ON THE FUNCTIONAL EQUATION

$f(x_1)+f(x_2)=f(y_1)+f(y_2)$

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1. In the theory of partial differential equations, namely at the homogeneous wave equation appears a functional equation. A necessary and sufficient condition — as it is well known — that a function $f(x, y)$ of the class $C^2(R^2)$ should be a solution of the above-mentioned differential equation in R^2, is that for all possible rectangles whose sides are segments of characteristics of the homogeneous wave equation with label the vertices V_i $(i=1,2,3,4)$ — V_1 and V_4 are denoted as opposite vertices — satisfy the functional equation $f(V_1)+$ $+f(V_4)=f(V_2)+f(V_3)$.

Our aim is to investigate a functional equation closely connected with the preceeding functional equation, but on a more general algebraic structure, namely on a semi-group and free of geometrical interpretation. Roughly spoken, we investigate a functional equation in which the sum of images are equal if the sum of arguments are equal. Theorem 2 states, that if this is true for two independent arguments, then it remains true for an arbitrary number of arguments. Such a functional equation is closely connected with the Cauchy and Jensen functional equations (Theorem 1) as well as with the Ryff ([5]) functional equation. We give also the most general solution of the investigated functional equation on the reals, as well as, the general solution of it if the unknown is not a function but a distribution. Also a method is given to solve the corresponding inhomogeneous functional equation.

2. Let $(X, +)$ be a commutative semi-groups with root functions $\gamma_n: X \to X$ for all $n \in N$, i.e.(see [4])

$$\gamma_n(x)+\gamma_n(y)=\gamma_n(x+y)$$

$$n\gamma_n(x)=x$$

for all $x, y \in X$.

Let A be a subset of X with the property that $x_i \in A$ $(i=1,2, \ldots, n)$ implies $\gamma_n(x_1+\ldots+x_n) \in A$ for all $n \in N$. Let $(Y, +)$ be a semi-group (we shall use the

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additive notation also in the noncommutative case). We consider the functional equation

$$f(x_1)+f(x_2)=f(y_1)+f(y_2) \tag{1}$$

for $f: X\to Y$ on restricted domain, for all $(x_1, x_2), (y_1, y_2)\in A\times A$ such that $x_1+x_2=y_1+y_2$.

THEOREM 1. If the function $f: X\to Y$ satisfies the equation (1), then f satisfies

$$f(x_1)+\ldots+f(x_n)=nf(\gamma_n(x_1+\ldots+x_n)) \tag{2}$$

for all $x_1, x_2, \ldots, x_n\in A$ and all $n\in N$. The left hand side of (2) is independent of the order of summation.

PROOF. We prove the theorem by induction on n. (2) is obviously true for $n=1$. Let us suppose its validity for all $n\leqq m-1$. For $n=m$ we take $x_2=x_3=\ldots=x_m$. Since

$$y_1=\gamma_m(x_1+(m-1)x_2)\in A$$
$$y_2=\gamma_m((m-1)x_1+x_2)\in A$$

and

$$x_1+x_2=y_1+y_2,$$

then by (1) we obtain

$$f(x_1)+f(x_2)=f(\gamma_m(x_1+(m-1)x_2)+f(\gamma_m((m-1)x_1+x_2)).$$

Hence

$$f(x_1)+(m-1)f(x_2)=f(x_1)+f(x_2)+(m-2)f(x_2)=$$
$$=f(\gamma_m(x_1+(m-1)x_2))+f(\gamma_m((m-1)x_1+x_2)+(m-2)f(x_2).$$

Now we apply the equality (2) for $n=m-1$ on the last $m-1$ factors in the preceding relation. In this way we obtain

$$\begin{aligned} &f(\gamma_m((m-1)x_1+x_2))+(m-2)f(x_2)= \\ &=(m-1)f(\gamma_{m-1}(\gamma_m((m-1)x_1+x_2)+(m-2)x_2)). \end{aligned} \tag{3}$$

We shall prove that

$$\gamma_{m-1}(\gamma_m((m-1)x_1+x_2)+(m-2)x_2)=\gamma_m(x_1+(m-1)x_2) \tag{4}$$

holds. Let us start from the left hand side

$$L=(\gamma_{m-1}(\gamma_m((m-1)x_1+x_2)+(m-2)x_2)=$$

$$=\gamma_{m-1}(\gamma_m((m-1)x_1))+\gamma_{m-1}(\gamma_m(x_2))+$$

$$+\gamma_{m-1}(\gamma_m(m(m-1)x_2)).$$

Since $\gamma_m \circ \gamma_{m-1}=\gamma_{m-1} \circ \gamma_m$ (see [4]), we can write

$$L=\gamma_m(\gamma_{m-1}((m-1)x_1)+\gamma_{m-1}(x_2+(m^2-2m)x_2))=$$

$$=\gamma_m(x_1+\gamma_{m-1}((m-1)^2x_2))=\gamma_m(x_1+(m-1)x_2),$$

and so we obtain the relation (4).

By (4) and (3) we have

$$f(x_1)+(m-1)f(x_2)=mf(\gamma_m(x_1+(m-1)x_2)). \tag{5}$$

Now we consider arbitrary elements $x_i \in A$ $(i=1, 2, \ldots, n)$. By (2) for $n=m-1$ and (5) we have

$$f(x_1)+\ldots+f(x_m)=f(x_1)+(f(x_2)+\ldots+f(x_m))=$$

$$f(x_1)+(m-1)f(\gamma_{m-1}(x_2+\ldots+x_m))=mf(\gamma_m(x_1+\ldots+x_m))$$

this is exactly the relation (2) for $n=m$.

By the commutativity of the operation $+$ in X, we have

$$f(\gamma_n(x_1+\ldots+x_n))=f(\gamma_n(x_{p(1)}+\ldots+x_{p(n)})),$$

where $p(\cdot)$ denotes any permutation of the numbers $1, 2, \ldots, n$. Hence by (2) we have

$$f(x_1)+\ldots+f(x_n)=f(x_{p(1)})+\ldots+f(x_{p(n)}).$$

This completes the proof.

REMARK 1. It is evident that every solution of the Cauchy functional equation is also a solution of the equation (1). If $(Y, +)$ is a commutative semi-group with the root function δ_n for all $n \in N$, then by (2) every solution of the equation satisfies also the generalized Jensen-equation

$$f(\gamma_n(x_1 + \ldots + x_n)) = \delta_n(f(x_1) + \ldots + f(x_n))$$

for all $x_i \in A$ $(i = 1, 2, \ldots, n)$ and all $n \in N$.

REMARK 2. Any homogeneous real solution of (1) satisfies the Ryff-equation ([5])

$$af(ax) + bf(bx + a) = bf(bx) + af(ax + b).$$

3. THEOREM 2. If $f: X \to Y$ is a solution of (1) for which the condition

$$f(2x) = 2f(x) \tag{6}$$

holds for all $x \in A$, then f satisfies

$$f(x_1) + \ldots + f(x_n) = f(y_1) + \ldots + f(y_m) \tag{7}$$

for all $x_i, y_k \in A$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$ and all $n, m \in N$ such that

$$x_1 + \ldots + x_n = y_1 + \ldots + y_m$$

holds.

PROOF. It is obvious that in the case $n = m$ Theorem 2 follows (without using (6)) from the Theorem 1. Let now be $n \neq m$. Then we have for arbitrary elements x_i, y_k $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$ fulfilling the condition

$$x_1 + \ldots + x_n = y_1 + y_2 + \ldots + y_m = z$$

by Theorem 1

$$f(x_1) + \ldots + f(x_n) = nf(\gamma_n(z))$$

$$f(y_1) + \ldots + f(y_m) = mf(\gamma_m(z)).$$

We shall now prove (7) in the equivalent form

$$nf(x) = mf(\gamma_m(nx)) \tag{8}$$

for $n \geqq m$, where $x = \gamma_n(z)$.

Let be first $n = m + 1$. By (6) and Theorem 1 we have

$$(m+1)f(x) = (m-1)f(x) + f(2x) = mf(\gamma_m((m+1)x))$$

since

$$\gamma_m((m+1)x) = \gamma_m((m-1)x + 2x).$$

(8) is proved for this particular case.

Let us now suppose that (8) is true for n fulfilling the inequalities $m+1 \leqq n \leqq m+k-1$. Hence

$$(m+k)f(x) = f(x) + (m+k-1)f(x) = f(x) + mf(\gamma_m((m+k-1)x).$$

For $n = m+1$ we have by (8)

$$(m+k)f(x) = mf(\gamma_m((m+k)x)),$$

i.e. the relation (8) is valid also for $n = m+k$. This completes the proof of Theorem 2.

4. It seems to be interesting to look for the most general solution $f: R \to R$ of

$$f(x_1) + f(x_2) = f(y_1) + f(y_2)$$

under the condition

$$x_1 + x_2 = y_1 + y_2 \qquad (x_1, x_2, y_1, y_2 \in R).$$

Introducing the new variables $x = x_1$, $y = y_1$, $z = x + x_2 = y + y_2$, we can write instead of our functional equation

$$f(x) - f(y) + f(z-x) - f(z-y) = 0. \tag{9}$$

For $x = z$ we have

$$f(x-y) - f(x) + f(y) = c, \tag{10}$$

where c is a constant. This is (in general) an inhomogeneous functional equation. Its general solution is the sum of a partial solution and the general solution of the corresponding homogeneous equation.

Obviously $f(x) = c$ $(x \in R)$ is a partial solution of (10) which also satisfies (9). The corresponding homogeneous functional equation is the following

$$f(x-y) = f(x) - f(y), \tag{11}$$

from this if $x = 0$

$$f(-y) = -f(y)$$

as (11) implies $f(0)=0$. We see at once that the most general solution of (11) is an odd additive function which also satisfies (9).

THEOREM 3. The most general solution of (9) has the form

$$f(x)=c+h(x) \qquad (x \in R),$$

where h is an arbitrary odd additive function and c an arbitrary real constant.

5. We can easily get also the general solution of the inhomogeneous functional equation

$$f(x-y)-f(x)+f(y)=\mathrm{d}(x, y) \qquad (x, y \in R) \tag{12}$$

if any exists. Here d is a given bounded function.

First let us calculate a partial solution, e.g. a bounded one. (12) implies

$$f(-x)+f(x)=\mathrm{d}(0, x)+f(0) \qquad (x \in R). \tag{13}$$

Now we substitute in (12) $y=2x$. Regarding (13) we get

$$f(2x)-2f(x)=\mathrm{d}(x, 2x)-\mathrm{d}(0, x)-f(0).$$

Putting $y=3x$ into (12) and apply (13) with $2x$ instead of x we get

$$f(3x)-3f(x)=\mathrm{d}(x, 2x)+\mathrm{d}(x, 3x)-\mathrm{d}(0, x)-\mathrm{d}(0, 2x)-2f(0).$$

By induction

$$f(nx)-nf(x)=\mathrm{d}(x, 2x)+\ldots+\mathrm{d}(x, nx)-\mathrm{d}(0, x)-\ldots-$$

$$-\mathrm{d}(x,(n-1)x)-(n-1)f(0).$$

From this follows as $f(0)=\mathrm{d}(x, x)$

$$\frac{f(nx)}{n}-f(x)=\frac{[\mathrm{d}(x, x)-\mathrm{d}(0, x)]+\ldots+[\mathrm{d}(x, nx)-\mathrm{d}(0, nx)]}{n}+\frac{\mathrm{d}(0, nx)}{n}-$$

$$-f(0).$$

By the assumption that f and d are bounded

$$\lim_{n\to\infty} f(nx)/n=0$$

$$\lim_{n\to\infty} \mathrm{d}(0, nx)/n=0$$

for every x, therefore

$$f(\mathrm{x})-f(0) = \lim_{n\to\infty} \frac{[\mathrm{d}(x, x)-\mathrm{d}(0, x)]+\ldots+[\mathrm{d}(x, nx)-\mathrm{d}(0, nx)]}{n}. \quad (14)$$

This means, if (12) has a solution at all, then the unique bounded solution has the form (14). Denoting by g an arbitrary odd additive function, $f+g$ is the most general solution of (12).

If we consider now the corresponding inhomogeneous functional equation

$$f(x)-f(y)+f(z-x)-f(z-y)=p(x, y, z) \quad (15)$$

and put $z=x$, then this overgoes into

$$f(x-y)-f(x)+f(y)=f(0)-p(x, y, x). \quad (16)$$

Let be $\mathrm{d}(x, y)=c-p(x, y, x)$ (c=constant) we get an equation of the form (10).

Obviously (10) is not equivalent to (14) and (15) not equivalent to (16). It would be interesting and useful to find conditions under which (14) is a solution of (10), respectively one of (15).

6. Looking at the relation between the wave equation and the functional equation (9) mentioned in the introduction, it seems to be of some interest to find the general solution of (9), if f is not a function but a distribution. Applying the distributional differential operator $D_z D_x$ to every term of (9), we get

$$L(1, -1)D^2 f=0,$$

where $L(1, -1)$ is the dual operator of

$$(l(1, -1)\varphi)(t)=\int_{-\infty}^{\infty} \varphi(x, x-t)\,\mathrm{d}x,$$

φ is an arbitrary Schwartz-testfunction (see [6], especially p. 48—49). But then $D^2 f = 0$ and therefore the most general solution of (9) is the distribution generated by the function $ax+b$ (a and b are arbitrary constants)*.

Summary

The paper investigates the functional equation $f(x_1)+f(x_2)=f(y_1)+f(y_2)$ under the condition $x_1+x_2=y_1+y_2$ on a semigroup. Also the general solution of it is given on the reals as well the general solution if the unknown is a distribution. A method is proposed which provides the solution of the corresponding inhomogeneous functional equation.

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