LUMPED NETWORK MODELS OF INHOMOGENEOUS DISTRIBUTED RC-LINES

By

L. Zombory

Department of Theoretical Electricity, Technical University, Budapest

Received October 5, 1979

Presented by Prof. Dr. G. Fodor

Introduction

The resistors and links of the integrated circuits are essentially distributed RC-lines. The computer-aided analysis of IC-s requires to integrate the models of distributed RC-lines into the program.

Among the commonly used models of the distributed RC-line the modelling by dominant poles gives the possibility to analyse transfer properties of the separate line itself. Its integration into the program is complicated, if it is possible at all, and usually requires to develop special subroutines performing a high number of matrix iterations [1].

Consequently it is advisable to replace the RC-line by a model containing lumped network elements, allowed by the program. It is obvious to apply RC ladder networks. But these types of networks are difficult to be identified. Therefore, this paper aims at the development of lumped equivalent networks which approximate the admittance parameters in the same order at a maximum flatness and can be extended by increasing the order of approximation — including the system of identifying equations as well.

In this paper the process of determining the coefficients of the admittance parameter series expansion is shown. Hereupon the structure of the networks approximating at a maximum flatness is determined, followed by the parameter identification.

1. Admittance characteristics of distributed RC-lines

In the following the determination of the coefficients of the admittance parameter series in s is shown for lines of parameters $r(x)=R_0/f(x)$, $c(x)=C_0f(x)$ but the procedure can be generalized without difficulty for lines having any bounded positive profile functions $r(x)$ and $c(x)$.

The governing equations of the RC-line along the axis x between 0 and L are the following:

$$-\frac{\mathrm{d}U}{\mathrm{d}x}=\frac{R_0}{f}I; \qquad -\frac{\mathrm{d}I}{\mathrm{d}x}=sC_0fU \tag{1.a.b}$$

from which the well-known second-order differential equation for the voltage is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\frac{\mathrm{d}U}{\mathrm{d}x}\right)-sR_0C_0fU=0 \tag{2}$$

The aim is to obtain the admittance parameter series of the form:

$$y_{ik}(s)=y_{ik}^{(0)}+sy_{ik}^{(1)}+s^2y_{ik}^{(2)}+\ldots \tag{3}$$

For this reason, first the coefficients of the series of the voltage

$$U(x,s)=U^{(0)}(x)+sU^{(1)}(x)+s^2U^{(2)}(x)+\ldots \tag{4}$$

are determined. Here the $U^{(n)}$ depend linearly both on $U_1=U(0,s)$ and on $U_2=U(L,s)$. The coefficients of series (3) can be calculated from the equations

$$y_{11}^{(n)}U_1+y_{12}^{(n)}U_2=-\frac{f}{R_0}\frac{\mathrm{d}U^{(n)}}{\mathrm{d}x}\bigg|_{x=0} \tag{5.a}$$

$$y_{21}^{(n)}U_1+y_{22}^{(n)}U_2=\frac{f}{R_0}\frac{\mathrm{d}U^{(n)}}{\mathrm{d}x}\bigg|_{x=L} \tag{5.b}$$

expressing the equality of the coefficients U_1 and U_2 on both sides.

In determining the coefficients of series (4) the direct solution of Eq. (2) is to be avoided. (The solution usually cannot be given in closed form.) For this reason, series (4) is substituted into Eq. (2) and the expression is ordered according to powers of s. After this the equality is required to be met by the coefficients of each power separately. (Essentially the procedure is a perturbation by parameter s.)

The following system of equations results:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\frac{\mathrm{d}U^{(0)}}{\mathrm{d}x}\right)=0; \qquad U^{(0)}(0)=U_1; \qquad U^{(0)}(L)=U_2 \tag{6.a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\frac{\mathrm{d}U^{(n)}}{\mathrm{d}x}\right)=R_0C_0fU^{(n-1)}; \qquad U^{(n)}(0)=U^{(n)}(L)=0; \qquad n>0 \tag{6.b}$$

The fundamental system of solutions for the homogeneous equation (6.a) satisfying the boundary conditions $\kappa_1(0)=\kappa_2(L)=1$; $\kappa_1(L)=\kappa_2(0)=0$:

$$\kappa_1(x)=\int_x^L \frac{dx}{f}\Big/\int_0^L \frac{dx}{f};\qquad \kappa_2(x)=\int_0^x \frac{dx}{f}\Big/\int_0^L \frac{dx}{f} \tag{7}$$

and the solution of the equation satisfying the boundary conditions:

$$U^{(0)}(x)=U_1\kappa_1+U_2\kappa_2 \tag{8}$$

The zero-order terms of the admittance characteristics according to Eq. (5) are:

$$y_{11}^{(0)}=y_{22}^{(0)}=-y_{12}^{(0)}=-y_{21}^{(0)}=1\Big/R_0\int_0^L \frac{dx}{f} \tag{9}$$

The zero-order approximation of the RC-line corresponds to a single series resistor.

The homogeneous equations corresponding to the inhomogeneous equations (6.b) identical with Eq. (6.a). Therefore, their solution is obtained by using the fundamental system (7) and applying the method of the variation of the coefficients:

$$U^{(n)}(x)=-R_0C_0\int_0^L \frac{dx}{f}\left[\kappa_1\int_0^x f\kappa_2U^{(n-1)}dx+\kappa_2\int_x^L f\kappa_1U^{(n-1)}dx\right] \tag{10}$$

where the Wronskian $W=\kappa_1\kappa_2'-\kappa_1'\kappa_2=1\Big/f\int_0^L \frac{dx}{f}$ has been substituted and the homogeneous boundary conditions have been satisfied.

The coefficients of series (3) are obtained from (10) by using (5).

The first-order approximation leads to directly realizable admittance parameters (Fig. 1) [2] [3]. These resulting so-called four-element approximat-

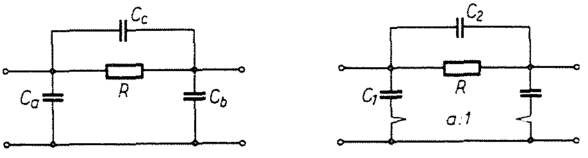

Fig. 1. Four-element equivalent networks

ing networks cannot, however, be consequently extended for higher-order approximations, because the higher-order approximating polynomials of the admittance parameters cannot be realized by lumped passive networks. A consequently extensible structure must be looked for in another way. For this reason the structure of the equivalent networks of distributed RC-lines, containing a countable infinity of lumped elements, is analyzed.

2. The structure of the equivalent network

A realizable network can be obtained by expanding the admittance parameters to the sum of partial fractions [4]. A tedious calculation results in:

$$y_{11}=\frac{1}{R_0\int\limits_0^L\frac{\mathrm{d}x}{f}}+\frac{1}{R_0^2C_0}\sum_{k=1}^{\infty}\frac{s}{(-s_k)(s-s_k)}\left[f\frac{\mathrm{d}\varphi_k}{\mathrm{d}x}\bigg|_{x=0}\right]^2 \tag{11.a}$$

$$-y_{12}=-y_{21}=\frac{1}{R_0\int\limits_0^L\frac{\mathrm{d}x}{f}}+\frac{1}{R_0^2C_0}\sum_{k=1}^{\infty}\frac{s}{(-s_k)(s-s_k)}\left[f\frac{\mathrm{d}\varphi_k}{\mathrm{d}x}\bigg|_{x=0}\right]\left[f\frac{\mathrm{d}\varphi_k}{\mathrm{d}x}\bigg|_{x=L}\right] \tag{11.b}$$

$$y_{22}=\frac{1}{R_0\int\limits_0^L\frac{\mathrm{d}x}{f}}+\frac{1}{R_0^2C_0}\sum_{k=1}^{\infty}\frac{s}{(-s_k)(s-s_k)}\left[f\frac{\mathrm{d}\varphi_k}{\mathrm{d}x}\bigg|_{x=L}\right]^2 \tag{11.c}$$

Here $\varphi_k(x)$ is the k-th eigenfunction of Eq. (2) satisfying the boundary conditions $\varphi_k(0)=\varphi_k(L)=0$. The eigenfunctions are normed by the formula

$$\int\limits_0^L f(x)\varphi_k^2(x)\mathrm{d}x=1 \tag{12}$$

s_k is the eigenvalue belonging to φ_k. Because of the properties of the Sturm—[illegible] type equation (2) all of the eigenvalues are negative real quantities. As another consequence of the properties of the equation, the terms of the infinite series (11.b) are of alternating sign. The infinite series converge uniformly for any finite s, except the poles s_k.

Two equivalent networks corresponding to the admittance parameters (11) are shown in Fig. 2. The network a) is the Π-equivalent, while that b) can be obtained by Cauer synthesis method [5]. The elements of both networks are uniquely determined by (11). It is worth to note that, because of the

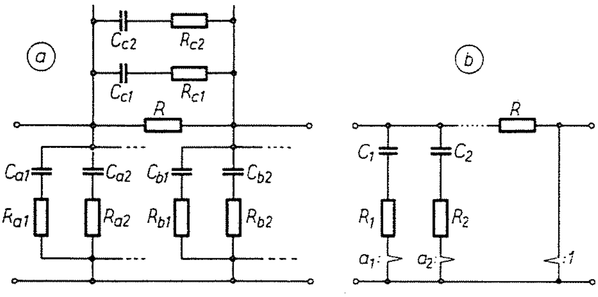

Fig. 2. Equivalent networks a) Π-type; b) Cauer-type

aforementioned behaviour of y_{12}, the Π-equivalent contains elements of negative value, too, while the transformer ratios in the Cauer-equivalent are of alternating sign. Furthermore it must be noted that

$$R_{ak}C_{ak}=R_{bk}C_{bk}=R_{ck}C_{ck}=-1/s_k; \qquad R_kC_k=-1/s_k. \tag{13}$$

The approximation of (3) will apply networks similar in structure to the infinite ones, but containing only finite number of elements. Thus, for higher-order approximation branches of the same structure have to be put between the same nodes.

3. The parameter identification of the approximating networks

In the case of maximally flat approximation there are two alternatives:

a) Several of the dominant poles are exactly represented by network elements satisfying (13) and further elements are used to fit the coefficients. This procedure usually results in larger bandwidth for an approximation of a given order, but contains larger number of elements and requires the determination of eigenvalues and ēigenfunctions (e.g. by variational methods).

b) All of the elements are used for maximally flat approximation and no satisfaction (13) is required. This procedure results in an approximation valid usually, in a narrower frequency band but there it is exacter [6].

In the following the latter procedure is shown in detail, but the statements are valid for the previous one, to the sense.

3.1. *Identification of the elements of the Π-type network*

In an element of the Π-type network k series RC sets are parallel coupled. The resulting admittance is:

$$Y = \sum_{\nu=1}^{k} \frac{1}{R_\nu} \frac{s}{s + 1/R_\nu C_\nu} \tag{14}$$

The coefficient of the i-th order term in s of the series of admittance (14):

$$\frac{1}{i!} \frac{\mathrm{d}^i Y}{\mathrm{d}s^i}\bigg|_{s=0} = (-1)^{i-1} \sum_{\nu=1}^{k} (C_\nu R_\nu)^{i-1} C_\nu; \quad i = 1, 2, \ldots \tag{15}$$

For the zero-order terms there is only one condition according to (9).

Comparing (15) to the corresponding coefficients of the series (3) after some ordering one gets the following system of equations:

$$y_{11}^{(i)} + y_{12}^{(i)} = (-1)^{i-1} \sum_{\nu=1}^{k_a} (C_{a\nu} R_{a\nu})^{i-1} C_{a\nu} \tag{16.a}$$

$$y_{22}^{(i)} + y_{12}^{(i)} = (-1)^{i-1} \sum_{\nu=1}^{k_b} (C_{b\nu} R_{b\nu})^{i-1} C_{b\nu} \qquad i = 1, 2, \tag{16.b}$$

$$-y_{12}^{(i)} = (-1)^{i-1} \sum_{\nu=1}^{k_c} (C_{c\nu} R_{c\nu})^{i-1} C_{c\nu} \tag{16.c}$$

Where k_a, k_b, k_c mean the numbers of the parallel coupled series RC sets in the admittances.

In the system of equations (16) the elements of indices a, b and c are seen to be totally decoupled. The number of unknowns being even in every independent system of equations, the admittance parameters can only be approximated in even order. If furthermore it is required that every parameter should be approximated in the same order, then $k_a = k_b = k_c = k$ must be valid. Therefore the total number of parameters to be identified is $1 + 6k$. Since the n-th order fit of all of the admittance parameters needs $1 + 3n$ network parameters, $k = n/2$.

The system of equations (16) is of momentum type and each equation can be reduced to a k-th order algebraic equation of one unknown [7].

3.2. *Parameter identification of the Cauer-type network*

Every cross branch admits 3 free network parameters to fit, consequently n cross branches are necessary for the n-th order approximation of every admittance parameter up to. The expansion of the admittance parameters of the network containing n cross branches and comparing the coefficients to those of the series (3) leads to:

$$y_{11}^{(i)}=(-1)^{i-1}\sum_{v=1}^{n}(C_vR_v)^{i-1}C_v \tag{17.a}$$

$$-y_{12}^{(i)}=(-1)^{i-1}\sum_{v=1}^{n}(C_vR_v)^{i-1}C_va_v \qquad i=1,2,\ldots \tag{17.b}$$

$$y_{22}^{(i)}=(-1)^{i-1}\sum_{v=1}^{n}(C_vR_v)^{i-1}C_va_v^2 \tag{17.c}$$

The system of equations (17) can be reduced to a system of symmetric momentum equations. It can be proven that the equations have a solution only for even n.

4. Comparison of the network models

In the following, some characteristics of the examined works and those of RC ladder networks approximating in the same (even) order have been tabulated.

Denomination	new nodal equations	state variables	branches
Cauer type network	$n(3n)$	n	$1+3n(1+4n)$
Π-type network	$3n/2$	$3n/2$	$1+3n$
RC ladder (assymm.)	$3n/2^{-1}_{+0}$	$3n/2^{+1}_{+0}$	$1+3n$
RC ladder (symm.)	$2n^{-1}_{+0}$	$2n^{+1}_{+0}$	$1+4n$

In the first value for Cauer-type network, the transformer and the resistor in every cross branch are supposed to be handled as a two-port. This type of network model is obviously superior for the purpose of nodal analysis in this case.

It is noteworthy at the same time that the Π-type network is of higher order for the same order of approximation. Therefore its behaviour is presumably better in the time-domain. The two type of networks (although they give an approximation of the same order) are not equivalent.

5. Examples

The second- and fourth-order Π-type and Cauer-type network models of the homogeneous RC-line have been determined by the method described above (Fig. 3). Already the fourth-order approximation is seen to realize quite correctly the first dominant poles.

In Fig. 4 the short circuit input impedance, in Fig. 5 the unloaded voltage-transfer function of the first and second order approximating network are shown, compared to the distributed RC-line and the RC ladder network of 5 elements, approximating in the first order.

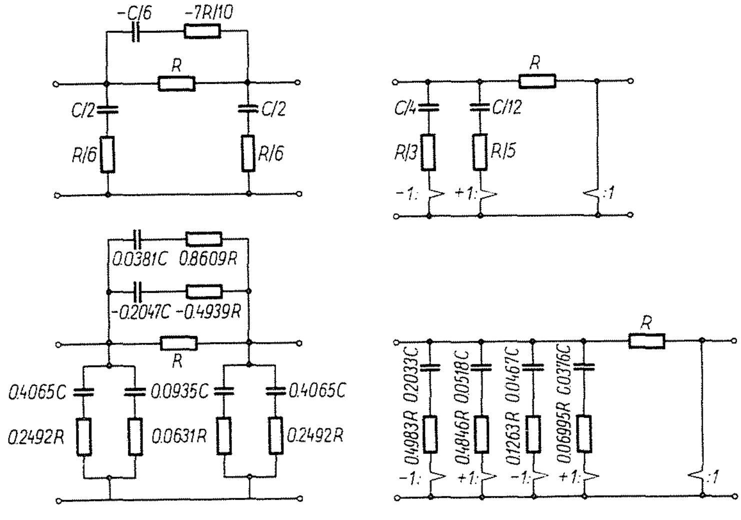

Fig. 3. Second and fourth order network models of homogeneous RC-line

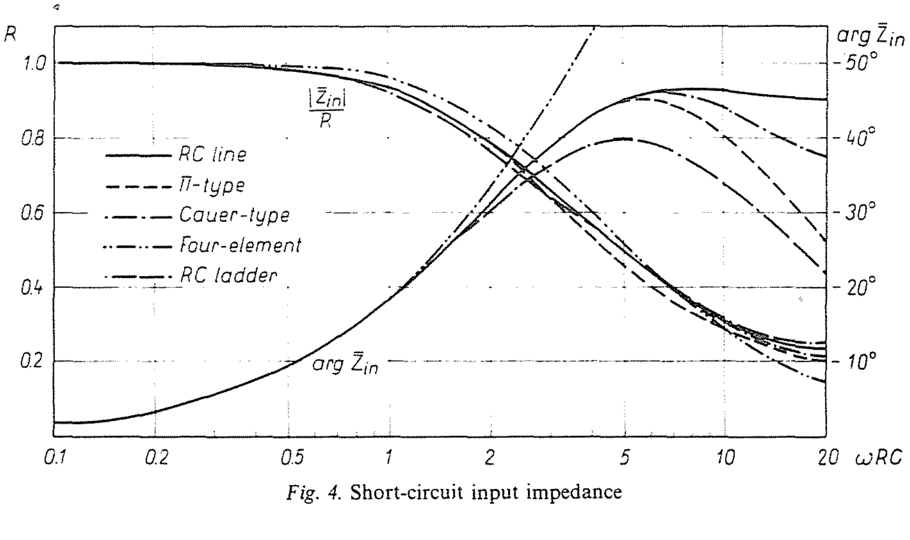

Fig. 4. Short-circuit input impedance

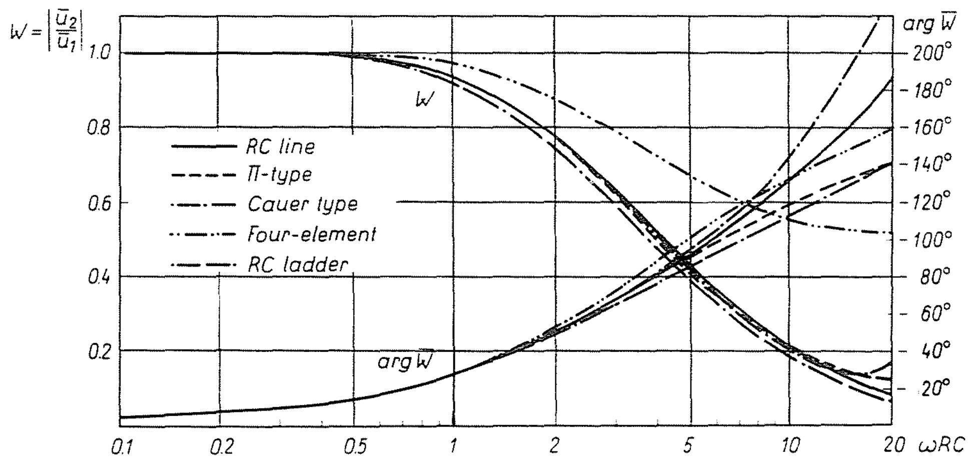

Fig. 5. Unloaded voltage-transfer function

Acknowledgements

The author is indebted to Prof. Dr. G. Fodor and dr. A. Magos, for their valuable advice.

Summary

The resistors and links of the integrated circuits are essentially distributed RC-lines. It is advisable to replace the RC-line by a model containing lumped network elements, allowed by the program. It is obvious to apply RC ladder networks. But these types of networks are difficult to be identified. This paper aims at the development of lumped equivalent networks which approximate the admittance parameters in the same order at a maximum flatness. The process of determining the coefficients of the admittance parameter series expansion is shown too.

References

1. Rajput, Y. V.: Modelling distributed RC lines for the transient analysis of complex networks. Int. J. Electron. **36**, 709 (1974)
2. Gurary, M. M.: Ob odnom metode posztrojenyija dinamicseszkih modelej komponentov integrálnih szhem. Mikroelektronika, vip. 4. M. „Szov. Radio". 1971.
3. Bhattacharyya, A. B.—Gupta M. L.: Single-section lumped models for integrated circuit resistors. Solid-St. Electron. **16**, 1506 (1973)
4. Korn, G. A.—Korn T. M.: Mathematical Handbook (Hungarian ed.) Bp. Műszaki K. 1975. Sec. 15. 3. 8.
5. Géher K.: Lineáris hálózatok (Linear Networks) Bp. Műszaki K. 1972. Sec. 5. 4. 1.
6. Magos A.: Lumped equivalent networks of transmission lines. Per. Pol. El. Eng. **16**, 235 (1972)
7. Magos A.: Private communication

dr. László Zombory, H-1521 Budapest