq$$

$$\leq\sqrt{\delta\psi'(0)\left(1+\frac{\varphi^2(0)}{(\psi'(0)+1)^2}\right)(2(\varphi(x_1)-\varphi(0))+\delta)} \tag{20}$$

If this inequality yields an estimate for $|x_1|$, then for a given δ, ρ_1 and ρ_2 can be determined. x_2 does not occur in the condition.

As a conclusion, under the conditions imposed upon the perturbed Linéard's equation (1), if

$$0<\eta<\frac{1-\sqrt{\dfrac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}}{1+\sqrt{\dfrac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}}\,\frac{\rho_1\rho_2}{2} \tag{21}$$

then for a given δ, from the interval

$$\left(0,\ \varphi(0)\left(1-\sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right)\right],$$

with the quadratic form

$$w(x)=(\varphi^2(0)+\psi'^2(0)+\psi'(0))x_1^2+2\varphi(0)x_1x_2+(\psi'(0)+1)x_2^2\,, \tag{22}$$

and with its eigenvalues

$$\lambda_{1,2}=\frac{1}{2}(\varphi^2(0)+(\psi'(0)+1)^2)\left(1\mp\sqrt{\frac{\varphi^2(0)+(\psi'(0)-1)^2}{\varphi^2(0)+(\psi'(0)+1)^2}}\right), \tag{23}$$

the uniform asymptotically stable invariant set of Eq. (1), ($\mathbb{R}^+\times A_\eta$) and its region of attractivity ($\mathbb{R}^+\times B$) can be estimated from (9).

Let us consider a numerical example.

$$\ddot{u}+(3-3u^4)\dot{u}+2u+3u^3=F(t,u,\dot{u})\,, \tag{24}$$

where F, F'_u, $F'_{\dot{u}}$ are continuous functions. From (20) we get

$$|x_1|\leq\frac{\sqrt{\delta}}{2.69}=\rho_1$$

and from (17),

$$\rho_2=0.765-\delta$$

if

$$0<\delta\leq 0.765$$

The quadratic form (22) is

$$w(x)=15x_1^2+6x_1x_2+3x_2^2$$

its eigenvalues are $\lambda_1=2.295$, $\lambda_2=15.705$. Now if

$$0<\eta<0.027\sqrt{\delta}\,(0.765-\delta) \tag{25}$$

then for a given δ

$$A_\eta=\left\{(x_1,x_2)\in\mathbb{R}^2:15x_1^2+6x_1x_2+3x_2^2\leq 429.885\frac{1}{(0.765-\delta)^2}\eta^2\right\}$$

$$B=\{(x_1,x_2)\in\mathbb{R}^2:15x_1^2+6x_1x_2+3x_2^2\leq 0.317\delta\}\,. \tag{26}$$

The upper bound of η has been maximized in (25).

$$\max\sqrt{\delta}\,(0.765-\delta)\,,\quad \text{for } \delta=0.255.\quad 0<\delta<0.765$$

Minimizing the attractive set A_η and maximizing its region of attractivity B, at the same time by maximizing

$$\frac{\delta}{\frac{1}{(0.765-\delta)^2}} = \delta(0.765-\delta)^2,$$

which is the square of the upper bound of η, we get

$$\max \delta(0.765-\delta)^2, \quad \text{for} \quad \delta=0.255. \quad 0<\delta<0.765$$

With $\delta=0.255$

$$A_\eta=\{(x_1, x_2)\in\mathbb{R}^2 : 15x_1^2+6x_1x_2+3x_2^2\le 1652.77\eta^2\}$$

$$B=\{(x_1, x_2)\in\mathbb{R}^2 : 15x_1^2+6x_1x_2+3x_2^2\le 0.08\},$$

where

$$0<\eta<0.007.$$

M. Farkas applied his estimates to van der Pol's equation. Considering van der Pol's equation as a special case of Liénard's one, (20) yields the estimate

$$|x_1|\le \frac{\sqrt{\delta}}{\sqrt{m\left(1+\sqrt{1+\frac{m^2}{m^2+4}}\right)}}.$$

For $m=0.20$ we got the same result as M. Farkas did.

Summary

Lienard's equation is valid in a wide range of applications, first of all, in electrical circuitry. It gives an explicit upper estimate of the attractor and a lower estimate to the region of attractivity of a non-autonomous system close to Liénard's equation.

References

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