# VOLTAGE SPECTRUM OF PULSE-WIDTH-MODULATED INVERTERS

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Received February 9, 1981

Presented by Prof. Dr. GY. RETTER

Several techniques are in use to control pulse-width-modulated inverters in frequency-changer asynchronous motor drives (Fig. 1). The first technique, so-called natural sampling was suggested as early as 1964 [1]. It consists in the comparison between a constant-frequency and constant-amplitude triangular signal and a changing-frequency, changing-amplitude sinusoidal reference signal (Fig. 2). The instantaneous intersections of the two waves determine the switching instants of the phase: if the triangular signal exceeds the sinusoidal signal, the phase is connected to the negative bar, and in the contrary case to the positive bar. Later, a technique called regular sampling was suggested [2], in which the triangular signal is being compared to a stepped approximation of the sine wave (Figs 3 and 4). Sampling can be carried out at both the positive and the negative peaks of the triangular signal (Fig. 4), or at the negative peak only (Fig. 3).

In the past years several papers appeared dealing with the analytical determination of the frequency components of the motor voltage. BOWES et al. [3] approached the problem by means of double Fourier series, BALESTRINO et

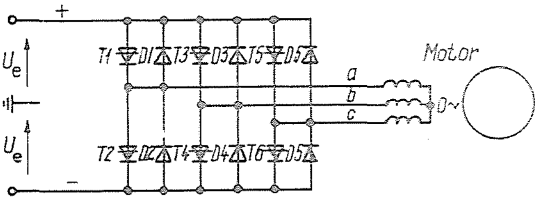


*Fig. 1*

al. [4] by means of Kapteyn series. Extremely complicated computations resulted in both cases, and on the other hand, the computations could not be made independent of the condition whether the ratio of the frequencies of the carrier wave and the reference wave

$$m = \frac{f_v}{f_a} \tag{1}$$

was an integer, a fraction or an irrational number.

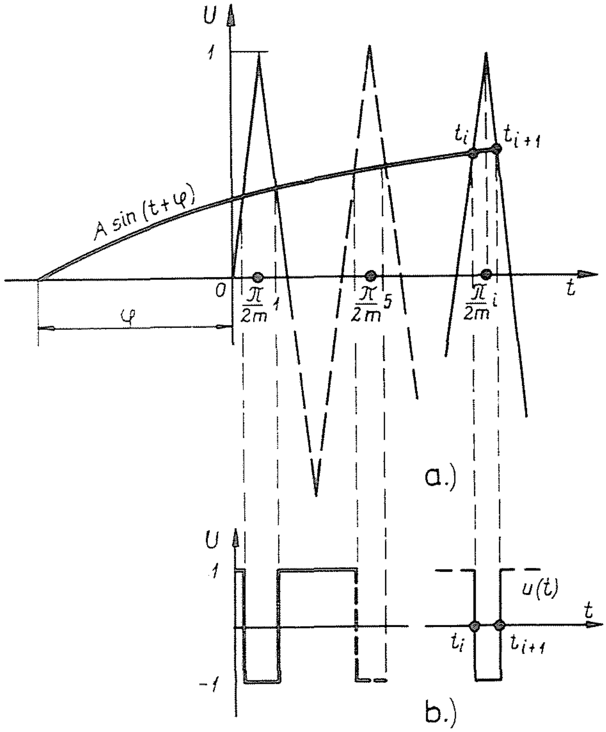


Fig. 2

## Fundamental relationships

Let us denote the voltage of the d.c. circuit with $2U_d$ and assume in the followings that $|U_d|=1$. We wish to find the Fourier series of the voltage $u(t)$ indicated in Figs 2b, 3b and 4b:

$$u(t)=\mathrm{Re}\sum_{\nu=0}^{\infty} U_\nu e^{j\nu t}, \tag{2}$$

where

$$U_\nu=\lim_{T\to\infty}\frac{2}{T}\int_0^T u e^{-j\nu t}\,\mathrm{d}t; \qquad U_0=\lim_{T\to\infty}\frac{1}{T}\int_0^T u\,\mathrm{d}t \tag{3}$$

and $\nu \geqq 0$. In the above formulae time $t$ is expressed in the angle of the fundamental component.

Since only zero-sequence components can arise between the star point of the motor and the zero point of the d.c. circuit, the voltage harmonics determined by the above expression will simultaneously represent the harmonics of the motor voltage, while the zero-sequence components will result only in a shift of the start point of the motor in relation to the zero point of the d.c. circuit.

## Regular sampling

*Sampling at the negative peak of the triangular signal*

The relationships are shown in Fig. 3. Using the symbol $\gamma=\frac{\pi}{2m}$, the switching times will be expressed in the following form ($A \leqq 1$):

$$t_i=\gamma i-\gamma[1-A\sin(\gamma i-2\gamma+\varphi)];$$
$$t_{i+1}=\gamma i+\gamma[1-A\sin(\gamma i-2\gamma+\varphi)] \tag{4}$$

where $i=1, 5, 9$, etc. is the serial number of summation.

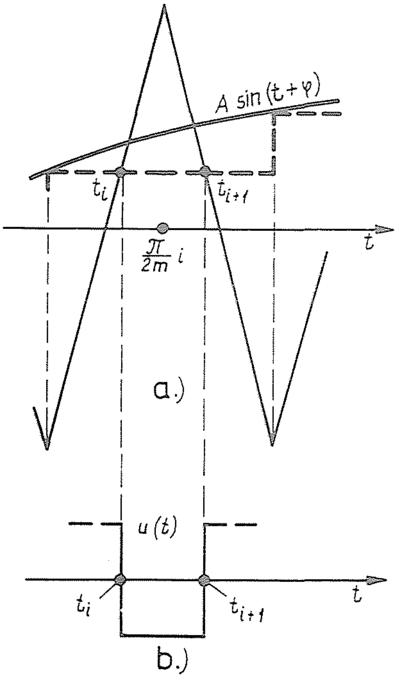


*Fig. 3*

By substituting this into Eq. (3), the following equation will result for the d.c. component:

$$U_0 = \lim_{T\to\infty} \frac{2}{T} \sum_{i=1}^{\infty} [2\gamma - 2\gamma A \sin(\gamma i - 2\gamma + \varphi)] - 1\,. \tag{5a}$$

Substituting $T = \gamma i_{max}$, where $i_{max} \to \infty$ is the highest serial number in the summation, we obtain

$$U_0 = \frac{-4A\gamma}{i_{max}\gamma} \sum_{i=1}^{\infty} \sin(\gamma i - 2\gamma + \varphi) = 0\,, \tag{5}$$

since the summation can only be different from zero if the sine function assumes identical values for all values of $i$, that is, $(\Delta i = 4)$

$$4\gamma = \frac{4\pi}{2m} = 2\pi K\,, \tag{6}$$

where $K = 0, \pm 1, \pm 2, \ldots$. Hence $mK = 1$, and therefore the d.c. component will not appear at any $m > 1$.

For determining the components $\nu \neq 0$, let us substitute Eq. (4) into Eq. (3):

$$U_\nu = \frac{4}{j\nu\gamma i_{max}} \sum_{i=1}^{\infty} \left\{ e^{-j\nu\gamma i} \left[ e^{-j\nu\gamma}\, e^{j\nu A\gamma \sin(\gamma i - 2\gamma + \varphi)} - e^{j\nu\gamma}\, e^{-j\nu A\gamma \sin(\gamma i - 2\gamma + \varphi)} \right] \right\}. \tag{6a}$$

Using the formula (M1) of the Appendix, and rearranging the equation we arrive to

$$\begin{aligned} U_\nu = \frac{4}{j\gamma\nu i_{max}} \Bigg\{ & J_0(A\nu\gamma)\left[e^{-j\nu\gamma} - e^{j\nu\gamma}\right] \sum_{i=1}^{\infty} e^{-j\nu\gamma i} + \\ & + \sum_{i=1}^{\infty} e^{-j\nu\gamma i} \sum_{n=1}^{\infty} J_n(A\nu\gamma) e^{j(n\gamma i - 2\gamma n + n\varphi)} \cdot \left[e^{-j\nu\gamma} - (-1)^n e^{j\nu\gamma}\right] + \\ & + \sum_{i=1}^{\infty} e^{-j\nu\gamma i} \cdot \sum_{n=1}^{\infty} J_n(A\nu\gamma) e^{-j(n\gamma i - 2\gamma n + n\varphi)} \left[(-1)^n e^{-j\nu\gamma} - e^{j\nu\gamma}\right] \Bigg\}. \end{aligned} \tag{7}$$

Let us now change the order of the summation and take into account that the summation of exponential functions yields values differing from zero only in the case if they will be in phase for each value of $i$, that is,

$$-4(\gamma\nu - n\gamma) = \pm 2\pi k\,, \tag{8}$$

and

$$-4(\gamma\nu+n\gamma)=-2\pi k\,, \tag{9}$$

resp., where $k=0, 1, 2, \ldots$. Since $\gamma=\frac{\pi}{2m}$, only harmonics of the following orders can appear:

$$\nu=\pm k\cdot m+n$$

and

$$\nu=k\cdot m-n,$$

respectively.

Taking this into account, and extending the value of $n$ to zero and negative integers:

$$n=\pm km+\nu; \tag{10}$$

we finally obtain, for a given pair of $k$ and $n$ values and $i=1$:

$$U_\nu=\frac{j}{\nu\gamma}J_n(A\nu\gamma)e^{jn\varphi}\,e^{-jn\gamma}\,[(-1)^n-e^{-j2\nu\gamma}]\,. \tag{11}$$

It should be noted here, and this will be valid in the followings too (except if $m$ is an irrational number), that a harmonic of a certain order will correspond to several pairs of $k$ and $n$ values, and hence harmonics with identical frequencies may yield positive, negative or zero sequence too. Since the shift angle between the sinusoidal reference signals of the individual phases is 120°, whereas the triangular signal is identical, a phase shift of $e^{\pm jn120°}$ will occur between the phase harmonics (based on (11)).

Therefore, in the case that $n$ is divisible by three, the corresponding harmonic will be zero order; if $n=1+3k_2$, it will be positive, if $n=2+3k_2$, it will be negative order ($k_2=0, \pm 1, \pm 2 \ldots$).

*Sampling at both peaks of the triangle*

From Figure 4, the following expressions can be writtten for the switching times ($A\leqq 1$):

$$t_i=\gamma i-\gamma[1-A\sin(\gamma i-2\gamma+\varphi)];$$

$$t_{i+1}=\gamma i+\gamma[1-A\sin(\gamma i+\varphi)\,. \tag{12}$$

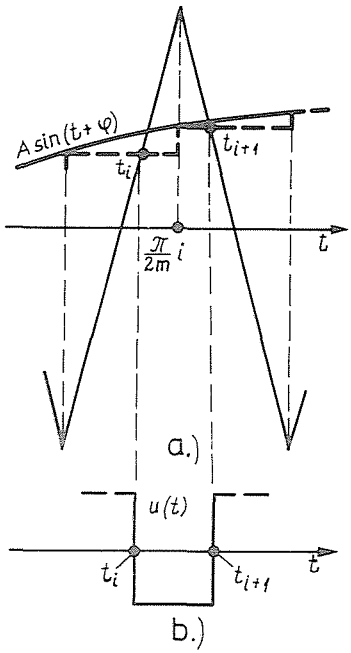


*Fig. 4*

By substituting these expressions into Eqs (3), analogously to the previous considerations we will obtain the followings [5]:

$$U_0=0, \qquad \text{if} \quad m>1; \tag{13}$$

$$\nu=\pm km+n>0$$

$$U_\nu=\frac{j}{\nu\gamma}J_n(A\nu\gamma)e^{jn\varphi}e^{-jn\gamma}[(-1)^n-(-1)^k]. \tag{14}$$

## Natural sampling

For natural sampling, the determination of the switching times $t_i$ and $t_{i+1}$ is more difficult. The following equations are valid ($A\leqq 1$, cf. Appendix):

$$t_i=\gamma i+\sum_{n=1}^{\infty}\frac{2}{n}J_n(nA\gamma)\sin(n\gamma i+n\varphi);$$

$$t_{i+1}=\gamma i+2\gamma+\sum_{n=1}^{\infty}\frac{2}{n}J_n(nA\gamma)\sin(n\gamma i+2n\gamma+n\varphi+n\pi), \tag{15}$$

where $i=0, 4, 8, \ldots$.

By substitution of these expressions into Eq. (3), we arrive to the equation of the d.c. component obtained in natural sampling:

$$U_0 = -\frac{2}{\gamma i_{\max}} \sum_{i=0}^{\infty} \left[ 2\gamma + \sum_{n=1}^{\infty} \frac{4}{n} J_n(A\gamma n) \cos\left( n\gamma i + n\varphi + n\gamma + n\frac{\pi}{2} \right) \times \right.$$
$$\left. \times \sin\left( n\gamma + n\frac{\pi}{2} \right) \right]. \qquad (15a)$$

When the summations are carried out, a result differing from zero is obtained only if

$$4n\gamma = 2\pi k_1, \quad \text{i.e.} \quad n = k_1 m$$

($k_1 = 1, 2, 3, \ldots$). Using this expression and assuming $i=0$:

$$U_0 = \frac{2}{\pi} \sum_{k_1=1}^{\infty} \left\{ \frac{1}{k_1} J_{k_1 m}\left( k_1 A \frac{\pi}{2} \right) [1 - (-1)^{k \cdot (1+m)}] \cdot \sin(k_1 m\varphi) \right\}. \qquad (16)$$

In this equation $k_1$ and $k_1 m$ must be integers.

To calculate further harmonics, we shall rewrite Eq. (3) into the following form:

$$U_\nu = \frac{4}{j\nu\gamma i_{\max}} \sum_{i=0}^{\infty} [e^{j\nu(\varphi+\pi)} e^{-j(\nu t_{i-1} + \nu\varphi + \nu\pi)} - e^{j\nu\varphi} e^{-j(\nu t_i + \nu\varphi)}]. \qquad (16a)$$

Subsequently, by using the expression (M4) from the Appendix (which, however, is not valid for $\nu = 1$), we obtain

$$U_\nu = \frac{4}{j\nu\gamma i_{\max}} \sum_{i=0}^{\infty} \left\{ e^{j\nu(\varphi+\pi)} \nu \sum_{p}^{\infty} \left[ \frac{1}{p} J_{p-\nu}(pA\gamma) e^{-jp(i\gamma + 2\gamma + \varphi + \pi)} - \right.\right.$$
$$\left. - \frac{1}{p} J_{p+\nu}(pA\gamma) e^{jp(i\gamma + 2\gamma + \varphi + \pi)} \right] - e^{j\nu\varphi} \nu \sum_{p}^{\infty} \left[ \frac{1}{p} J_{p-\nu}(pA\gamma) e^{-jp(i\gamma + \varphi)} - \right. \qquad (17)$$
$$\left.\left. - \frac{1}{p} J_{p+\nu}(pA\gamma) e^{jp(i\gamma+\varphi)} \right] \right\},$$

where $p \pm \nu$ must be an integer, and the summations must be carried out for all possible $p \pm \nu$ integers.

Following the derivation, applied above, we will obtain a value differing for zero only for $p$ for which

$$4p\gamma = 2\pi k,$$

and hence the terms $p = mk$ must be summed ($k = 1, 2, 3, \ldots$).

Let $\nu$ be

$$\nu = \pm km + n\,, \tag{18}$$

that is, $n$ can be a positive or negative integer or zero. By this the above equation will assume the following form for a given pair of $k$ and $n$ (substituting $i = 0$):

$$\nu = \pm km + n > 0$$
$$U_\nu = \frac{2j}{\pi k} J_n\left(kA\frac{\pi}{2}\right) e^{jn\varphi}\left[(-1)^n - (-1)^k\right]. \tag{19}$$

For the fundamental $\nu = 1$ we will obtain yet another complementary term:

$$U_1 = -Aje^{j\varphi}\,. \tag{19a}$$

The remark made earlier is again valid that different pairs of $k$ and $n$ can yield a harmonics with the same order $\nu$, except the case when $m$ is an irrational number.

## Conclusions

The relationships deduced for the different sampling techniques allow to make important conclusions:

(i) It is advantageous from the view of both the operation of the motor and the construction of open and closed control loops if the amplitude of the fundamental voltage is proportional to the amplitude of the sinusoidal reference signal (Figs 2—4). This condition is satisfied best by natural sampling: for $m = 6$, e.g., the linearity error is around 0.3%, at higher $m$ values even less. For regular sampling, the linearity error is somewhat higher: with $m = 6$ it is around 5% (Fig. 3) and 1% (Fig. 4), these values, however, being still satisfactory. What is more disadvantageous is the fact that with regular sampling, a change in the value of $m$ will shift the phase of the fundamental voltage relative to the sinusoidal reference signal.

(ii) The voltage spectrum can be characterized in all three cases with $\nu = \pm km + n$, where $k = 1, 2, 3, \ldots$ but for regular sampling $k$ may have zero value too. However, in the case of natural sampling and the regular sampling technique with sampling of both peaks of the triangular signal, $(k+n)$ must be uneven. The regular sampling technique using sampling of the negative peak only has the disadvantage that it spoils the symmetry of sampling, and therefore the voltage spectrum will be very rich. E.g. a harmonic $\nu = 2$ will also appear, whose amplitude at $m = 10$ is about 2.4%, at $m = 6$ about 6.1%.

(iii) From the view of motor operation it is greatly disadvantageous if subharmonics and harmonics with low orders $(1 < \nu < 5)$ appear. Fig. 5 demonstrates the maximum amplitude of these troublesome subharmonics and harmonics *vs* the value of $m$. This figure indicates that regular sampling with both peaks of the triangular signal is more advantageous from this view; it may, however, be seen from the figure that at $m > 10$ the amplitude of these unwelcome harmonics will become negligible.

(iv) The order of the importance harmonics for motor operation are as follows: $m \pm 2$; $2m \pm 1$; $3m + 2$; $3m \pm 4$; $4m \pm 1$ and $4m \pm 5$. Among these, $n = 1 \pm 3k_2$ will have positive sequence, $n = 2 \pm 3k_2$ will have negative sequence $(k_2 = 0, 1, 2, \ldots)$. Since a harmonic with a given order can be obtained for different pairs of $k$ and $n$, on principle, positive, negative and zero sequence can result from identical-order harmonics. This might cause asymmetry in the stator current. However, for $m > 6$, only the harmonics belonging to the smallest value of $n$ can be of any importance.

(v) The initial phase angle $\varphi$ can only affect the values of the voltage harmonics with the most significant orders, till a harmonic with one order is obtained with an appreciable value for at least two pairs of $k$ and $n$. Since this is

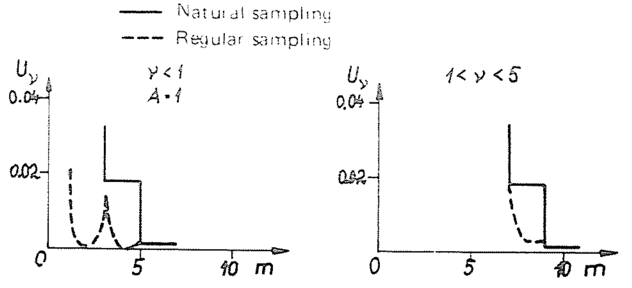


*Fig. 5*

impossible for $m > 6$, from this value of $m$ on the initial phase angle has practically no effect on the absolute values of the important harmonics.

(vi) With all three sampling techniques, the result is identical at $m \to \infty$. However, already at $m > 20$ the voltage harmonics important from the practical view are identical both regarding their order and their absolute value. Thus, there can be no difference between the sampling techniques in regard to the operation of the motor.

(vii) The regular sampling technique using both peaks of the triangular signal is advantageous as compared to natural sampling at $m<15$. With this technique, the value of the harmonics with the orders $(m-2)$, $(2m+1)$ is appreciably lower than with natural sampling. This has a favourable effect on the harmonic currents of the motor and on its warming.

(viii) The smallest-ordinal significant voltage harmonic is $(m-2)$ (positive order), and hence the lowest pulsation frequency in the moment of the motor will be $(m-3)f_a=f_v-3f_a$, this value being practically independent from the fundamental feed frequency $f_1=f_a \ll f_v$ of the motor. As a result, only relatively high-frequency pulsating moments will arise in the total operation range of the motor, and the rotor will be unable to follow these pulsations, so that the motor will run uniformly and smoothly. This is of particular importance at low rpm values, where otherwise there would be a risk of mechanical resonance. (Luckily, in this range the amplitudes of voltage harmonics are much reduced: the $(m-2)$ component, e.g., is of no significance either.)

## Appendix

From the theory of Bessel-functions it is well known that

$$e^{\pm jz \sin \Theta}=J_0(z)+\sum_{n=1}^{\infty}(\pm 1)^n[e^{jn\Theta}+(-1)^n e^{-jn\Theta}]\,J_n(z)\,, \tag{M1}$$

where $J_0(z)$ and $J_n(z)$ are first-kind Bessel-functions of the zero and $n$ order, resp.

According to WATSON [6, p. 554], if

$$M=E-\varepsilon \sin E\,, \tag{M2}$$

then

$$E=M+\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\varepsilon)\sin nM\,, \tag{M3}$$

and $(\nu \neq 1)$:

$$e^{j\nu E}=\nu\sum_{n=1}^{\infty}\left\{\frac{1}{n}\,[J_{n-\nu}(n\varepsilon)\,e^{jnM}-J_{n+\nu}(n\varepsilon)\,e^{-jnM}]\right\}. \tag{M4}$$

The equation corresponding to Eq. (M2) can be set up for the intersections $t_i$ and $t_{i+1}$ (Fig. 2):

$$\begin{aligned} \gamma i+\varphi &= t_i+\varphi-A\gamma \sin(t_i+\varphi); \\ \gamma i+2\gamma+\varphi+\pi &= t_{i+1}+\varphi+\pi-A\gamma\sin(t_{i+1}+\varphi+\pi). \end{aligned} \tag{M5}$$

According to [6], Eq. (M4) is valid for positive and integer values of $\nu$. It may, however, be assumed that the validity of Eq. (M4) can be extended to fractions and irrational numbers by performing the summation for all integers $n\pm\nu$. E.g. for $\nu=0, 1, n$ may have the values 0.1, 0.9, 1.1, 1.9, . . . :

$$\begin{aligned} e^{j0.1E}=0.1\Big[&\frac{1}{0.1}J_0(0.1\,\varepsilon)\,e^{j0.1M}-\frac{1}{0.9}J_1(0.9\,\varepsilon)\,e^{-j0.9M}+\frac{1}{1.1}J_1(1.1\,\varepsilon)\,e^{j1.1M}- \\ &-\frac{1}{1.9}J_2(1.9\,\varepsilon)e^{-j1.9M}+\dots\Big]. \end{aligned} \tag{M}$$


## Summary

In induction motor drives with variable frequency several methods came into general use for the control of PWM. invertors. The so-called natural sampling method and two kinds of so-called regular samplings are investigated in the paper. The amplitudes of the harmonics are analytically determined by the author, should the ratio of the frequencies of the triangle wave and of the reference be either integer, fraction or irrational number. Finally, conclusions are drawn as to the orders and amplitudes of the voltage components.


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