ON A FUNCTIONAL INTEGRAL EQUATION

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1. In a paper of R. A. Horn [1] the following integral-functional equation was investigated in connection with a problem in probability:

$$f(x)=\frac{1}{2}\int_{x-1}^{x+1} f(t)\,\mathrm{d}t\,. \tag{1}$$

Horn proved that the general solution of (1) for every real x in the class of continuous functions of polynomial growth is: $f(x)=c_0+c_1x$ with arbitrary c_0 and c_1 (Lemma 3.7 in [1]). To get this result the technic of Fourier-transformation was used. In a further paper of R. A. Horn and R. D. Meredith [2] the following statement was proved: Let us denote by $S(\alpha)$ the set of functions $f\colon R\to R$, locally integrable for which $f(x)=0(\exp\alpha|x|)$ $(|x|\to\infty)$, $\alpha\in R_+$; $S=\bigcup_{\alpha\in R_+} S(\alpha)$. The general solution of (1) in the class S has the form $f(x)=c_0+c_1x$. This theorem was proved by reducing (1) to a system of homogeneous linear equations in infinitely many unknowns, which system could be solved by elementary method.

We will give now the general solution of (1) without making any restriction on the grows, supposing only the local integrability of the unknown function f.

2. Let us denote by L_{loc} the set of all functions defined for every real value of the argument and integrable on every bounded interval.

Theorem 1. The general solution of (1) in the class of functions L_{loc} is of the form

$$f(x)=c_0+c_1x$$

with arbitrary c_0 and c_1.

Proof. Observe first, that any locally integrable solution of (1) is necessarily continuous and has derivatives of every order. If f is a solution of (1) in L_{loc} and $F(x)$ is an arbitrary primitive function of f, then (1) is equivalent to

$$f(x)=\frac{F(x+1)-F(x-1)}{2}$$

for every real x. From this we see at once F is a continuously differentiable function with the property, that for every interval of the length 2, $\vartheta=1/2$ in the Lagrange middle-value theorem. By a well-known theorem of R. Rothe [3] and J. V. Gonçalves [4] $F(x)$ can have only the following form:

$$F(x)=a_0+a_1x+a_2x^2 \text{ or } F(x)=a_0+a_1x+a_2\exp(cx).$$

If the first case occurs, the theorem is just proved.

In the second case due to linearity it is enough to look for the possible value of c. Would $\exp(cx)$ be a solution, then we would have

$$ce^{cx}=\frac{e^{c(x+1)}-e^{c(x-1)}}{2}=e^{cx}\frac{e^c-e^{-c}}{2}$$

this implies

$$c=\frac{e^c-e^{-c}}{2}=\sinh c.$$

From this follows $c=0$ and this completes the proof.

If f is a solution of (1) then f satisfies in the same time the following functional-differential equations

$$f^{(n)}(x)=\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+n-2k) \tag{2}$$

for every real x and for every positive integer n. In connection with this, Theorem 1 means that only the linear functions satisfy all the functional differential equations (2) without making any assumption on the grows of f.

3. We can also put (1) in the form

$$f(x)=\int_0^1 f(t-(x-1))\,dt$$

and this equation can be considered as a special case of the following functional-integral equation:

$$f\left(\frac{u+v}{2}\right)=\int_0^1 f(ut+v(1-t))\,dt \tag{3}$$

where u and v are independent real variables.

Theorem 2. The general solution of (3) in the class L_{loc} is: $f(x)=c_0+c_1x$.
Proof. If $u\neq v$ then (3) implies

$$f\left(\frac{u+v}{2}\right)=\frac{1}{u-v}\int_u^v f(t)\,dt$$

and we see immediately that every solution $f\in L_{loc}$ of (3) is necessarily continuous and has derivatives of any order. If again F denotes a primitive function of f, then

$$f\left(\frac{u+v}{2}\right)=\frac{F(u)-F(v)}{u-v}.$$

this means F is a function possessing the property that in the middle-value theorem of Lagrange $\vartheta=1/2$ independently of the length and position of the interval. Therefore by the theorem of Rothe [3] and Gonçalves [4] F is a quadratic polynomial and therefore $f(x)$ is linear. As every linear function satisfies (3) Theorem 2 is proved.

Summary

The paper investigates a functional-integral equation and its general solution is given.

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