DETERMINATION OF THE STABILITY REGIONS OF NONLINEAR FEEDBACK SYSTEMS

By

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1. Introduction

The analysis of nonlinear systems stands, at present too, in the centre of interest; the article of Lyapunov [1] published in 1892 is even today the starting point of examinations. At nearly the same time Poincaré, too, came to decisive results in determination of the qualitative properties of differential equations [2]. Since the 1950s the methods of Lyapunov and the ones similar to his have their renaissance [1], [3].

For nonlinear systems, stability is not a "global" property, but—with continuous excitation—depends also on the initial conditions. It is of practical importance to find the set H of the initial conditions, starting from which the state variables tend to determine finite values in the case of $t \to \infty$. The present paper undertakes to find such sets for the case of second-order systems having given properties. Our examinations will be performed on the state plane (xy) where the behaviour of the second order system is described by a plane curve (the state trajectory), and the set H consists of one or more portions of the plane (Fig. 1).

The Direct Method of Lyapunov [1] and the methods presented in [3] and [4] have the common feature that the domain obtained in the non-systematic way is, in general, only a subset of the stability region.

The different methods yield differing stability region for the same system and the size of the region obtained considerably depends on the subjectivity of the person using the method (Fig. 10).

Our considerations are aimed at defining the curve SX—a separatrix—bounding the real stability region. The conclusions have been drawn from mathematical considerations and from a great number of results obtained on analog and digital computers for the dynamic model described by the equation system

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= g(x(t), y(t), a) \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} &= f(x(t), y(t), z) \end{aligned} \qquad (1)$$

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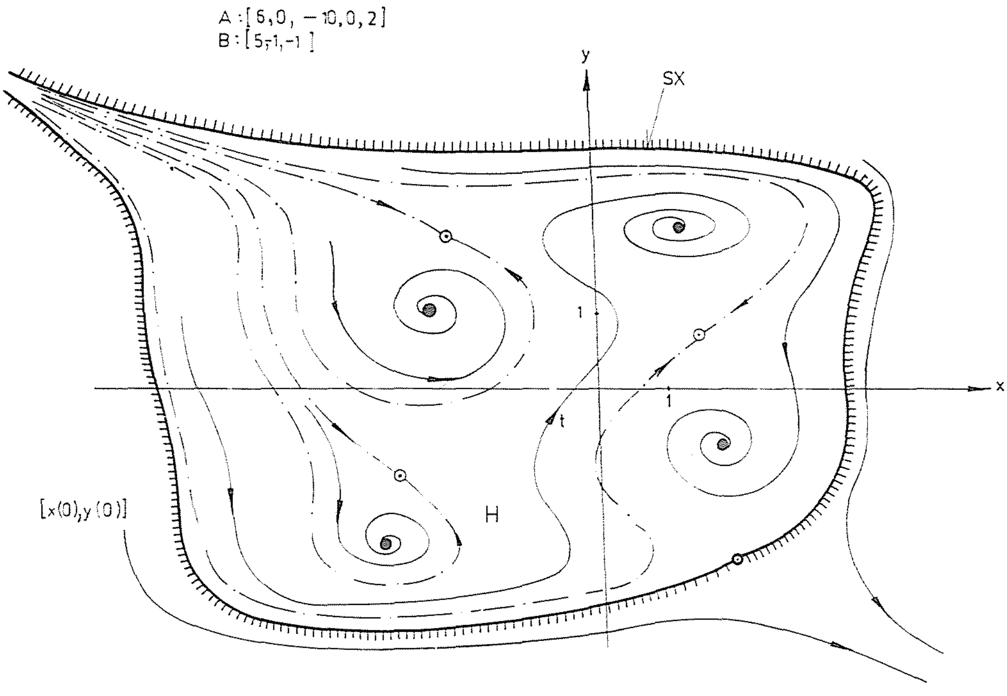


Fig. 1. Division of the state plane (*xy*) into stable and labile portions

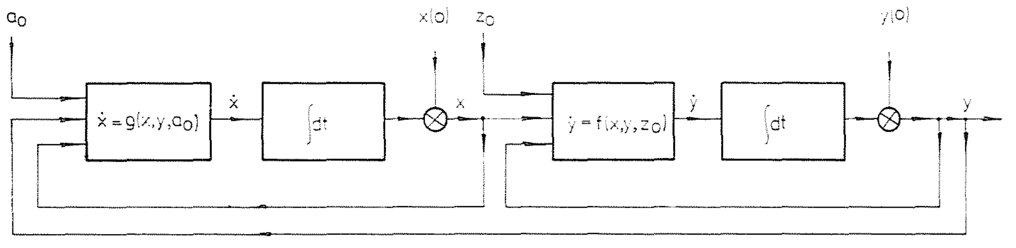


Fig. 2. Flow chart of the second-order nonlinear system examined

with functions f and g belonging to the class of polynomials. In the description of the properties of the flow chart structure corresponding to (1) (see Fig. 2), the terminological notions of control theory will be used,—since a similar topology is characteristic also of the control system working on the feedback principle.

2. Notations

As in control engineering, the conventional notations are used. The dot over the symbols denotes the derivative of the varying time. The simple dash means a column vector, and the double dash a matrix. The upper index T means transposition, while "Δ" is the sign of change. In the figures, the stable equilibrium point in the intersection of the static characteristics is indicated by a full circle "●" and the labile equilibrium point by an empty circle "○".

t	: time
$x(t)$	: state variable (modified signal)
$y(t)$	: state variable (controlled signal)
(xy)	: state plane
$[x(0), y(0)]$	: initial state plane condition
H	: domain of asymptotic stability of the state plane
SX	: separatrix bounding the domain of asymptotic stability
$g(x, y, a_0)=0$	: static characteristic curve of the controller
$f(x, y, z_0)=0$	: static characteristic curve of the controlled process
$A: [A_0, \ldots, A_n]$	: coefficient vector
$B: [B_0, \ldots, B_m]$	: coefficient vector
S	: stable singular point
L	: labile singular point
$\bar{q}(t)=[x(t), y(t)]^T$	: state vector
$\bar{u}=[a_0, z_0]^T$	: excitation vector
k	: notation for arbitrary singular point
$\operatorname{tg} \alpha$, $\operatorname{tg} \beta$	: slope of the static characteristic curves of controller and controlled process
$\bar{\bar{A}}_k$	: state matrix
$\bar{\bar{I}}$	: unity matrix
n, m	: degree numbers of the static characteristic curves of controller and controlled process
λ_i	: eigenvalue of the state matrix
$D(\lambda)=\det(\lambda\bar{\bar{I}}-\bar{\bar{A}})=0$	: characteristic equation of the linearized system
$\bar{i}, \bar{j}$	: unity vectors
$\bar{W}=g(x, y)\bar{i}+f(x, y)\bar{j}$	
$\operatorname{div} \bar{W}$	: divergence of function $\bar{W}$
(x_k, y_k)	: coordinate of the singular point marked k
P, Q	: points of the separatrix in the vicinity of the equilibrium point
P', Q'	: points approaching the points P and Q
$\varepsilon, \delta, \rho$	: positive real numbers

3. The structure examined

The mathematical model of the system consisting of the controller and of the controlled process described by first-order differential equations is the differential equation system (1) or the flow chart structure containing the feedback corresponding to it (Fig. 2). Also the vector differential equation

$$\dot{\bar{q}}(t)=\bar{F}(\bar{q}(t), \bar{u})$$

is suitable for describing the system, and thus expression (1) can be considered a two-dimensional special case of the general state space.

For functions g and f in expression (1) the following assumptions are made:

— Function g describing the control system and function f characterizing the controlled process are of the kind that both the control equipment and the controlled process have the property of selfregulation. This means that, separated from the feedback structure, and with signals constant in time given to their inputs, their output signals will tend to constant values in the case of $t \to \infty$ (Figures 3a, b).

— As a consequence of self-regulation, there exists a static characteristic defining the relationship between the steady-state signals of the controller and of the controlled process. This static characteristic is a graph plotted in the

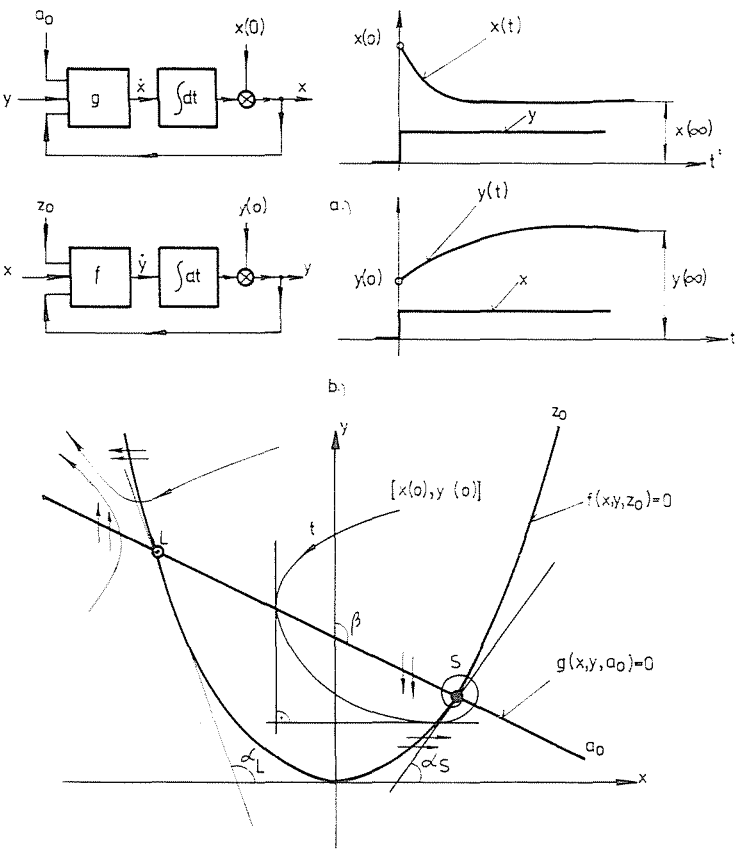


Fig. 3. Characteristic properties of the function g describing the control equipment and of the function f characterizing the controlled process

plane (xy) and parametered in a_0 and z_0 corresponding to the equations $g(x, y, a_0)=0$ and $f(x, y, z_0)=0$, respectively (Fig. 3c).

— The real solutions of the equation system

$$\begin{aligned} g(x, y, a_0)=0 \\ f(x, y, z_0)=0 \end{aligned} \tag{2}$$

(i.e. the intersections of the static characteristic curve of the controller and of the process) are the equilibrium (singular) points of the nonlinear system. The system under consideration has a finite number of singular points (points S, L in Fig. 3).

— In the state plane (x, y), based on differential equation (1) of the state trajectory satisfying an arbitrary initial assumption $x(0)$, $y(0)$:

$$\frac{dy}{dx} = \frac{f(x, y, z_0)}{g(x, y, a_0)} \tag{3}$$

Consequently, the functions $f(x, y, z_0)=0$ and $g(x, y, a_0)=0$ furnish—in addition to defining the static curves of both the controller and the process—determine also the locations of those points of the plane (x, y) where the state trajectories proceed with the horizontal and vertical slope, respectively (Fig. 3c).

— The functions g and f are continuous and single-valued, they have partial derivatives with respect to the variables x, y, and these are continuous.

4. Stability of a position of equilibrium

Lyapunov demonstrated (First Method of Lyapunov) that the stability conditions of phenomena taking place in the vicinity of an arbitrary singular point k of a nonlinear system can be determined from the eigenvalue distribution of the state matrix of the system linearized in the vicinity of the equilibrium position. The linearized state equations of nonlinear system (1) are with $(\Delta a \equiv \Delta z \equiv 0)$:

$$\begin{aligned} \Delta \dot{x} &= \left(\frac{\partial g}{\partial x}\right)_k \Delta x + \left(\frac{\partial g}{\partial y}\right)_k \Delta y \\ \Delta \dot{y} &= \left(\frac{\partial f}{\partial x}\right)_k \Delta x + \left(\frac{\partial f}{\partial y}\right)_k \Delta y \end{aligned} \tag{4}$$

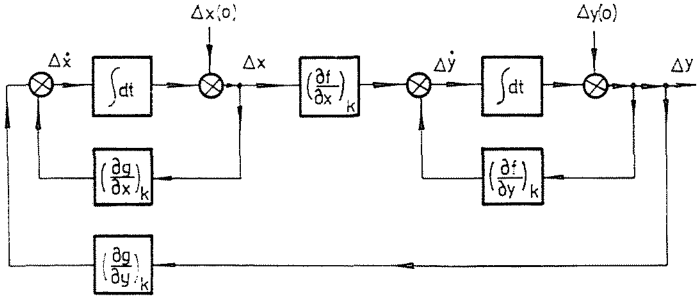


Fig. 4. Flow chart of the linearized system corresponding to the nonlinear second-order system

The flow charts corresponding to expression (4) can be seen in Figure 4. As it shows, the integrators must operate in negative feedback, owing to the assumption of self-regulation [5]. The condition of this is that

$$\left(\frac{\partial g}{\partial x}\right)_k < 0$$
$$\left(\frac{\partial f}{\partial y}\right)_k < 0 \tag{5}$$

have to hold for the neighbourhood of the equilibrium point. That domain of the plane (x, y) for which conditions (5) are satisfied, is termed self-regulation domain.

The stability of an arbitrary equilibrium point k of the plane (x, y) depends on the eigenvalues of the Jakobi-type state matrix.

$$\bar{A}_k = \begin{bmatrix} \left(\frac{\partial g}{\partial x}\right)_k & \left(\frac{\partial g}{\partial y}\right)_k \\ \left(\frac{\partial f}{\partial x}\right)_k & \left(\frac{\partial f}{\partial y}\right)_k \end{bmatrix} \tag{6}$$

As the eigenvalues satisfy the characteristic equation $D(\lambda) = 0$, it follows for the condition of stability that the characteristic equation

$$D(\lambda) = \left(\lambda - \left(\frac{\partial g}{\partial x}\right)_k\right)\left(\lambda - \left(\frac{\partial f}{\partial y}\right)_k\right) - \left(\frac{\partial f}{\partial x}\right)_k \cdot \left(\frac{\partial g}{\partial y}\right)_k = 0 \tag{7}$$

must have roots with negative real part. This is fulfilled if, in the intersection point in question of the static curves $g(x, y, a_0)=0$ and $f(x, y, z_0)=0$ (which is a possible equilibrium point of the system), the slopes of the tangents drawn to the curves satisfy the condition

$$\left(-\frac{\left(\frac{\partial g}{\partial y}\right)_k}{\left(\frac{\partial g}{\partial x}\right)_k}\right)\cdot\left(-\frac{\left(\frac{\partial f}{\partial x}\right)_k}{\left(\frac{\partial f}{\partial y}\right)_k}\right)=\operatorname{tg}\alpha_k\cdot\operatorname{tg}\beta_k<1 \tag{8}$$

Thus, in the second-order systems considered, the stability of the equilibrium point can be determined from the slopes of the tangents tan α_k and tan β_k drawn to the intersection points of the static characteristics (Fig. 3c) [5].

5. Distribution of the equilibrium points in the state plane (x, y)

In further examinations the functions $g(x, y, a_0)$ and $f(x, y, z_0)$ will be chosen from the class of polynomials in the way that the entire state plane (x, y) satisfies self-regulation conditions (5). Such requirements will be satisfied by the differential equation system

$$\begin{aligned}\dot{x}&=g(x, y, a_0)=-x+\sum_{i=0}^{n} A_i y^i\\ \dot{y}&=f(x, y, z_0)=-y+\sum_{j=0}^{m} B_j x^j\end{aligned} \tag{9}$$

The equations of the static curves of the controller and the process[1,2] are

$$\begin{aligned}x&=\sum_{i=0}^{n} A_i y^i\\ y&=\sum_{j=0}^{m} B_j x^j\end{aligned} \tag{10}$$

[1] The state equations may have the form (9) either because the original system itself is characterized by this type of the equations of material and energy equilibrium or because the otherwise arbitrary functions g and f have been approximated by polynomials in finite ranges.

[2] Since the summarization in (9) is started from $i=j=0$, and the excitation signals a_0 and z_0 enter the closed loop generally in additive way, also the correspondences $A_0 y^0=A_0=a_0$ and $B_0 x^0=B_0=z_0$ can be used.

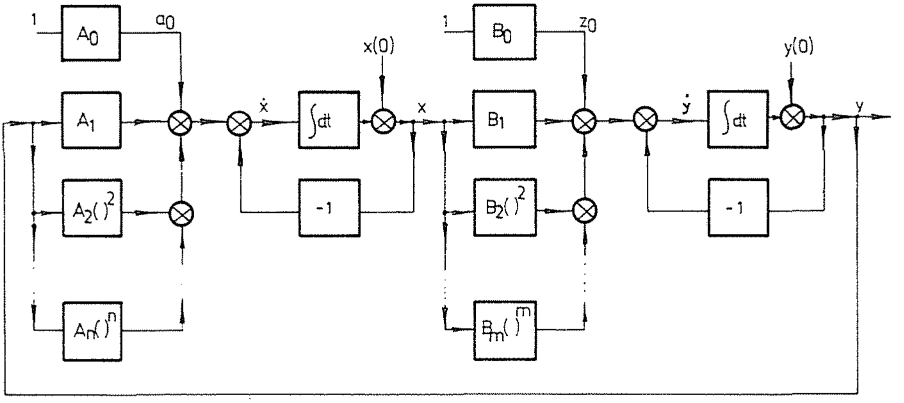


Fig. 5. Flow chart of the second-order system containing a nonlinearity of polynomial type

The flow chart of the nonlinear system corresponding to state equations (9) are shown in Figure 5.

An analysis of the static curves of the controller and the process with the use of expression (10) leads to the results summarized in Table 1, in which the static curves of the controller and the controlled process have been plotted for the case of polynomials (n, m: 1, 2, 3, 4) having differing degree numbers. For simplicity it has been assumed that the polynomials have coefficients A_i ($i = 0, 1, 2, 3, 4$) and B_j ($j = 0, 1, 2, 3, 4$) such that the static curves corresponding to them be symmetrical and intersect each other in the highest number of points possible in principle. Thus, the maximum number of the equilibrium points amounts to $n \cdot m$.

The state matrix of the linearized system corresponding to the nonlinear model (9)—from which the stability of the singular points having the coordinates x_k, y_k ($k = 1, 2, \ldots n \cdot m$)—will now be

$$\bar{\bar{A}}_k = \begin{bmatrix} -1 & \sum_{i=1}^{n} iA_i y_k^{i-1} \\ \sum_{j=1}^{m} jB_j x_k^{j-1} & -1 \end{bmatrix} \tag{11}$$

For the given system, $n \cdot m$ state matrices of type (11) must be written in order that the state of stability or lability of all the equilibrium points can be judged from the eigenvalues of (11). Characteristic equation of the linearized system is

$$\lambda^2 + 2\lambda + 1 - \sum_{i=1}^{n} \sum_{j=1}^{m} ijA_i B_j y_k^{i-1} x_k^{j-1} = 0 \tag{12}$$

Let the following cases be distinguished from each other:

a. $n=1$ and $m=1$ constitute a special case of the linear system having a single equilibrium point. In this case the characteristic equation (12) will be

$$\lambda^2+2\lambda+1-A_1B_1=0$$

and condition of the equilibrium point stability: $A_1B_1=\text{tg}\,\alpha_1\cdot\text{tg}\,\beta_1<1$. (Table 1, 1st line, 1st column).

b. If $n>1$ and $m\geqq 1$ or $n\geqq 1$ and $m>1$, the system is nonlinear and has $n\cdot m$ singular points. The stability of the individual equilibrium points can be decided with the use of the characteristic equation (12). A considerably simpler procedure is the one presented in [5]; if it is known of an arbitrary singular point what type an equilibrium point is from the view-point of stability, the stability conditions of the rest of equilibrium points can simply be decided, for —progressing along one of the static curves in arbitrary direction—the equilibrium points may follow each other exclusively in the order ... stable (S), labile (L), stable (S)...etc.[3]

The characteristic curves corresponding to $n=m=4$ are plotted separately in Fig. 6. It is easy to find a stable equilibrium point, since it is

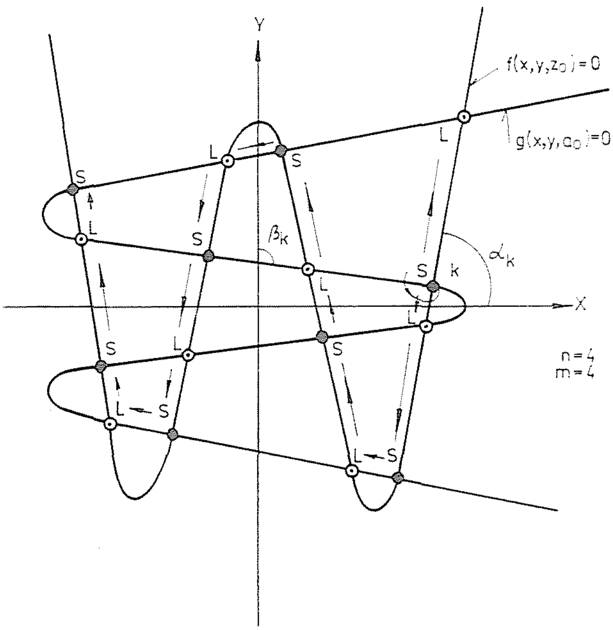
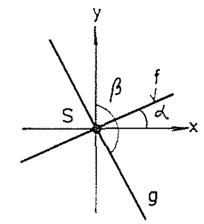
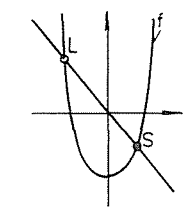
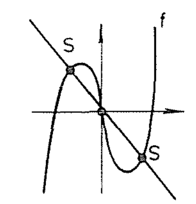
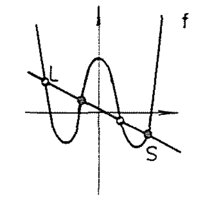
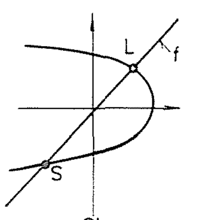
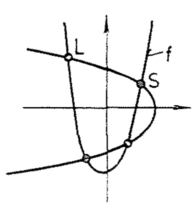
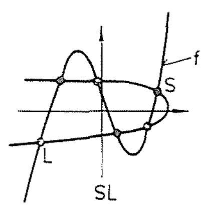
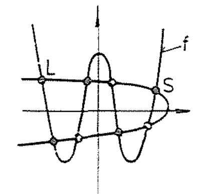
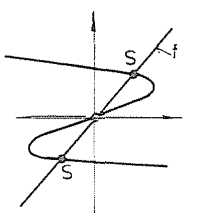
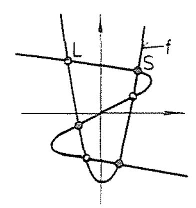
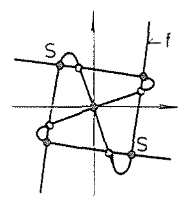
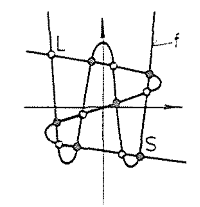
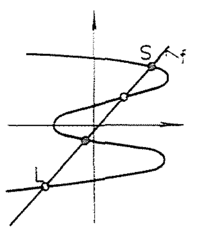
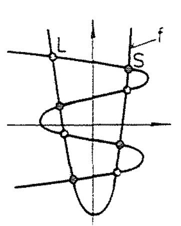
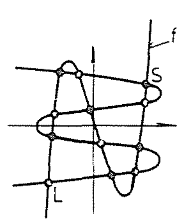
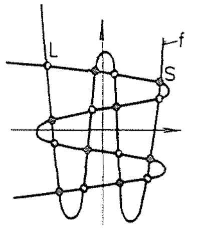


Fig. 6. Distribution of the equilibrium points in the case of an *LS*-type system

[3] If the static curves $g(x, y, a_0)=0$ and $f(x, y, z_0)=0$ are tangential to each other in an arbitrary point, then this is to be considered as the coincidence of two equilibrium points (of type S and L).

Table 1

n \ m	1	2	3	4
1	 SS	 SL	 SS	 SL
2	 SL	 SL	 SL	 SL
3	 SS	 SL	 SS	 SL
4	 SL	 SL	 SL	 SL

necessary to find an intersection point in which the slopes of the curves are of signs opposite to each other. Such a point is, e.g., the S-type equilibrium point of mark k, since here the condition $\tan\alpha_k \tan\beta_k < 0 < 1$ is sure to be satisfied $(0 < \alpha_k < \frac{\pi}{2} < \beta_k < \pi)$. This means, on the one hand, that the state trajectory starting from the vicinity of this point runs to it, and on the other hand—e.g. processing along the curve f—the other equilibrium points follow each other in the order labile, stable, labile, etc.

6. Division of the state plane (xy) into stable and labile regions

After having decided whether the equilibrium points are stable or labile, we only know that the state trajectories starting from the vicinity of the stable points end in stable equilibrium points.

The types of the equilibrium points, however, do not give any conclusion to how large exactly this vicinity is and how the entire plane (xy) can be divided into stable and labile domains.

From an analysis of Table 1 it can be seen that—progressing in either direction along the curve f—the first and last equilibrium points will be of differing type (SL systems) when $n \cdot m$ is an even number, and of identical type (SS or LL systems) when $n \cdot m$ is an odd number. Of the latter, the SS type systems are included in Table 1. However, e.g., if $n=3, m=3$ and $A_3<0, B_3<0$, the static curves will have the shapes shown in Fig. 7. Here—in contrast with the case corresponding to $n \cdot m = 9$ shown in Table 1—the first and last equilibrium points are of type L, and thus the system is an LL system.

The classification of dynamic systems into SS, SL and LL types leads to the mechanical interpretation shown in Figure 8, and this has been supported also experimentally by actual analyses. According to these, in the case of an SS system, the whole plane consists of stable domains. This means that a trajectory started from any point of the plane (xy) remains in the finite range in contrast with the LL or SL systems, where the plane has a domain, from which the started trajectory tends to the infinity. This recognition creates a newer possibility of grouping the systems. Let the nonlinear functions in differential equation (9) be approximated with their components having the highest exponents. The state equations obtained in this way will be[4]

$$\begin{aligned}\dot{x} &= -x + A_n y^n \\ \dot{y} &= -y + B_m x^m\end{aligned} \tag{13}$$

[4] State equation (13) corresponds to such an approximation of (9) when the vicinity of the origo is circumscribed by a circle having radius r, which includes all the singular points; then contraction is performed by coordinate transformation to make the domain with radius r very small.

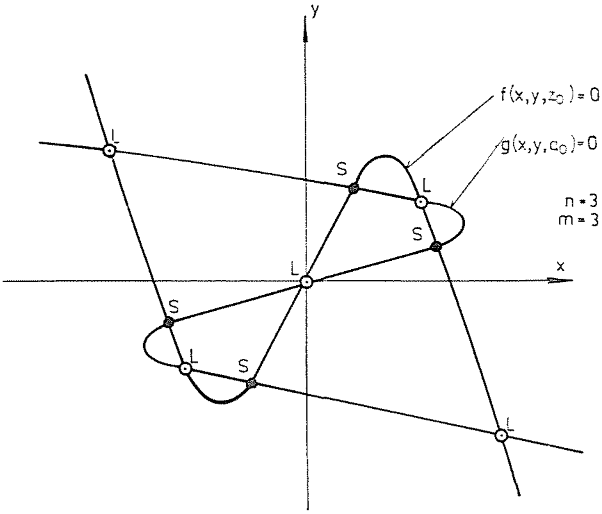


Fig. 7. Static curves and singular points of an *LS*-type system

Then the algebraic equation system determining the equilibrium points will have the forms:

$$x = A_n y^n \\ y = B_m x^m \tag{14}$$

Table 2 summarizes the possible cases characteristic of system (13).

The system described by differential equation (13) has either one, or two, or three equilibrium points. As the state matrix of the linearized system is now

$$\bar{\bar{A}}_k = \begin{bmatrix} -1 & nA_n y_k^{n-1} \\ mB_m x_k^{m-1} & -1 \end{bmatrix} \tag{15}$$

and thus, according to (14), the origo $x=0$, $y=0$ of the state plane (xy) is one of the equilibrium points whose vicinity is sure to be stable (S-type origo). From this follows that the second or the other two equilibrium points of systems having two or three equilibrium points, respectively, are certainly L-type (labile) points. This means, at the same time, that of the cases included in Table 2, in any of the systems satisfying the conditions

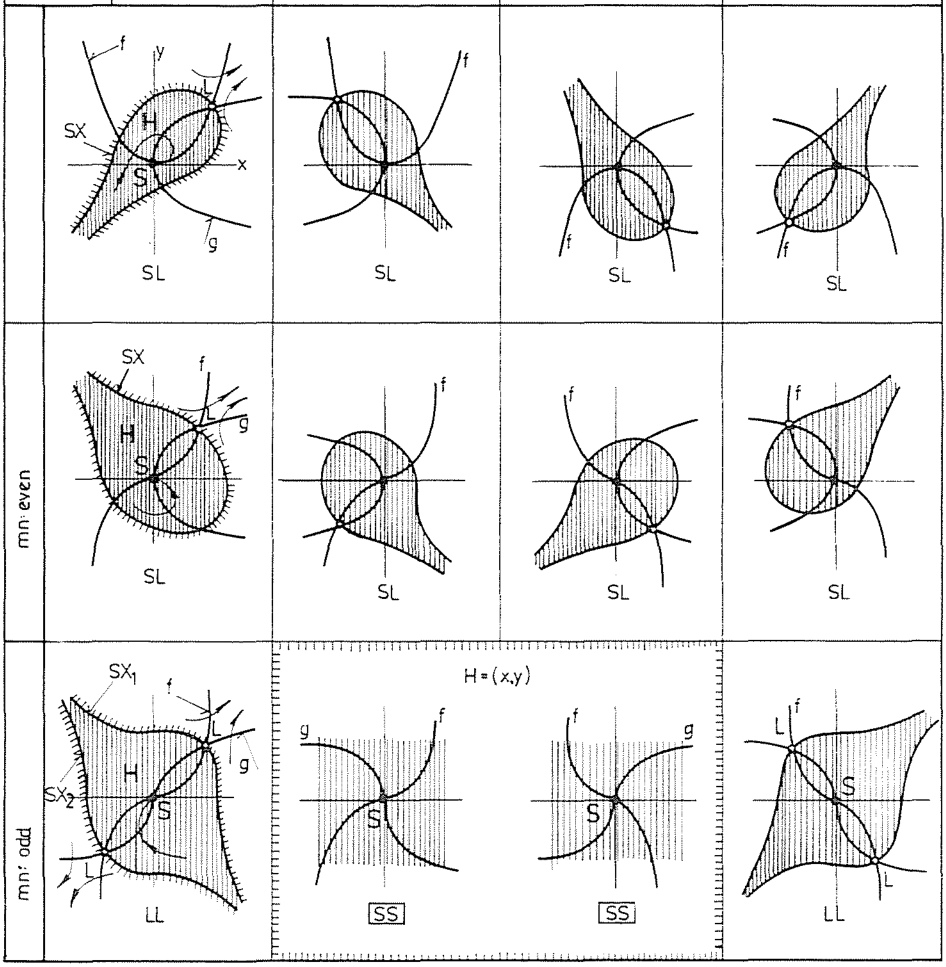
$n \cdot m$: odd number

$\operatorname{sign} A_n \cdot \operatorname{sign} B_m < 0$

the plane (xy) consists of stable domains. In all the other cases, only in a given part of (xy) is the feedback system stable, and the stable and labile parts are separated from each other by the separatrix marked SX.

Table 2

sign An	1	1	−1	−1
sign Bm	1	−1	1	−1



As the original system (9) has been approximated by means of (13), Table 2 can be considered to have general validity. From this follows, e.g., that in the case of $n=3$, $m=3$, $A_3>0$, $B_3<0$ (or $A_3<0$ and $B_3>0$)—irrespective of the values of the coefficients A_0, A_1, A_2 and B_0, B_1, B_2—the entire plane (xy) is stable, and the separatrices separate here only the stable portions from each other (Fig. 9a). On the contrary, with sign $A_3 \cdot$ sign $B_3>0$, the plane—similarly to the cases of $n \cdot m =$ an even number—is divided into stable and labile regions.

Within this, the stable domain is divided into several portions, depending on how many stable equilibrium points there are in it (Fig. 9b).

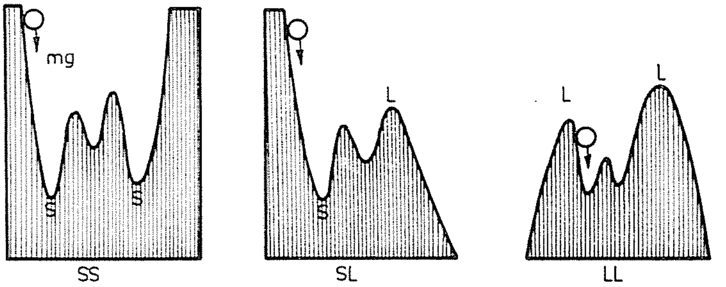


Fig. 8. Mechanical illustration of the movement conditions of *SS*, *SL* and *LL* systems

7. Determination of the stability regions

A number of papers have dealt with determining the domain of asymptotic stability of nonlinear system (1). The Direct Method of Lyapunov [1] offers the possibility of directly estimating the size of the stability region. Also the method elaborated by Infante and Clark and based on Bendixon's criterion determines only a subset of the whole stability region [3].[5]

The present work is aimed at elaborating a numerical method of determining the limiting contour (Separatrix) of the stability region.

The singular points of the system described by differential equation (9) may be stable focal or nodal points, or labile saddle points [10]. An asymptotic stability region belongs to each of the stable singular points.

[5] As an example, consider the system

$$\dot{x} = -x + (1 + y)$$
$$\dot{y} = -y - (4 + 3x + x^2).$$

The static curves and the singular points can be read off from Figure 10. On the basis of the foregoing it is easy to see that the point having coordinates $(-1, -2)$ is a stable focal point, while the equilibrium point of coordinates $(-3, -4)$ is a labile saddle point. A possible Lyapunov function belonging to the stable focal point is

$$V(x, y) = \frac{1}{2}(y+2)^2 - (x+1)(y+2) + \frac{3}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 < \frac{4}{3}$$

The corresponding stability region is indicated in Fig. 10. (Of course, another region would belong to another Lyapunov function.)

The stability region calculated by the method of Infante and Clark would be bounded by the curves

$$H(x, y) = \frac{1}{3}(x+1)^3 + 3(x+1)^2 + 2(x+1)(y-x+1) + \frac{1}{2}(y-x+1)^2 = \frac{28}{3}$$

$$x = -3$$

The domain obtained in this way is considerably larger than the previous one, but smaller than the stability region bounded by *SX* and similarly indicated in the Figure. Details of the calculation are contained in [6].

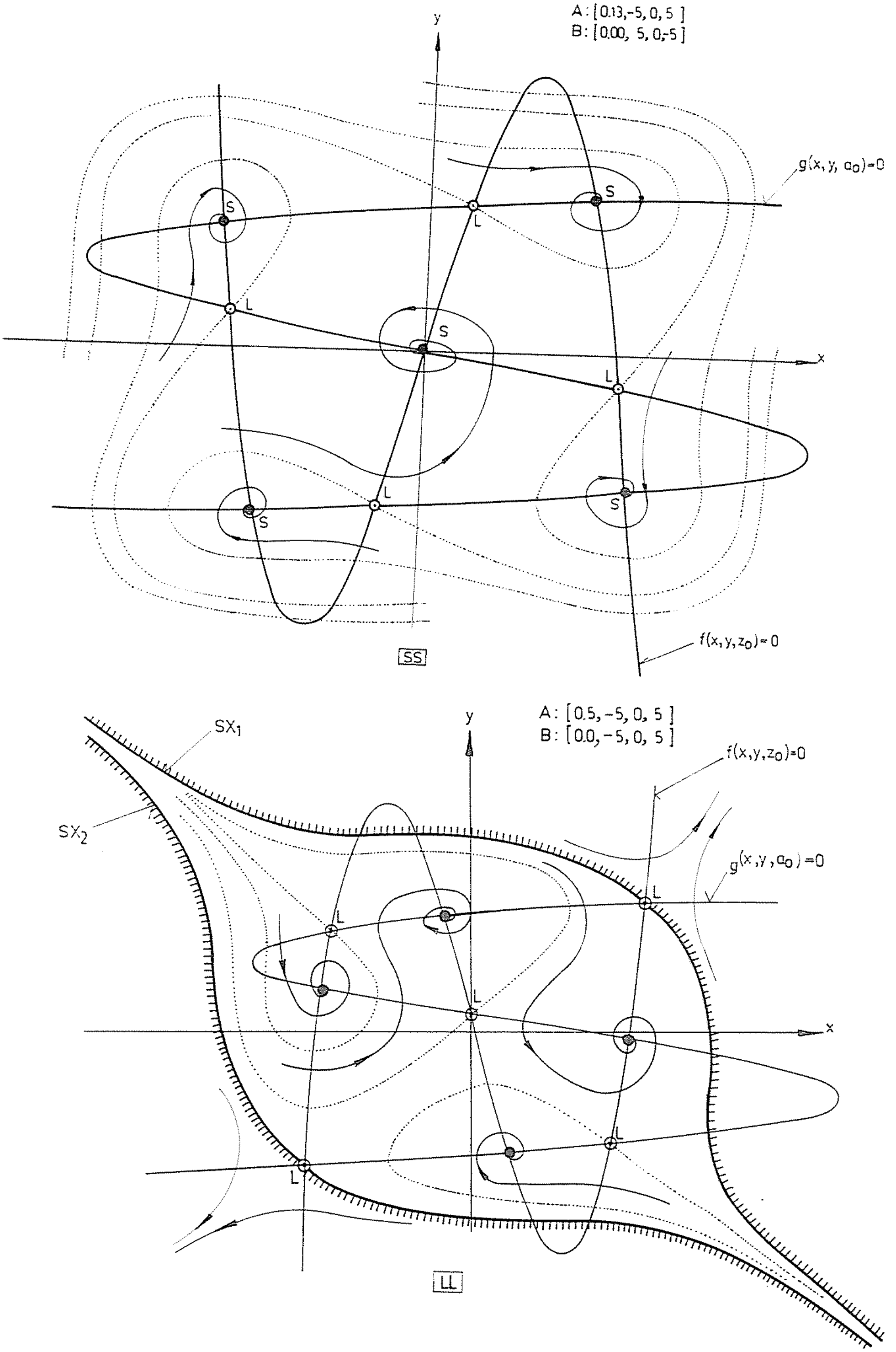


Fig. 9. Stable regions of the *SS*-type system and labile and stable regions of the *LL*-type system

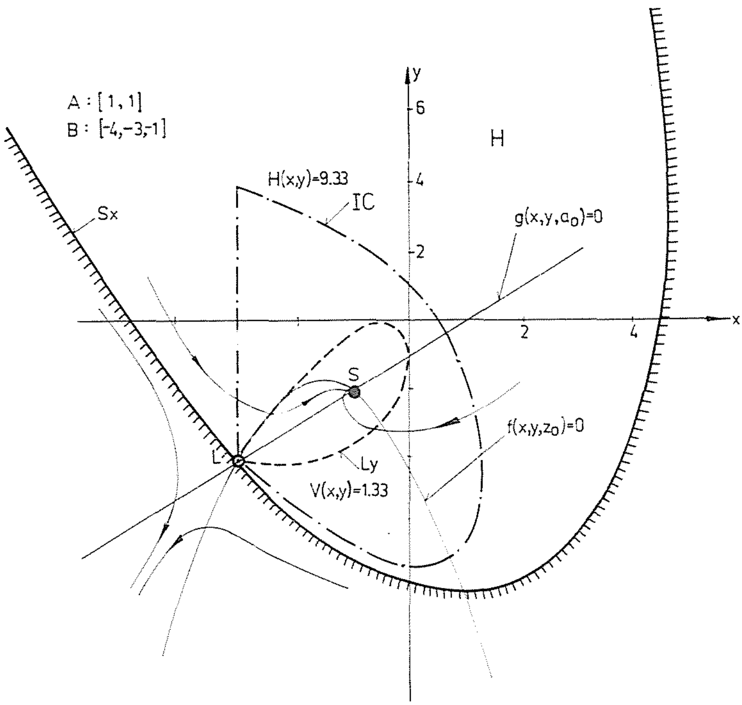


Fig. 10. Comparison of stability region obtained by different methods

The given nonlinear system (9) has no limited or closed regions of asymptotic stability [6]. It is easy to see that the Bendixon criterion [10] is fulfilled for state equations (9) in the whole plane, for div $\bar{W} = -2$, and thus there is no limit cycle or closed trajectory in our system. This means that the special trajectories limiting the regions of asymptotic stability cannot close in the finiteness, i.e., the stability regions extend necessarily to the infinite (Table 2).

In the case of the system examined, the direct and complete determination of the stability region is made possible through the recognition based on practical experience according to which, for the system having state equations (9), the limiting contours enveloping the regions of asymptotic stability are the special trajectory pairs that run to the labile saddle points of the system, when $t \to \infty$. In cases of SL systems the stability region is simple, the separatrix SX is the trajectory passing through the extreme point L (Fig. 10). In cases of LL-type systems, H is a compound set (Fig. 9).

The desired limit contours are special trajectory pairs satisfying differential equation (3), passing through the saddle-type singular points or ending there in the case of $t \to \infty$. The numerical determination by computer of the trajectory passing through such a singular point involves difficulties. The

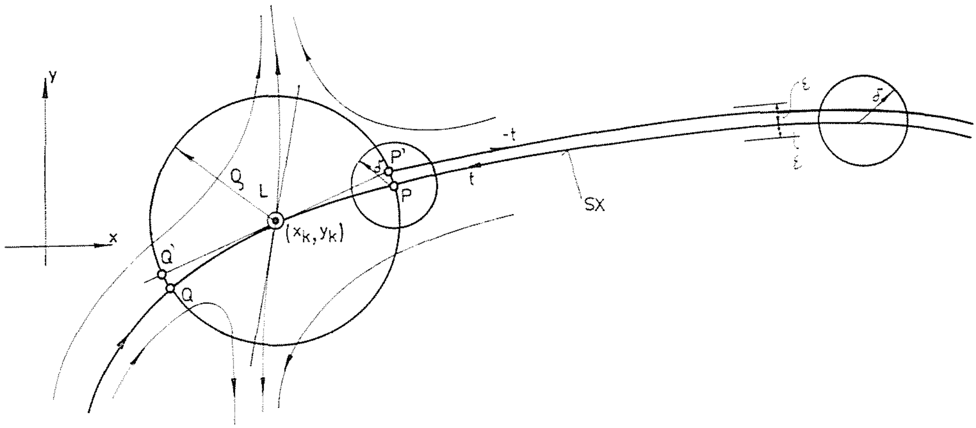


Fig. 11. Numerical determination of the starting point of the separatrix

reason for this is that no trajectory can be started from a singular point. The procedures using the method of the type Runge-Kutta substitute the coordinates of the singular point (x_k, y_k) as initial conditions into state equations (9). As the functions g and f give zero values in the singular point, the integration does not start at all. From all this one can conclude that other points P, Q (differing from the singular point) of the trajectory ending in the saddle point are necessary, from where the integration can be started. Such points P and Q can be approximated by the following recommended method (Fig. 11).

The method is based on the fact that in the vicinity of its saddle-type singular point with small radius ρ the nonlinear system (9) can be approximated well with the use of the linearized state equations (4). The desired points P and Q will be approximated with points P' and Q' on the trajectory traversing the saddle point of the linearized system, since these trajectories are straight lines and their equations can be determined. In Figure 11, we have plotted the movement conditions characteristic of the neighbourhood of a saddle point of the nonlinear system (9), and also the points P, Q and P', Q'.

Using the notations of Figure 11, the following theorem will be employed [7]: For each $\varepsilon>0$ there exists a real number $\delta>0$ if $\|(x(0), y(0))-(x(0), y(0))^*\|<\delta$, then the trajectories starting from the two initial conditions $(x(0), y(0))$ and $(x(0), y(0))^*$ of differential equation (9) differ from each other by a value smaller than ε in a finite and closed time interval.

In a finite region of the plane (xy), in the case of a finite time interval, such a value δ can be chosen to the given error ε that can be estimated with the value ρ from above. Thus it can be stated that there exists such a value ρ with which, by inverting the time, the trajectory started from points P', Q' with a value $\Delta t<0$ differs from the trajectories traversing the real points P and Q in the whole range by a value not higher than ε (disregarding the formula error of the numerical integration method and the truncation error of the computer). The concrete value of ρ can be decided only by a thorough investigation of the given system and by repeated trial runs on the computer.

After this, the numerical computerized calculation of the limit contours enveloping the regions of asymptotic stability of the nonlinear system described by state equations (9) will be performed in the following way:

For all the labile singular points, the following steps have to be executed:

a. The coordinates of the labile saddle point (x_k, y_k) are furnished by equation system (10). The numerical method of finding the polynomial roots has been used to determine the roots of equation system (10).

b. The eigenvalues λ_1, λ_2 of state matrix (11) are calculated in the place $x = x_k$, $y = y_k$ by the solution of (12).

c. In the case $t \to \infty$ of the linearized system the slope of the trajectories ending in the saddle point is given by the expression

$$m_1 = \frac{\lambda_1 + 1}{\sum_{i=0}^{n} i A_i y_k^{i-1}}$$

where λ_1 is the negative eigenvalue of the matrix $\bar{A}$ [9].

d. The coordinates of points P' and Q' in the case of given x_k, y_k are furnished by the expressions

$$P': \left(x_k + \frac{\rho}{\sqrt{1+m_1^2}},\ y_k + \frac{\rho m_1}{\sqrt{1+m_1^2}} \right)$$

$$Q': \left(x_k - \frac{\rho}{\sqrt{1+m_1^2}},\ y_k - \frac{\rho m_1}{\sqrt{1+m_1^2}} \right)$$

e. The SX limit contours are determined by "retrograde integration", starting from points P', Q' with the choice of $\Delta t < 0$[6].

[6] The trajectory pictures of the present paper were made by the second order Runge—Kutta method. The choice of the step space $\Delta t < 0$ was made automatically, with taking into account the local radius of curvature of the trajectory.

Examples

The static curves, the state trajectories of the systems (1) discussed in the present paper, the coordinates of their singular points and the separatrix SX limiting the stable region H of the state plane (xy) have been determined by means of the computer program based on the numerical procedure described. If there are more than one singular points inside the region H, the stable region will be divided into as many internal regions as there are S-type singular points in H. The trajectories limiting the regions traverse the L-type points present in H. The program system plots also these curves.

Example 1

$$\dot{x} = -x + 6 - 10y^2 + 2y^4$$
$$\dot{y} = -y + 5 - x - x^2$$

From the functions one can read: $n=4$, $m=2$ and sign A_4 · sign $B_2 < 0$.

There are 8 singular points, and the systems is of SL-type. The limiting contour SX, the singular points and the internal division of the region H are shown in Fig. 1.

Example 2

$$\dot{x} = -x + 0.13 - 5y + 5y^3$$
$$\dot{y} = -y + 5x - 5x^3$$

As $n=m=3$ and sign A_3 · sign $B_3 < 0$, H is the entire plane (xy). The stable regions are separated from each other by the trajectories ending in points L. The system is of the SS-type (Fig. 9.a).

Example 3

$$\dot{x} = -x + 0.50 - 5y + 5y^3$$
$$\dot{y} = -y - 5x + 5x^3$$

From the differential equation one can read: $n=m=3$, and sign A_3 · sign $B_3 > 0$. The system is of the LL-type. The stable region is bounded by the separatrices SX_1 and SX_2 traversing the two "extreme" points L, and H is divided into regions (Fig. 9.b).

Summary

The study discusses the determination of the regions of asymptotic stability of second-order nonlinear feedback dynamic systems. The system is described by a first order nonlinear vector differential equation (state equation).

The stability region of the state plane means the set H of the initial conditions from which — with the assumption of excitation constant in time, — after starting a movement process, the state variables tend to determined finite values when $t \to \infty$. The trajectories starting from this set of the initial conditions converge into the stable singular points of the state plane. The limiting contour (separatrix) determining the stability region was plotted with a computer program.

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