FREE VIBRATION ANALYSIS OF A ROTATING SHAFT OF VARIABLE CROSS SECTION USING THE FINITE ELEMENT METHOD

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Introduction

Among references on non-rigid rotors as gyroscop continua mainly recapitulative works by Meirovitch will be relied on in the attempt likely to help computing upper limits for the natural circular frequencies in bending vibration of a rotating shaft of variable circular cross-section. *Meirovitch* [6] introduces a stationary principle, on the analogy of the Rayleigh principle, for finitizing continua in gyroscopic movement and presents a numerical example to illustrate the Rayleigh—Ritz method. As the problem described in the title can well be handled as a one dimensional problem, the finite element method using the line element is suitable. Its application by-passes the intuitive steps necessary in other methods of finitization.

To solve the Lagrange's equations resulting from finitization the procedure shown in [4] and [5] will be applied.

The finite element

The rotor of variable cross section will be analyzed on the basis of the following initial assumptions:

1. The rotor is modelled as a Bernoulli-type shaft;
2. Only small deformations are examined.

The model conform to the above assumptions is the so-called line element, presented in Figs 1a and 1b.

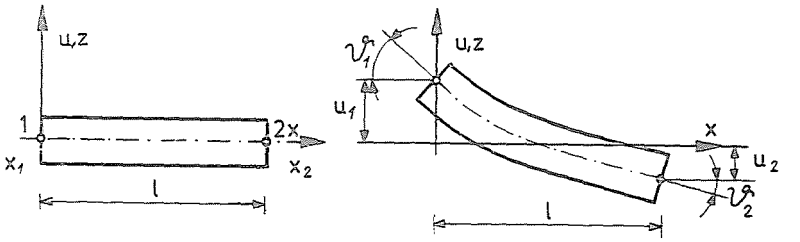

Fig. 1. The coordinate system of the line element

The displacements of the points of the element type shown in Figs 1a and 1b can be approximated by: [1]

$$u(x,t)=\alpha_1+\alpha_2 x+\alpha_3 x^2+\alpha_4 x^3 \tag{1}$$

and the rotation by

$$\vartheta(x,t)=\frac{\partial u(x,t)}{\partial x}=\alpha_2+2\alpha_3 x+3\alpha_4 x^2$$

where α designates time dependent generalized coordinates.

The defined model has four degrees of freedom, so the displacement and the rotation at the ends of the element can be the elements of the so-called nodal displacement vector u^e:

$$u^{eT}=\{u_1,\vartheta_1,u_2,\vartheta_2\}$$

(T means transposition).

Using the matrix of basic functions $\mathbb{A}$ and the generalized coordinate vector $\boldsymbol{\alpha}$ the nodal displacement vector can be expressed in the form

$$\mathbf{u}^e=\mathbb{A}\boldsymbol{\alpha}\,. \tag{2}$$

The introduced displacement model is compatible and complete as it is understood by [1].

To characterize the displacements and rotations in one element let us introduce the generalized displacement vector

$$w^T=\{u(x,t),\ \vartheta(x,t)\}\,.$$

By means of $\boldsymbol{\alpha}=\mathbb{A}^{-1}\mathbf{u}^e$ from (2) the generalized displacement vector will be obtained from expressions

$$\underset{2\times 1}{w}=\underset{2\times 4}{\mathbb{V}}\cdot\underset{4\times 1}{\boldsymbol{\alpha}}\quad\text{and}\quad \underset{2\times 1}{w}=\underset{2\times 4}{\mathbb{V}}\ \underset{4\times 4}{\mathbb{A}^{-1}}\ \underset{4\times 1}{\mathbf{u}^e}\,. \tag{3}$$

Or expanded:

$$\mathbb{V}=\begin{bmatrix}1 & x & x^2 & x^3\\ 0 & 1 & 2x & 3x^2\end{bmatrix},\qquad \mathbb{A}^{-1}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3/l^2 & -2/l & 3/l^2 & -1/l\\ 2/l^3 & 1/l & -2/l^2 & 1/l^2\end{bmatrix},$$

$$\mathbf{A}\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vartheta_1 \\ u_2 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} u_{x=0} \\ \vartheta_{x=0} \\ u_{x=l} \\ \vartheta_{x=l} \end{bmatrix} = \mathbf{u}^e .$$

As the generalized nodal displacement coordinates depend only on time, in their knowledge, relationship

$$w(x, t) = \mathbf{N}(x)\, \mathbf{u}^e(t)$$

permits to approximate the displacement field in side the element by means of the approximation matrix $\mathbf{N}$:

$$\mathbf{N} = \mathbf{V}\mathbf{A}^{-1}, \qquad \underset{4\times 2}{\mathbf{N}^T} = [\mathbf{N}_1^T, \mathbf{N}_2^T]. \tag{4}$$

The vector $\mathbf{N}_1$, to be used later, is as follows: [8]

$$\mathbf{N}_1^T(x) = \{L_1^2(3-2L_1),\, L_1^2 L_2,\, L_2^2(3-2L_2),\, -L_2^2 L_1\},$$

$$L_1 = 1 - \frac{x}{l}, \qquad L_2 = \frac{x}{l}.$$

The elements of $\mathbf{N}_1$ are Hermite polinoms of order 3.

The potential and kinetic energy of one element

A bar supported both ends in the prescribed manner (Fig 2) rotating around the fixed z_0 axis at an angular velocity ω_A will be examined. Let s be the arc length coordinate along the bar length.

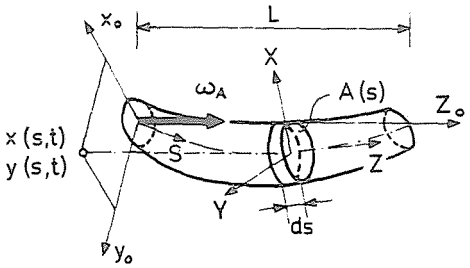

Fig. 2

Let us assign the local coordinate system X, Y, Z rotating together with the system to the centre of gravity of an arbitrary bar element of length ds. The position of the centre of gravity of the bar element in the fixed coordinate system is described by values $x(s, t)$ and $y(s, t)$. The vector of angular velocity of the bar element in general position in its local coordinate system, is (see [4]):

$$\mathbf{\Omega} = -\frac{\partial^2 y}{\partial s \partial t} \cdot \mathring{\mathbf{X}} + \frac{\partial^2 x}{\partial s \partial t} \mathring{\mathbf{Y}} + \left(\omega_A - \frac{\partial^2 y}{\partial s \partial t} \cdot \frac{\partial x}{\partial s} \right) \mathring{\mathbf{Z}},$$

where $\mathring{\mathbf{X}}$, $\mathring{\mathbf{Y}}$, and $\mathring{\mathbf{Z}}$ are unit vectors of the axes of the local coordinate system. In the local coordinate system second-order moment of inertia matrix $\mathrm{d}\mathbb{I}$ of the bar element is

$$\mathrm{d}I = \mathrm{d}m \begin{bmatrix} R^2/4 & & \\ & R^2/4 & \\ & & R^2/2 \end{bmatrix}$$

Where R is the cross-sectional radius function of s and $\mathrm{d}m = \rho A \mathrm{d}s$. Accordingly, the kinetic energy of the bar element of length ds is as follows:

$$\mathrm{d}W_{\mathrm{kin}} = \frac{1}{2} \mathrm{d}m \left[\left(\frac{\partial x}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial t} \right)^2 \right] + \frac{1}{2} \mathbf{\Omega}^T d\mathbb{I} \mathbf{\Omega}, \tag{5}$$

and the potential energy:

$$\mathrm{d}W_{\mathrm{pot}} = \frac{1}{2} EI \left[\left(\frac{\partial^2 x}{\partial s^2} \right)^2 + \left(\frac{\partial^2 y}{\partial s^2} \right)^2 \right] \mathrm{d}s. \tag{6}$$

Let us divide the bar into n elements. The cross sectional characteristics of the i-th element along the length of the bar element are given by the functions $R_i(s)$, $I_i(s)$ and $A_i(s)$ of radius, moment of inertia and cross sectional area, respectively).

$$\sum_{j=1}^{i-1} l_j \leqq s \leqq \sum_{j=1}^{i} l_j,$$

(l_j is the length of the j-th element). The expressions approximating the field of displacement of the element will be written again for the i-th bar element:

$$x_i(s, t) = \mathbb{N}_1^T(s) \mathbb{X}_i(t), \qquad y_i(s, t) = \mathbb{N}_1^T(s) \mathbb{Y}_i(t),$$

$$\mathbb{u}_i^T(s, t) = \{ \mathbb{X}_i^T, \mathbb{Y}_i^T \},$$

$$\mathbb{X}_i^T = \left\{ x_i(0, t), \quad \frac{\partial x_i(0, t)}{\partial s}, \quad x_i(l_i, t), \quad \frac{\partial x_i(l_i, t)}{\partial s} \right\}, \tag{7}$$

$$\mathbf{Y}_i^T = \left\{ y_i(0,t), \quad \frac{\partial y_i(0,t)}{\partial s}, \quad y_i(l_i,t), \quad \frac{\partial y_i(l_i,t)}{\partial s} \right\}.$$

The energies of the i-th element of length l_i are, using (5), (6) and (7):

$$W_{\text{pot}_i} = \frac{1}{2} E \int_0^{l_i} I_i \left(\mathbf{X}_i^T \frac{\partial^2 \mathbf{N}_1}{\partial s^2} \frac{\partial^2 \mathbf{N}_1^T}{\partial s^2} \mathbf{X}_i + \mathbf{Y}_i^T \frac{\partial^2 N_1}{\partial s^2} \frac{\partial^2 \mathbf{N}_1^T}{\partial s^2} \mathbf{Y}_i \right) \mathrm{d}s ,$$

$$W_{\text{kin}_i} = \frac{1}{2} \int_0^{l_i} \rho A_i \left\{ \left(\frac{\partial x_i}{\partial t} \right)^2 + \left(\frac{\partial y_i}{\partial t} \right)^2 + \frac{R_i^2}{4} \left[\left(\frac{\partial^2 y_i}{\partial s \partial t} \right)^2 + \left(\frac{\partial^2 x_i}{\partial s \partial t} \right)^2 \right] + \right.$$

$$\left. + \frac{R_i^2}{2} \left(\omega_A^2 - 2\omega_A \frac{\partial^2 y_i}{\partial s \partial t} \frac{\partial x_i}{\partial s} + \left[\frac{\partial^2 y_i}{\partial s \partial t} \right]^2 \left[\frac{\partial^2 x_i}{\partial s} \right]^2 \right) \right\} \mathrm{d}s$$

Term $\left(\frac{\partial^2 y_i}{\partial s \partial t} \right)^2 \left(\frac{\partial x_i}{\partial s} \right)^2$ as a small quantity of second order will be neglected below.

Let us introduce the following symbols:

$$\mathbf{K}_i = \frac{1}{2} E \int_0^{l_i} I_i \frac{\partial^2 \mathbf{N}_1}{\partial s^2} \frac{\partial^2 \mathbf{N}_1^T}{\partial s^2} \mathrm{d}s, \qquad \mathbf{G}_i = \int_0^{l_i} A_i \frac{R_i^2}{2} \omega_A \frac{\partial \mathbf{N}_1}{\partial s} \frac{\partial \mathbf{N}_1^T}{\partial s} \mathrm{d}s ,$$

$$\mathbf{M}_i = \rho \int_0^{l_i} A_i \left(\mathbf{N}_1 \mathbf{N}_1^T + \frac{R_i^2}{4} \frac{\partial \mathbf{N}_1}{\partial s} \frac{\partial \mathbf{N}_1^T}{\partial s} \right) \mathrm{d}s .$$

Yielding for the energies

$$2W_{kin_i} = \dot{\mathbf{X}}_i^T \mathbf{M}_i \dot{\mathbf{X}}_i - \dot{\mathbf{Y}}_i^T \mathbf{M}_i \dot{\mathbf{Y}}_i - 2\dot{\mathbf{Y}}_i^T \mathbf{G}_i \dot{\mathbf{X}}_i ,$$

$$W_{\text{pot}_i} = \mathbf{X}_i^T \mathbf{K}_i \mathbf{X}_i - \mathbf{Y}_i^T \mathbf{K}_i \mathbf{Y}_i .$$

(Dot indicates differentiating with respect to time).

The equations of motion and the solution possibilities:

Having use of the available energy expressions, some variation principle (e.g. the Hamilton principle) may be applied to set up the Lagrange's equation system. Substitution into the equation in the general form:

$$\mathbf{E}_1\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W_{\mathrm{kin}}}{\partial \dot{\mathbf{u}}}\right)-\frac{\partial W_{\mathrm{kin}}}{\partial \mathbf{u}}-\frac{\partial W_{\mathrm{pot}}}{\partial \mathbf{u}}\right]=\mathbf{0}$$

($\mathbf{E}_1$ being the matrix fixing the boundary conditions and the connection conditions of the elements) yields the equation system:

$$\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{X}}\\ \ddot{\mathbf{Y}}\end{bmatrix}+\begin{bmatrix}\mathbf{0} & \mathbf{G}\\ -\mathbf{G} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{X}}\\ \dot{\mathbf{Y}}\end{bmatrix}+\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{8}$$

Let the coefficient matrices be of the order $2m$. Dealing with the equations of type (8), LANCASTER [10] and later MEIROVITCH [5] have shown that searching the solution of the equation system in the form $\mathbf{u}(t)=\mathbf{u}e^{\lambda t}$ (where $\mathbf{u}$ is a constant vector of constant elements and λ is a complex number) the eigenvalues form purely imaginary pairs conjugated to each other. Substituting the solution of the form above mentioned into (8) results in a generalized eigenvalue problem, where the matrix of the determinant of the characteristic equation is not symmetrical, so because of the expected complex eigenvalues, complex algebraic methods have to be used. It is possible [5] to transform the problem into one involving only real eigenvalues. This way any of the well-known procedures of real algebra, such as the Jacobi, the Householder and QR algorithms are convenient. If, however, the coefficient matrices are already of high order, its doubling due to transformation is disadvantageous, [7] offers a modified version of the Jacobi method for eigenvalue problems of not symmetrical coefficient matrices (with good numerical results during the calculations related to be numerical example), so thus the procedure presented by [4] can be used which results in coefficient matrices of half order. [9] deals with a solution different from those in [4] and [5].

Starting from Eq. (8), let the expected form of the solution be:

$$\mathbf{u}(t)=\mathbf{u}e^{i\omega t}, \qquad \mathbf{u}^T=\{\mathbf{X}^T,\mathbf{Y}^T\}.$$

Here $\mathbf{u}$ is a constant vector of complex elements. Substituted into (8) yields two equations:

$$(\mathbf{K}-\mathbf{M}\omega^2)\mathbf{X}+\omega\mathbf{G}i\mathbf{Y}=\mathbf{0} \tag{9a}$$

$$(\mathbf{K}-\mathbf{M}\omega^2)i\mathbf{Y}+\omega\mathbf{G}\mathbf{X}=\mathbf{0}\,. \tag{9b}$$

Introducing vectors, $\mathbf{Z}_1 = \mathbf{X} + i\mathbf{Y}$, and $\mathbf{Z}_2 = \mathbf{X} - i\mathbf{Y}$ moreover adding and subtracting Eqs (9a) and (9b) leads to:

$$(\mathbf{K} - \mathbf{M}\omega^2)\mathbf{Z}_1 + \omega\mathbf{G}\mathbf{Z}_1 = \mathbf{0}$$

$$(\mathbf{K} - \mathbf{M}\omega^2)\mathbf{Z}_2 - \omega\mathbf{G}\mathbf{Z}_2 = \mathbf{0}\,.$$

In the two equations above, ω is seen to differ only by sign. Therefore it is enough to deal with the solution of one of the equations. Let our new starting equation be

$$(\mathbf{K} - \mathbf{M}\omega^2)\mathbf{Z} + \omega\mathbf{G}\mathbf{Z} = \mathbf{0} \tag{10}$$

Let a fictitious differential equation be assigned to the equation above, so that its solution in the form $\tilde{\mathbf{Z}} = \mathbf{Z}e^{\omega t}$ yields (10). Such may be the equation

$$-\mathbf{M}\ddot{\tilde{\mathbf{Z}}} + \mathbf{K}\tilde{\mathbf{Z}} + \mathbf{G}\dot{\tilde{\mathbf{Z}}} = \mathbf{0}\,. \tag{11}$$

Introducing vector $\mathbf{T}^T = \{\tilde{\mathbf{Z}}^T, \dot{\tilde{\mathbf{Z}}}^T\}$ and substituting it, the solution of (10) can be reduced to:

$$\mathbf{M}^{-1}\mathbf{K}\tilde{\mathbf{Z}} + \mathbf{M}^{-1}\mathbf{G}\dot{\tilde{\mathbf{Z}}} - \ddot{\tilde{\mathbf{Z}}} = \mathbf{0} \tag{12}$$

somewhat transformed from (11). Using matrix form of representation for the numerical computation, yields the equations in final form:

$$\begin{bmatrix} \mathbf{0} & \mathbf{E} \\ \mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{G} \end{bmatrix} \mathbf{T} - \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \dot{\mathbf{T}} = \mathbf{0}$$

($\mathbf{E}$ is a unit matrix of order m.) Let the solution of the differential equation system be $\mathbf{T}(t) = \mathbf{T}e^{\omega t}$. Substitution and division by $e^{\omega t}$ results in the algebraic eigenvalue problem:

$$\begin{bmatrix} \mathbf{0} & \mathbf{E} \\ \mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{G} \end{bmatrix} \mathbf{T} - \omega \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} = \mathbf{0}\,.$$

The coefficient matrix being unsymmetrical the solution must rely on complex algebra. It is an advantage, however, that the volume of numeric computations decreases by avoiding doubling of the coefficient matrix order. The advantages of the latter procedure should be illustrated by the following numerical example.

Numerical example

The rotor under examination is a bar of variable circular cross section, with hinged supports both ends. Let the longitudinal axis of the bar be coincident with the z_0 axis in Fig. 1. The length of the bar is $L=1$ m. The variation of the diameter is given by the function $d(z)=(4-z)\cdot 10^{-2}$ m. The substance parameters are regarded to be constant assuming homogeneous substance distribution all over. The values of the substance parameters are:

$$E=2\cdot 10^{11}\ \mathrm{Nm}^{-2} \text{ and } \rho=7850\ \mathrm{kgm}^{-3}.$$

Computations involved the determination of the natural circular frequencies of the bar alone. The results for the bar of variable cross section, based on different ω_A values are shown in the next table:

ω_A	390	460	490	1600
ω_1	469.96	469.98	470	470.1
	469.64	469.63	469.61	469.4
ω_2	1814	1814.15	1814.2	1814.6
	1813.5	1813.39	1813.35	1812.9
ω_3	4175.43	4176.15	4176.92	4178.3
	4175.41	4175	4174.31	4173.25

(Hz)

Two values are seen for each ω_A in the vicinity of natural circular frequencies in the case $\omega_A=0$. The higher of both results if the direction of the vector ω_A is the same as that of the unit vector of the z_0 axis in Fig. 1. This is the

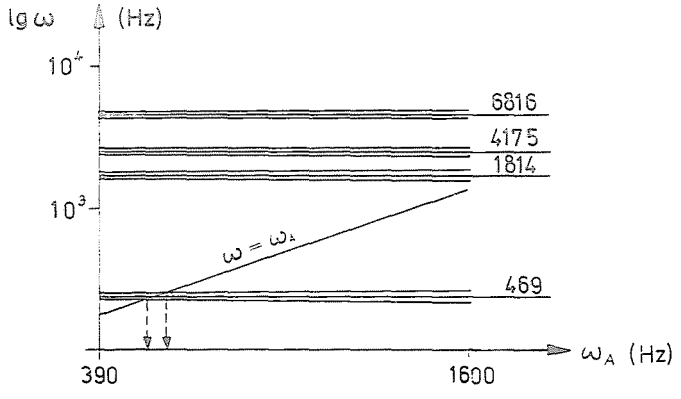

Fig. 3

case of the so-called backward precession. A detailed analysis of the precessions in both directions is found e.g. in [2], [3]. The critical angular velocities of the rotating shaft may be found graphically from Fig. 3.

Critical angular velocities have been demonstrated experimentally [2] to belong to $\omega = \omega_A$.

In adopting the graphic solution, the curve set $\omega(\omega_A)$ has to be plotted true to scale (Fig. 3) and to determine the ω_A values where the line $\omega = \omega_A$ intersects this set. Remark that the obtained critical values, approximate the real values from above [6]. The eigenvalue problem for the selected shaft was solved by using the norm reducing method by JACOBI [7], hardly less fast than the usual Jacobi method, but is suitable to determine complex eigenvalues as well. The numeric computations could be performed on the CDC 3300 type central computer of the Hungarian Academy of Sciences.

Summary

Publications on small deformations of rotating shafts or rotors report doubling of natural frequencies of bending vibration due to the gyroeffect as demonstrated by the finite element method developed for the case of variable cross sections.

The present continuum model is easiest finitized by using the so-called line-element. The presented numerical example illustrates the computations using this kind of elements. The procedure permits to halve the order of the equation system obtained by finitization, and to improve thereby the numerical efficiency. The computations suit to determine the critical revolution numbers as well. The obtained approximate values are the upper limits of the correct ones.

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