FINITE ELEMENT ANALYSIS OF BENDING VIBRATIONS OF A STRAIGHT-AXED BEAM UNDER CONSIDERATION OF ROTARY INERTIA AND SHEAR DEFORMATION

By

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Symbols

$X \equiv x, y, z$	space co-ordinates
$\mathbf{u}(X, t)$	generalized displacement vector
$\mathbf{N}(X)$	approximation matrix
$\mathbf{r}(t)$	nodal generalized displacement vector
$\boldsymbol{\varepsilon}(X, t)$	vector of deformations
$\boldsymbol{\partial}(X)$	differential operator matrix
$\boldsymbol{\sigma}(X, t)$	stress vector
$\mathbf{D}(X)$	matrix of material constants
$\mathscr{L}$	Lagrangian function
T	kinetic energy
U	deformation energy
V	volume
$\mathbf{M}$	mass matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{R}$	amplitude vector
α_i	i-th circular natural frequency
$\rho(X)$	density
$E(x)$	Young's modulus
$G(x)$	shear modulus
$I(x)$	moment of inertia
$A(x)$	cross-sectional area
$v(x)$	total displacement
$v_b(x)$	displacement due to bending
$v_s(x)$	displacement due to shear
M	bending moment
Q	shear force
k	shear coefficient
ν	Poisson's ratio
L	length of the beam element
t	time

1. Application of the finite element method (FEM) to elastic continua performing free vibrations

The finite element method suits to describe the movement of elastic continua, by reducing the continuum of infinite degrees of freedom to a mechanical system of finite degrees of freedom. For solving dynamic problems mostly the displacement method is used.

After the division of the continuum to finite elements let us approximate the field of displacement in the following form [1]:

$$\mathbf{u}=\mathbf{N}\mathbf{r} \tag{1}$$

Deformations inside the element are obtained from the relationship

$$\varepsilon=\partial\mathbf{u}=\partial\mathbf{N}\mathbf{r}=\mathbf{B}\mathbf{r}\,; \qquad \mathbf{B}=\partial\mathbf{N}\,, \tag{2}$$

and the stresses from:

$$\sigma=\mathbf{D}\varepsilon=\mathbf{D}\mathbf{B}\mathbf{r}\,. \tag{3}$$

The matrix differential equation relative to the element can be obtained e.g. by means of Lagrange's principle:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{\mathbf{r}}}-\frac{\partial\mathscr{L}}{\partial\mathbf{r}}=\mathbf{0}\,; \qquad \mathscr{L}=T-U\,. \tag{4}$$

The kinetic energy of the element is:

$$T=\frac{1}{2}\cdot\int\limits_{(V)}\rho\,\frac{\partial\mathbf{r}^T}{\partial t}\cdot\frac{\partial\mathbf{r}}{\partial t}\,dV. \tag{5}$$

Substituting (1) into (5):

$$T=\frac{1}{2}\,\dot{\mathbf{r}}^T\cdot\int\limits_{(V)}\rho\mathbf{N}^T\mathbf{N}dV\cdot\dot{\mathbf{r}}=\frac{1}{2}\,\dot{\mathbf{r}}^T\mathbf{M}\dot{\mathbf{r}}\,, \tag{6}$$

from this the mass matrix of the element is

$$\mathbf{M}=\int\limits_{(V)}\rho\mathbf{N}^T\mathbf{N}\,dV.$$

The deformation energy of the element is:

$$U=\frac{1}{2}\cdot\int_{(V)}\boldsymbol{\varepsilon}^T\cdot\boldsymbol{\sigma}\,\mathrm{d}V. \tag{7}$$

Substituting (2) and (3) into (7),

$$U=\frac{1}{2}\mathbf{r}^T\cdot\int_{(V)}\mathbf{B}^T\mathbf{D}\mathbf{B}\,\mathrm{d}V\cdot\mathbf{r}=\frac{1}{2}\mathbf{r}^T\mathbf{K}\mathbf{r}. \tag{8}$$

$$\mathbf{K}=\int_{(V)}\mathbf{B}^T\mathbf{D}\mathbf{B}\,\mathrm{d}V$$

is the stiffness matrix of the element.

Substituting (6) and (8) into (4), the matrix differential equation relevant to the element will have the following form:

$$\mathbf{M}\ddot{\mathbf{r}}+\mathbf{K}\mathbf{r}=\mathbf{0}.$$

The differential equation relevant to the whole system can be obtained from the usual coupling conditions of the elements [1], thus the blocks representing the relationship between nodal points i and j of the mass matrix and the stiffness matrix can be calculated with the relationship

$$\mathbf{M}_{ij}=\sum_{e\in i,j}\mathbf{M}_{ij}^e;\qquad \mathbf{K}_{ij}=\sum_{e\in i,j}\mathbf{K}_{ij}^e,$$

in the summation taking only elements containing nodal points i and j alike into consideration. When using the displacement method, only the kinematical boundary conditions must be satisfied. This is done by omitting from the matrix differential equation the rows and columns belonging to zero displacements. For the sake of simplicity let us write this system of differential equations similarly in the form

$$\mathbf{M}\ddot{\mathbf{r}}+\mathbf{K}\mathbf{r}=\mathbf{0}. \tag{9}$$

Looking for a standing wave solution, nodal displacements will be found in the form

$$\mathbf{r}=\mathbf{R}\sin(\alpha t+\varphi). \tag{10}$$

Substituting (10) into (9), we obtain the algebraic system of equations

$$(\mathbf{K}-\alpha^2\mathbf{M})\mathbf{R}=\mathbf{0},$$

which has a non-trivial solution only in the case

$$\det(\mathbb{K}-\alpha^2\mathbb{M})=0\,. \tag{11}$$

From (11) the values of α_i^2 $(i=1, 2, \ldots, n)$ can be determined, that is to say the approximate values of the first n circular natural frequencies of the elastic continuum, where n is the degree of freedom of the finite element model. $\mathbb{K}$ and $\mathbb{M}$ being positive definite, α_i^2 values will be positive.

2. Timoshenko beam model

Simpler theories for the study of bending vibrations of beams or structural elements which can be modelled as beams have been developed in the following main steps:

1. Classical or Euler-Bernoulli's theory,
2. Rayleigh's theory,
3. TIMOSHENKO's theory [2].

In the classical theory only effects resulting in pure bending are taken into consideration, the behaviour of prismatic beams being described by the following equation:

$$EI\,\frac{\partial^4 v}{\partial x^4}+\rho A\,\frac{\partial^2 v}{\partial t^2}=0\,.$$

The resulting circular natural frequencies give acceptable results, in satisfactory agreement with measured values only for slender beams and only the first few modes, but they much exceed the exact values for higher modes. This difference rapidly increases with increasing order due to secondary effects ignored by this theory in establishing the model. One of these effects is the so-called rotary inertia of the cross-sections of the beam. Besides pure bending, Rayleigh's theory takes also this into consideration, and leads, similarly for prismatic beams, to the following equation:

$$EI\,\frac{\partial^4 v}{\partial x^4}+\rho A\,\frac{\partial^2 v}{\partial t^2}-I\rho\,\frac{\partial^4 v}{\partial x^2\partial t^2}=0\,.$$

This theory gives already a better approximation of circular natural frequencies, but still does not take the other important secondary effect, shear stress into account.

TIMOSHENKO, taking rotary inertia *and* shear stress into consideration, evolves a theory, the results of which show very good agreement with measured natural frequencies, even for higher modes and not only in the case of slender beams.

Fig. 1 picks out a deformed state of the beam element of length dx during vibration, in the plane of vibration. In the rest state the axis of the beam has been the x axis. Presuming small deformations, this may be assumed to have developed by the following steps. First, the rigid body-like displacement by angle Φ, caused by pure bending, occurs.

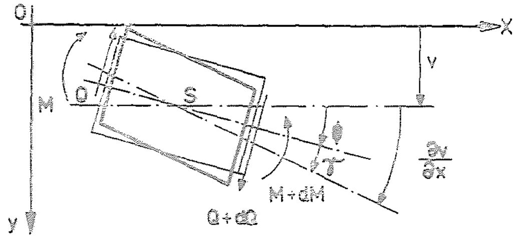

Fig. 1

The elastic line of the beam element after angular displacement is perpendicular to the cross-sections of the beam elements, in accordance with Kirchhoff's hypothesis. The second step of deformation results from the deforming effect of shear forces. The phenomenon can best be illustrated by the cards of a pack slipping on one another. No further angular displacement takes place, but mid-line will change, and will not be perpendicular to the cross-sections of the beam element any more. According to superposition, for the mid-line position:

$$\frac{\partial v}{\partial x} = \Phi + \gamma .$$

Applying Kirchhoff's hypothesis for the bending moment, and Timoshenko's hypothesis for the shear force:

$$M = -EI \frac{\partial \Phi}{\partial x} ; \qquad Q = kGA \left(\frac{\partial v}{\partial x} - \Phi \right). \tag{12a-b}$$

The coefficient k depends on Poisson's ratio and on the cross-section of the beam [3]. Substituting (12b) into the impulse theorem and (12a) into the moment of momentum theorem and eliminating Φ, we obtain for a beam of constant cross-section:

$$EI \frac{\partial^4 v}{\partial x^4} + \rho A \frac{\partial^2 v}{\partial t^2} - I\rho \left(\frac{E}{kG} + 1 \right) \frac{\partial^4 v}{\partial t^2 \partial x^2} + \frac{I\rho^2}{kG} \frac{\partial^4 v}{\partial t^4} = 0 . \tag{13}$$

The circular natural frequencies can be obtained from the transcendental equation—to be derived from (13) and difficult to handle—,though the beam is of constant cross-section. For non-prismatic beams a system of two differential equations is obtained instead of (13), seldom solved under utilization of special functions of mathematical physics.

3. Stiffness and mass matrices of Timoshenko beam element

The circular natural frequencies of Timoshenko beam, performing free transverse vibrations are determined therefore with the process outlined under Section 1.

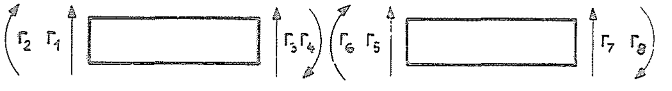

Fig. 2
Nodal generalized displacements due to bending and shear

Let the field of displacement be characterized by the vector $\mathbf{u}^T = [v_b, v_s]$. Total displacement will be evidently $v = v_b + v_s$. The nodal generalized displacement co-ordinates are the deflections and the rotations of the two end points of the beam element, separately from bending and shear [4] (see Fig. 2).

Both v_b and v_s will be expediently approximated by third-degree polynomials, each containing four constants. Determining the constants from the boundary conditions

$$v_b|_{x=0} = r_1 \,; \qquad -\left.\frac{\partial v_b}{\partial x}\right|_{x=0} = r_2 \,; \qquad \ldots ; \qquad -\left.\frac{\partial v_s}{\partial x}\right|_{x=L} = r_8 \,,$$

the matrix $\mathbf{N}$ contained in (1) is obtained:

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{22} \end{bmatrix} \tag{14}$$

$$\mathbf{N}_{11} = \mathbf{N}_{22} = \left[\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \,; \quad -\frac{x^3}{L^2} + \frac{2x^2}{L} - x ; \right.$$

$$\left. -\frac{2x^3}{L^3} + \frac{3x^2}{L^2} ; \quad -\frac{x^3}{L^3} + \frac{x^2}{L^2} \right].$$

A beam model being considered, it is sufficient to calculate, instead of the stresses, with the bending moment and the shear force, and instead of the déformations, with the second derivate of v_b and the first derivate of v_s with respect to x. With symbols in Section 1, quantities in (2) and (3) are as follows:

$$\varepsilon^T = \left[\frac{\partial^2 v_b}{\partial x^2}; \frac{\partial v_s}{\partial x}\right], \qquad \partial = \left\langle \frac{\partial^2}{\partial x^2}; \frac{\partial}{\partial x} \right\rangle, \qquad \mathbb{D} = \langle IE,\ kAG \rangle, \tag{15}$$

and from these $\mathbb{K}$ can be obtained according to (8), only now the integration will be performed with respect to x. Substituting (15) into (7), using (1), (2) and (3) actually the strain energy of the Timoshenko beam element

$$U = \frac{1}{2} \int_{x=0}^{L} \left[IE \left(\frac{\partial^2 v_b}{\partial x^2}\right)^2 + kAG \left(\frac{\partial v_s}{\partial x}\right)^2 \right] \mathrm{d}x$$

is obtained, where naturally IE and AG may be quantities varying along the length.

For the determination of the mass matrix, (5) and (6) cannot be directly used, because $\mathbf{u}$ comprises in our case only the displacements of the elastic line, so that (5) does not take the kinetic energy arising from rotary inertia into consideration.

The kinetic energy of Timoshenko beam element is

$$T = \frac{1}{2} \int_{x=0}^{L} \left[\rho A \left(\frac{\partial (v_b + v_s)}{\partial t}\right)^2 + \rho I \left(\frac{\partial^2 v_b}{\partial x \partial t}\right)^2 \right] \mathrm{d}x . \tag{16}$$

Using (1), (14) and (16), and introducing notations

$$\mathbb{T} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \qquad \partial_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & 0 \end{bmatrix}; \qquad \mathbb{B}_1 = \partial_1 \mathbb{N},$$

the expression

$$T = \frac{1}{2} \dot{\mathbf{r}}^T \int_{x=0}^{L} (\rho A \mathbb{N}^T \mathbb{T} \mathbb{N} + \rho I \mathbb{B}_1^T \mathbb{B}_1)\, \mathrm{d}x \cdot \dot{\mathbf{r}} = \frac{1}{2} \dot{\mathbf{r}}^T \mathbb{M} \dot{\mathbf{r}}$$

is obtained, from which the mass matrix can be calculated.

In case of a uniform beam, the stiffness matrix $\mathbb{K}$ of the beam element has the following form:

$$\mathbb{K} = \langle \mathbb{K}_{11}, \mathbb{K}_{22} \rangle \cdot \frac{EI}{L^3};$$

$$\mathbb{K}_{11} = \begin{bmatrix} 12 & & & \text{Symm} \\ -6L & 4L^2 & & \\ -12 & 6L & 12 & \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix};$$

$$\mathbb{K}_{22} = \begin{bmatrix} 36 & & & \text{Symm} \\ -3L & 4L^2 & & \\ -36 & 3L & 36 & \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \cdot S;$$

where $S = \dfrac{kGAL^2}{30\,EI}$.

The mass matrix $\mathbb{M}$ of the beam element has the following form:

$$\mathbb{M} = \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} \cdot \frac{A\rho L}{420};$$

$$\mathbb{M}_{11} = \begin{bmatrix} 156+36H & & & \\ -(22+3H)L & 4(1+H)L^2 & \text{Symm} & \\ 54-36H & (-13+3H)L & 156+36H & \\ (13-3H)L & -(3+H)L^2 & (22+3H)L & 4(1+H)L^2 \end{bmatrix};$$

where $H = 14\dfrac{I}{AL^2}$;

$$\mathbb{M}_{12} = \mathbb{M}_{21} = \mathbb{M}_{22} = \begin{bmatrix} 156 & & & \text{Symm} \\ -22L & 4L^2 & & \\ 54 & -13L & 156 & \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}.$$

4. Numerical example for determining the circular natural frequencies of a uniform beam under different boundary conditions

The numerical example given to illustrate the process determines the first fourcircular natural frequencies of a beam with annular cross-section.

Data: $D=0.296$ m; $d=0.122$ m; $l=1$ m; $\nu=\frac{1}{3}$; $k=\frac{2}{3}$; $E=2.1\cdot 10^{11}$ Pa, $\frac{G}{E}=\frac{3}{8}$; $\rho=7.8\cdot 10^{3}$ kgm^{-3}.

Number of element	i	$10^{-7}\alpha_i$ [s^{-1}]			
		FEM	EXACT	FEM	EXACT
1	1	3.867		1.356	
	2	12.68		7.989	
2	1	3.565	3.554	1.351	1.350
	2	11.75	11.08	6.404	6.368
	3	21.46	19.60	15.21	14.10
	4	30.52	28.26	25.34	22.06
		Simply supported beam		Cantilever beam	

Summary

The finite element displacement method is used for the approximate determination of the circular natural frequencies of Timoshenko beam performing free bending vibrations under various boundary conditions.

References

1. ZIENKIEWICZ, O. C.—CHEUNG, Y. K.: The finite element method in structural and continuum mechanics. McGraw Hill (1967).
2. TIMOSHENKO, S. P.; On the correction for shear of the differential equations for transverse vibrations of prismatic bars. Philosophical Magazine (1921) 41, 744—746.
3. COWPER, G. R.: The shear coefficient in Timoshenko's beam theory. Journal of Applied Mechanics (1966) 33, 335—340.
4. DAVIS, R.—HENSHELL, R. D.—WARBURTON, G. B.: A Timoshenko beam element. Journal of Sound and Vibration (1972) 22, 475—487.
5. THOMAS, J.—ABBAS, B. A. H.: Finite element model for dynamic analysis of Timoshenko beam. Journal of Sound and Vibration (1975) 41, 291—299.

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