ELECTROMAGNETIC FIELD OF COUPLED CONDUCTORS BY VARIATIONAL CALCULUS

By

I. BÁRDI

Department of Theoretical Electricity, Technical University, Budapest
Received July 8, 1981

Presented by Prof. Dr. I. VÁGÓ

Introduction

The practice of electrical engineering presents numerous problems involving the determination of the electromagnetic field and impedance of adjacent, parallel, sinusoidally excited, coupled conductors with arbitrary cross section. There is no analytical method for taking either the arbitrary cross section or the effect of proximity into account.

Calculation of the electromagnetic field of even a single conductor of other than circular cross section presents difficulties, a problem solved by Altshuler [11] using numerical approximation, a collocational method. Unfortunately, approximate functions applied in the field outside the conductor, behaved unsuitably in infinity. For the case of a single conductor, a more exact solution by variational method is found in [8]. The analytic method fails in taking the proximity effect, i.e. the influence of the current of one conductor on the current density of the other conductor into account. The non-uniform current density in a single conductor, is due to the magnetic field of the current of the conductor. In the case of two or more adjacent conductors, the current density of one conductor depends not only on the magnetic field of its own current, but also on those of the other conductors. No solution by analytical methods is found in the literature even for the case of circular cross section.

This paper, a development of [8], presents a numerical method based on variational calculus, suitable for calculating the electromagnetic field and the inner impedance of conductors with arbitrary cross section, subject to steady-state excitation, taking the proximity effect into account. Fundaments of the applied variational calculus are found in [1, 3, 6, 8].

Field equations

Let us consider two adjacent, parallel, coupled conductors, with arbitrary cross section and finite conductivity σ, carrying sinusoidal current of angular frequency ω (Fig. 1).

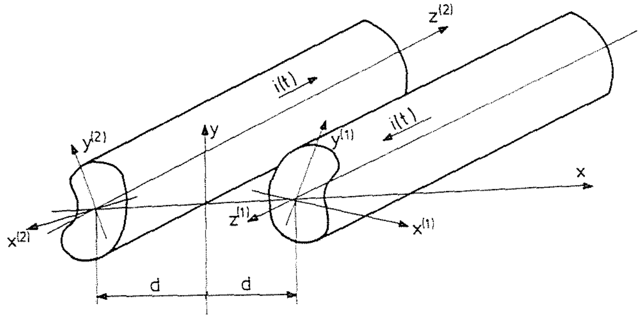

Fig. 1

Maxwell's equations are written for a charge-free field, using complex peak values of the field intensities:

$$\text{rot}\,\mathbf{H} = \mathbf{J}^e \tag{1}$$

$$\text{rot}\,\mathbf{E} = -j\omega\mathbf{B} \tag{2}$$

$$\text{div}\,\mathbf{D} = \mathbf{0} \tag{3}$$

$$\text{div}\,\mathbf{B} = \mathbf{0} \tag{4}$$

$$\mathbf{B} = \mu\mathbf{H}\,; \quad \mathbf{D} = \varepsilon\mathbf{E} \tag{5}$$

$$\mathbf{J}^e = \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E} \tag{6}$$

where $\mathbf{J}^e$ is electrical current density. Using vector potential

$$\mathbf{B} = \text{rot}\,\mathbf{A} \tag{7}$$

as usual, leads to the differential equation:

$$\text{rot rot}\,\mathbf{A} + j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A} - \text{grad div}\,\mathbf{A} = \mathbf{0}\,, \tag{8}$$

taking Lorentz's condition

$$\operatorname{div} \mathbf{A} = -(\mu\sigma + j\omega\mu\varepsilon)\varphi \tag{9}$$

into account (φ is the scalar potential). We note that in the literature the equation

$$\Delta\mathbf{A} - j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A} = \mathbf{0} \tag{10}$$

is usual, which can be derived from (8), using

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \Delta\mathbf{A} \tag{11}$$

Equation (8) is convenient for the variational calculus.

Solving Eq. (8) to meet the boundary conditions yields the vector potential, leading, in turn, to the magnetic induction vector from Eq. (7), the electric field intensity being:

$$\mathbf{E} = \frac{1}{\mu\sigma + j\omega\mu\varepsilon}\operatorname{grad}\operatorname{div}\mathbf{A} - j\omega\mathbf{A}\,. \tag{12}$$

The application of numerical methods for low frequencies presents a computing difficulty. Namely, inside the conductor the current density is:

$$\mathbf{J} = -j\omega\sigma\mathbf{A} \tag{13}$$

($\operatorname{div}\mathbf{A} = 0$). Since $\mathbf{J}$ is finite and $\mathbf{J} \neq 0$ throughout the cross section for $\omega \to 0$, vector potential $\mathbf{A}$ tends to infinity. For solving this problem, the electromagnetic field is considered as sum of two fields:

$$\mathbf{A} = \mathbf{A}' + \mathbf{A}'' \tag{14}$$

$$\varphi = \varphi' + \varphi'' \tag{15}$$

$$\mathbf{B} = \mathbf{B}' + \mathbf{B}'' \tag{16}$$

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}''\,. \tag{17}$$

This decomposition should meet the following relationships:

$$\text{rot}\,\mathbf{H}'=\mathbf{J}^{e'};\quad \text{rot}\,\mathbf{E}'=\mathbf{0} \tag{18}$$

$$\text{div}\,\mathbf{D}'=0;\quad \text{div}\,\mathbf{B}'=0 \tag{19}$$

$$\mathbf{D}'=\varepsilon\mathbf{E}';\quad \mathbf{B}'=\mu\mathbf{H}';\quad \mathbf{J}^{e'}=\sigma\mathbf{E}'+j\omega\varepsilon\mathbf{E}' \tag{20}$$

and

$$\text{rot}\,\mathbf{H}''=\mathbf{J}^{e''};\quad \text{rot}\,\mathbf{E}''=-j\omega\mathbf{B}''-j\omega\mathbf{B}' \tag{21}$$

$$\text{div}\,\mathbf{D}''=0;\quad \text{div}\,\mathbf{B}''=0 \tag{22}$$

$$\mathbf{D}''=\varepsilon\mathbf{E}'';\quad \mathbf{B}''=\mu\mathbf{H}'';\quad \mathbf{J}^{e''}=\sigma\mathbf{E}''+j\omega\varepsilon\mathbf{E}'' \tag{23}$$

Eqs (18) to (20) are seen to contain magnitudes with a single comma superscript and the electrical field to be irrotational as against Eqs (21) to (23). The irrotational electric field generates a time-dependent, but steady-state magnetic field, since the timely variation of the magnetic field does not generate stationary electrical field $\mathbf{E}'$. Magnetic field $\mathbf{H}'$ generaters electric field $\mathbf{E}'$ (it is called eddy-current field) which modifies the distribution of both field $\mathbf{E}'$ and of the current density. Field characteristics with comma superscript yield the direct current solution at zero frequency gradually modified by the eddy-current field $\mathbf{E}''$ for increasing frequencies. The arising field can be imagined superposition of frequency-varying eddy-current field on the zero frequency stationary field at each frequency. Obviously, the eddy-current field tends to zero at low frequencies. In reality, the two fields do not exist separately the above conception is merely a physical interpretation of the mathematical decomposition.

Accordingly, the problem is solved in two steps. The first step is to solve the problem of a time-dependent, but stationary electromagnetic field (Eqs (18) to (20)). The second step is to solve the eddy-current field generated by magnetic field intensity $\mathbf{H}'$ (excitation to be called below magnetic displacement current density).

Differential equation of the vector potential $\mathbf{A}'$:

$$\text{rot rot}\,\mathbf{A}'=\mu\mathbf{J}^{e'}, \tag{24}$$

involving

$$\varphi'=0. \tag{25}$$

The current density is uniform inside and zero outside the conductors. So $\mathbf{J}^{e'}$ can be considered as a known scalar parameter, the stationary and the eddy-current fields linearly depend on.

Differential equation for the eddy-current field is (similar to (8)):

$$\text{rot rot } \mathbf{A}'' + j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A}'' - \text{grad div } \mathbf{A}'' =$$

$$= -(\mu\sigma + j\omega\mu\varepsilon)j\omega\mathbf{A}', \tag{26}$$

taking Lorentz's condition

$$\text{div } \mathbf{A}'' = -(\mu\sigma + j\omega\mu\varepsilon)\varphi'' \tag{27}$$

into account. Equation (26) is an inhomogeneous wave equation for $\mathbf{A}''$, involving excitation $-j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A}'$. For $\omega \to 0$ also vector potential $\mathbf{A}''$ tends to zero. (This fact also shows that the current density tends to uniform at low frequencies.)

The previous considerations are applied to calculate the field of the conductors shown in Fig. 1. A travelling wave in direction Z with propagation coefficient γ is presumed. Supposing the vector potential to point to Z and separating to products of a part depending on Z and of another part depending on the coordinates of the transversal plane we get:

$$\mathbf{A}' = \mathbf{A}'_t e^{-\gamma z}; \quad \mathbf{A}'' = \mathbf{A}''_t e^{-\gamma z} \tag{28}$$

yielding the differential equation of vector potential $\mathbf{A}'_t$:

$$\text{rot}_t \, \text{rot}_t \mathbf{A}'_t = \mu \mathbf{J}^{e'}, \tag{29}$$

where

$$\mathbf{A}'_t = \mathbf{e}_z A'_t(x_1, x_2) \tag{30}$$

$$\mathbf{J}^{e'} = \mathbf{e}_z J_0, \tag{31}$$

x_1, x_2 are coordinates of the transversal plane and J_0 is uniform inside and zero outside the conductor. rot_t denotes a special curl operation: it contains only derivatives with respect to x_1 and x_2. Similarly, the differential equation for vector potential $\mathbf{A}''$ is:

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}''_t + j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A}''_t - \gamma^2 \mathbf{A}''_t =$$

$$= -j\omega(\mu\sigma + j\omega\mu\varepsilon)\mathbf{A}'_t, \tag{32}$$

where

$$\mathbf{A}_t'' = \mathbf{e}_Z A_t''(x_1, x_2) \tag{33}$$

Introducing the square of the propagation coefficient g^2 as follows:

$$g^2 = p^2 + \gamma^2, \tag{34}$$

$$p^2 = -j\omega(\mu\sigma + j\omega\mu\varepsilon), \tag{35}$$

we get:

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t'' - g^2 \mathbf{A}_t'' = p^2 \mathbf{A}_t' \tag{36}$$

where g is the propagation coefficient in the transversal plane.

Inside the conductor

$$\gamma^2 \approx 0, \tag{37}$$

so

$$g^2 \approx p^2 \tag{38}$$

and

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t'' - p^2 \mathbf{A}_t'' = p^2 \mathbf{A}_t' \tag{39}$$

Outside the conductor, except at very high frequencies:

$$g^2 \approx 0, \tag{40}$$

hence

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t'' = \mathbf{0} \tag{41}$$

Outside the conductor the displacement current density is neglected, therefore the approximation holds for relatively low frequencies. In heavy current engineering this is suitable because in air the relationship $\omega^2\mu\varepsilon \ll \omega\mu\sigma$ holds at relatively high frequencies.

As a conclusion, presuming uniform current density J_0 inside the conductor, differential equations

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t' = \mu J_0 \mathbf{e}_Z \tag{42}$$

and

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t' = \mathbf{0} \tag{43}$$

have to be solved for the vector potential $\mathbf{A}_t'$ and for the field outside the conductor, respectively. Knowing $\mathbf{A}_t'$, differential equations

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t'' - p^2 \mathbf{A}_t'' = p^2 \mathbf{A}_t' \tag{44}$$

and

$$\text{rot}_t \, \text{rot}_t \, \mathbf{A}_t'' = \mathbf{0} \tag{45}$$

have to be solved for vector potential $\mathbf{A}_t''$ inside and outside the conductor, respectively. In the knowledge of $\mathbf{A}_t'$ and $\mathbf{A}_t''$ the resultant vector potentials and the field intensities can be determined as:

$$\mathbf{A}_t = \mathbf{A}_t' + \mathbf{A}_t'' \tag{46}$$

and

$$\mathbf{A} = \mathbf{A}_t e^{-\gamma Z} \tag{47}$$

as well as:

$$\mathbf{B} = \text{rot}\, \mathbf{A}\,, \tag{48}$$

$$\mathbf{E} = \frac{J_0}{\sigma}\mathbf{e}_Z - j\omega \mathbf{A} \tag{49}$$

The solution has to satisfy the continuity conditions on the boundary between inner and outer regions both for the stationary and the eddy-current fields, requiring continuity between tangential components of the electric and the magnetic field intensities. We note that owing to the approximations applied to the propagation coefficients, continuity of the tangential component of the electrical field intensity is equivalent to the continuity of the vector potential. In the following this latter will be used.

Solution by variational method

According to the above, differential equations of the form

$$\text{rot}_t\, \text{rot}_t\, \mathbf{A}_t - g^2 \mathbf{A}_t = \mu \mathbf{J} \tag{50}$$

are seen to refer to both $\mathbf{A}_t'$ and $\mathbf{A}_t''$ inside and outside the conductors. Inside the conductor $g^2 = p^2$ and outside the conductor $g^2 = 0$. For the stationary field $\mathbf{J} = \mathbf{e}_Z \mathbf{J}_0$ and for the eddy-current field $\mathbf{J} = p^2 \mathbf{A}_t'$. The Fourier separation applied, reduces the problems to two-dimensional.

The studied region is divided into two subregions: those inside and outside the conductor. The effect of the outer field upon the inner field is taken with a surface current density **K** into account, regarded to be known for the inner region. So, inside the conductor an inhomogeneous Neumann boundary condition problem is to be solved. In the outer region a homogeneous Laplace equation is to be solved under Dirichlet boundary condition such as prescribed by the inner potential function for the boundary line between the two regions.

According to variational calculus, the solution of differential equation (50) under an inhomogeneous Neumann boundary condition for the tangential component of the magnetic field intensity is the vector potential for which the first variation with respect to $\mathbf{A}_t^*$ of the functional

$$W=\int_s\left[-\frac{1}{2}\operatorname{rot}_t\mathbf{A}_t\operatorname{rot}_t\mathbf{A}_t^*-g^2\mathbf{A}_t\mathbf{A}_t^*+\mu\mathbf{J}\mathbf{A}_t^*\right]\mathrm{d}s+ \tag{51}$$
$$+\oint_l\mu\mathbf{K}\mathbf{A}_t^*\,\mathrm{d}l$$

is zero. (See references [1, 3, 6, 8]. The mark "*" denotes conjugation.)

The vector potential is approximated by the linear combination of the elements of an entire real function set:

$$\mathbf{A}_t=\sum_{k=1}^{n}f_k a_k=\mathbf{f}^+\mathbf{a} \tag{52}$$

where **f** and **a** are column vectors of n elements, the k-th element of **f** is the k-th element of the function set and **a** contains the complex parameters to be determined. Substituting into functional (51) it becomes a multi-parameter function of **a**:

$$W(\mathbf{a})=\int_s[-\operatorname{rot}_t\mathbf{f}^+\mathbf{a}\operatorname{rot}_t\mathbf{f}^+\mathbf{a}^*-g^2\mathbf{f}^+\mathbf{a}\mathbf{f}^+\mathbf{a}^*+ \tag{53}$$
$$+\mu\mathbf{J}\mathbf{f}^+\mathbf{a}^*]\,\mathrm{d}s+\oint_l\mu\mathbf{K}\mathbf{f}^*\mathbf{a}^*\mathrm{d}l$$

Formal extremization of functional (53) with respect to $\mathbf{a}^*$ leads to the approximate vector potential function, closest satisfying differential equation (50) and the boundary conditions for the given number of parameters in **a** [1, 4, 6, 7, 8], yielding, in turn, a set of linear equations for **a**:

$$\mathsf{A}\mathbf{a}=\mathbf{b} \tag{54}$$

where

$$\mathsf{A}=-\int_s\operatorname{rot}_t\mathbf{f}\operatorname{rot}_t\mathbf{f}^+\mathrm{d}s-\int_s g^2\mathbf{f}\mathbf{f}^+\mathrm{d}s \tag{55}$$

$$\mathbf{b}=-\int_s\mu\mathbf{J}\mathbf{f}\mathrm{d}s-\oint_l\mu\mathbf{K}\mathbf{f}\mathrm{d}l \tag{56}$$

Solving Eq. (54), the approximate vector potential function can be derived using Eq. (52).

The above are applied to a symmetrical pair of coupled conductors seen in Fig. 1. The field outside the conductor can be written according to the superposition principle, as sum of the fields generated by the two conductors. Owing to symmetry, functions of both vector potential generated by currents of the two conductors coincide in both coordinate systems. The arrangement is symmetrical about the x axis, so it is sufficient to study the region $y > 0$. Owing to the symmetry to the y axis, it is sufficient to study one of the conductors.

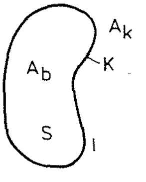

Fig. 2

For the outer region, a homogeneous equation is valid, therefore, it is expedient to choose an approximate set of functions where all terms satisfy the differential equation (harmonic functions) and behave suitable in infinity. Thus, it is not necessary to apply the variational method to the outer region. The coefficients of the set of functions are determined from the continuity condition of the vector potential. In the inner region according to the above considerations, the effect of the outer field on the inner one is taken into account by a surface current density **K** generating the inner field, equalling the tangential component of the magnetic field on the surface of the conductor (Fig. 2). So the vector potential in the outer region is:

$$\mathbf{A}_k = (\mathbf{f}^{(1)+} + \mathbf{f}^{(2)+})\mathbf{a}_k \tag{57}$$

where $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ are column vectors containing the elements of the approximate function set, in the coordinate systems (1) and (2), respectively. (Subscript k refers to the outer region.) For the inner region, the vector potential is:

$$\mathbf{A}_b = \mathbf{f}_b^+ \mathbf{a}_b \tag{58}$$

where column vector $\mathbf{f}_b$ contains the elements of the approximate function set in coordinate system (1). (Subscript b refers to the inner region.) The functional to be extremized is for the inner region:

$$W(\mathbf{a}_b, \mathbf{a}_k) = \int_S [-\mathrm{rot}_t \mathbf{f}_b^+ \mathbf{a}_b \, \mathrm{rot}_t \mathbf{f}_b^+ \mathbf{a}_b^* -$$

$$-p^2 \mathbf{f}_b^+ \mathbf{a}_b \mathbf{f}_b^+ \mathbf{a}_b^*] \mathrm{d}s + \int_S \mu \mathbf{J} \mathbf{f}_b^+ \mathbf{a}_b^* \mathrm{d}s +$$

$$+ \oint_l \mathrm{rot}_t (\mathbf{f}^{(1)+} + \mathbf{f}^{(2)+}) \times \mathbf{n} \mathbf{a}_k \mathbf{f}_b^+ \mathbf{a}_b^* \mathrm{d}l \tag{59}$$

(S is the conductor cross section). The condition of the formal extremum is:

$$\frac{\partial W}{\partial \mathbf{a}_b^*} = \mathbf{0} \tag{60}$$

From Eq. (60) a set of linear equations is derived:

$$\mathbf{A} \mathbf{a}_b + \mathbf{B} \mathbf{a}_k = \mathbf{b} \tag{61}$$

where

$$\mathbf{A} = -\int_S [\mathrm{rot}_t \mathbf{f}_b \mathrm{rot}_t \mathbf{f}_b^+ + p^2 \mathbf{f}_b \mathbf{f}_b^+] \mathrm{d}s \tag{62}$$

$$\mathbf{B} = \oint_l \mathbf{f}_b \mathrm{rot} (\mathbf{f}^{(1)+} + \mathbf{f}^{(2)+}) \times \mathbf{n} \mathrm{d}l \tag{63}$$

$$\mathbf{b} = \int_S \mu \mathbf{J} \mathbf{f}_b \mathrm{d}s \,. \tag{64}$$

Equation (61) yields condition for $\mathbf{a}_b$. Coefficients $\mathbf{a}_k$, are determined from the boundary condition of vector potential continuity on curve l. This can be approximated by using the least square error integral method, such as:

$$\oint_l [(\mathbf{f}^{(1)+} + \mathbf{f}^{(2)+}) \mathbf{a}_k - \mathbf{f}_b^+ \mathbf{a}_b] [(\mathbf{f}^{(1)+} + \mathbf{f}^{(2)+}) \mathbf{a}_k^* - \mathbf{f}_b^+ \mathbf{a}_b^*] \mathrm{d}l =$$

$$= \min_{\mathbf{a}_k^*} . \tag{65}$$

From the minimum condition (65), with respect to $\mathbf{a}_k^*$:

$$\mathsf{F}_k \mathbf{a}_k + \mathsf{F}_b \mathbf{a}_b = \mathbf{0} \tag{66}$$

where

$$\mathsf{F}_k = \oint_l (\mathbf{f}^{(1)} + \mathbf{f}^{(2)}) (\mathbf{f}^{(1)+} \mathbf{f}^{(2)+}) \mathrm{d}l \tag{67}$$

$$\mathsf{F}_b = -\oint_l (\mathbf{f}^{(1)} + \mathbf{f}^{(2)}) \mathbf{f}_b^+ \, \mathrm{d}l \tag{68}$$

Eqs (61) and (66) yield parameters $\mathbf{a}_b$ and $\mathbf{a}_k$.

Accordingly, the course of the solution is: 1. A uniform current density $J_0 = 1$, $(p^2 = 0)$ inside the conductor is presumed to obtain stationary field intensities. 2. For calculating the eddy-current components, the value of $\mathbf{J}$ has to be calculated from the stationary vector potential:

$$\mu \mathbf{J} = -j\omega\mu\sigma \mathbf{f}_b^+ \mathbf{a}_b') \tag{69}$$

Hence

$$\mathbf{b} = -\int_s j\omega\mu\sigma \mathbf{f}_b \, \mathbf{f}_b^+ \mathbf{a}_b') \mathrm{d}s \tag{70}$$

instead of (56) and

$$p^2 = -j\omega\mu\sigma \tag{71}$$

delivers the eddy-current field. Characteristics of the resultant field are:

$$\mathbf{a}_b = \mathbf{f}_b^+ (\mathbf{a}_b' + \mathbf{a}_b'') \tag{72}$$

$$\mathbb{A}_k = (\mathbf{f}_k^{(1)+} + \mathbf{f}_k^{(2)+}) (\mathbf{a}_k' + \mathbf{a}_k'') \tag{73}$$

$$\mathbb{B}_b = \mathrm{rot} \, [\mathbf{f}_b^+ (\mathbf{a}_b' + \mathbf{a}_b'')] \tag{74}$$

$$\mathbb{B}_k = \mathrm{rot} \, [(\mathbf{f}_k^{(1)+} + \mathbf{f}_k^{(2)+}) (\mathbf{a}_k' + \mathbf{a}_k'')] \tag{75}$$

$$\mathbb{E}_b = \mathbf{e}_Z \frac{1}{\sigma} - j\omega \mathbf{f}_b^+ (\mathbf{a}_b' + \mathbf{a}_b'') \tag{76}$$

The resultant current of the conductor:

$$I = \int_s [\mathbf{e}_Z - j\omega \mathbf{f}_b^+ \sigma (\mathbf{a}_b' + \mathbf{a}_b'')] \mathbf{d}s \tag{77}$$

Knowing the field intensities, the inner impedance of the conductor can be derived by using the Poynting vector as:

$$Z_b = \frac{\oint_l (\mathbf{E}_b \times \mathbf{H}_b^*) \mathrm{d}l}{|I|^2} \tag{78}$$

Example

These results have been applied to a pair of coupled conductors with rectangular cross section shown in Fig. 3. In the inner region power series have been applied for the numerical approximation:

$$f_{bij} = \left(\frac{x}{a}\right)^i \left(\frac{y}{b}\right)^j \qquad \begin{matrix} i=0,1,2,\ldots \mathrm{NI} \\ j=0,2,4,\ldots \mathrm{NJ} \end{matrix} \tag{79}$$

In the outer region, a harmonic function set has been applied:

$$f_{ko}^{(1)} = \ln \frac{R}{r^{(1)}}, \quad f_{ki}^{(1)} = \left(\frac{R}{r^{(1)}}\right)^i \cdot \cos i\varphi ,$$

$$i=1,2,3\ldots \mathrm{NK} \tag{80}$$

where R is constant.
The dimensions are:

$$a=5 \text{ mm}; \quad b=10 \text{ mm}; \quad d=10 \text{ mm};$$

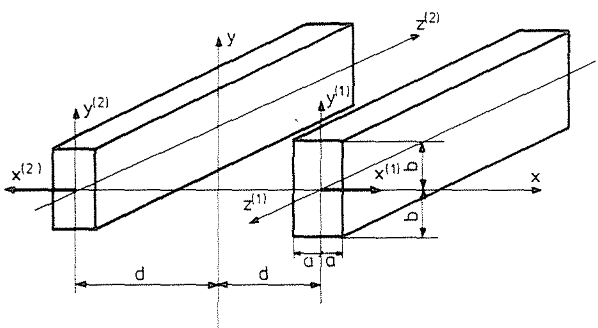

Fig. 3

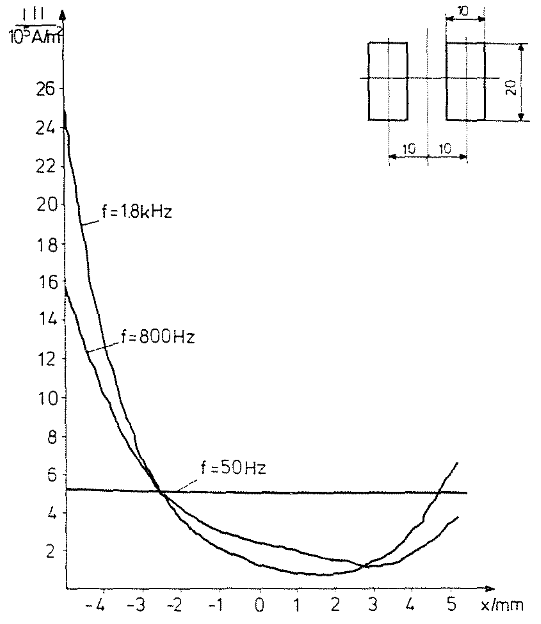
|H| / 10^5 A/m
26
24
22
20
18
16
14
12
10
8
6
4
2
f = 1.8kHz
f = 800Hz
f = 50Hz
-4 -3 -2 -1 0 1 2 3 4 5 x/mm
10
20
10 10

Fig. 4

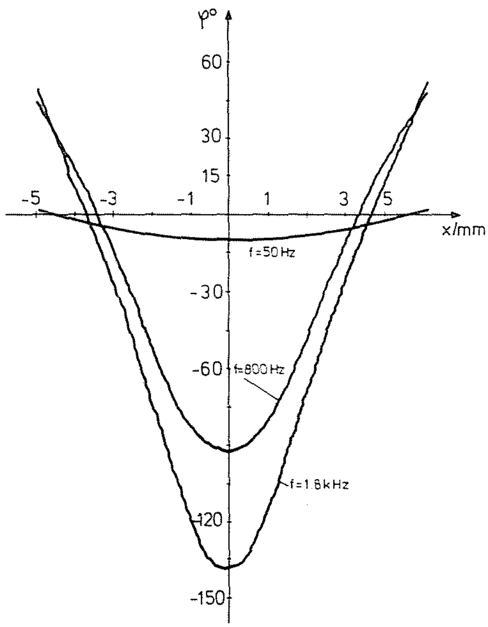
φ°
60
30
15
-5 -3 -1 1 3 5
x/mm
f = 50 Hz
-30
-60
f=800 Hz
f = 1.8 kHz
-120
-150

Fig. 5

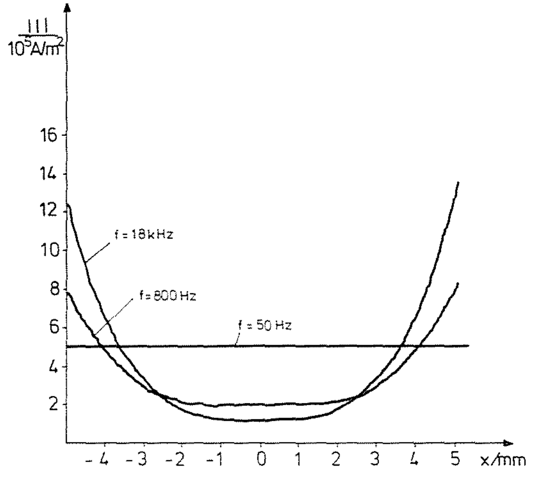
|I|
10^5 A/m^2
16
14
12
10
8
6
4
2
f = 18 kHz
f = 800 Hz
f = 50 Hz
- 4 - 3 - 2 - 1 0 1 2 3 4 5 x/mm

Fig. 6

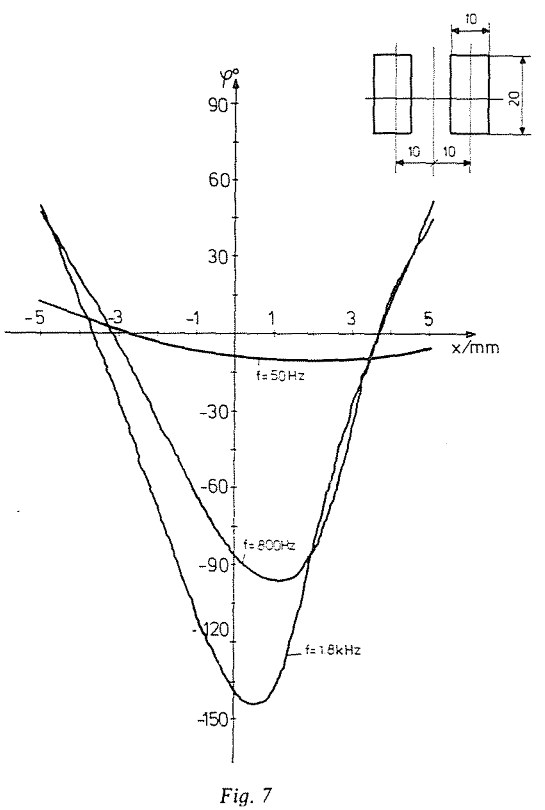
φ°
90
60
30
-30
-60
-90
-120
-150
-5 -3 -1 1 3 5
x/mm
f=50Hz
f=800Hz
f=1.8kHz
10
20
10 10

Fig. 7

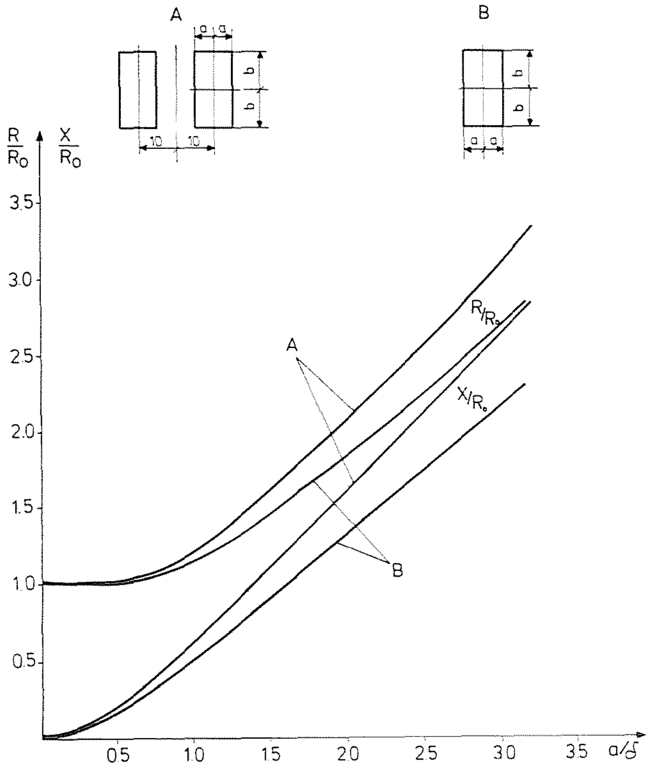

Fig. 8

The current density was calculated inside the conductor along the axis $x^{(1)}$ at different frequencies. Figs 4 and 5 show the amplitude and the phase angle of the current density, respectively. For the sake of comparison, the calculation was performed for the case of a single conductor as well. (Figs 6 and 7). In the case of a single conductor, these figures point out the skin effect. Namely, with increasing frequency, there is a higher rate of skin effect. In the case of a pair of coupled conductors, the proximity effect appears from the non-symmetrical distribution of current density about the $y^{(1)}$ axis.

The inner impedance was calculated both for the case of a single conductor and a pair of coupled conductors. The real and imaginary parts of the impedance have been plotted in Fig. 8 *vs.* a/δ, where the skin depth

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}. \tag{81}$$

In accordance with the above, both real and imaginary parts of the inner impedance of a pair of coupled conductors are seen to exceed those of a single conductor.

Summary

A numerical method based on variational principle is presented for calculating the electromagnetic field and inner impedance of adjacent, coupled conductors of arbitrary cross section. Neither the arbitrary cross section nor the proximity effect can be taken into account analytically, imposing to apply a numerical variational method. Application of the method is illustrated on a pair of conductors with rectangular cross section. The inner impedance and the current density inside the conductors have been calculated and plotted at different frequencies.

References

1. BÁRDI, I.—BIRÓ, O.—GYIMESI, M.: Calculation of static and stationary fields in inhomogeneous, anisotropic media by the variational method. Per. Polytech. Electrical Engineering Vol. 20. No. 2, pp. 103—118. Budapest 1976.
2. BÁRDI, I.: Contribution to the variational methods for solving static and stationary electromagnetic field problems. Per. Polytech. Electrical Engineering Vol. 21. No. 2, pp. 197—206. Budapest 1977.
3. BÁRDI, I.: The least action and the variational calculus in electrodynamics. Per. Polytech. Electrical Engineering Vol. 23. No. 2, pp. 149—162. Budapest 1979.
4. BÁRDI, I.—BIRÓ, O.: Calculation of electromagnetic field in the air gap of asynchronous machines by variational method. Per. Polytech. Electrical Engineering Vol. 23. No. 3—4, pp. 217—224. Budapest 1979.
5. SIMONYI, K.: Theoretical Electrotechniques (in Hungarian) Tankönyvkiadó, Budapest 1967.
6. MIKHLIN, S. G.: Variational methods in mathematical physics. (In Russian) Publisher "Hauka", Moscow 1970.
7. MIKHLIN, S. G.—SMOLITSKIY, K. L.: Approximate Methods for Solution of Differential and Integral Equations. American Elsevier Publishing Company Inc. New York 1967.
8. BÁRDI, I.: Variational method for numerical calculation of quasistationary electromagnetic field in conductive medium. 23. Intern. Wiss. Koll. TH Ilmenau, 1978. (DDR) "Theoretische Elektrotechnik".
9. FODOR, GY.—VÁGÓ, I.: Electrotechniques (Booklet 12.) (in Hungarian) Tankönyvkiadó, Budapest 1976.
10. IVÁNYI, A.: Application of variational calculus to two-dimensional problems of electrostatics. Elektrotechnika 71. (1978) 21—25. sz. Budapest.
11. ALTSHULER, I. B.: Calculation of electromagnetic field in electrical machines (In Russian) Publisher "Energia" Moscow 1968.
12. ANTAL, J. (Ed.): Handbook of Physics (in Hungarian) Müszaki Könyvkiadó, Budapest, 1980.

István BÁRDI H-1521 Budapest