THE CONVERGENCE OF THE CONSTANT INTERVAL DIGITAL SIMULATION

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In the digital simulation of a control system numerical methods are used to solve the system differential equation. The one-step integration formulas—especially the Runge-Kutta algorithms—are widely used. In special circumstances, however, they give inaccurate or instable solution. The cases where the one-step algorithms of constant interval lead to satisfactory solution are determined in a very simple and clear way, based on the linear control circuits of constant parameters.

The digital simulation replaces the continuous part of the system by a discrete model, and the analogous operations by numerical procedures, constructed from algebraic relations. In determining the discrete model to start with the state variable form of the system is desirable, for the numeric programing of the differential equations transforms the other forms into state variable form. Let the differential equation of the continuous linear system with constant parameters be:

$$\dot{X}(t)=AX(t)+BU(t) \tag{1}$$

$$Y(t)=CX(t)+DU(t)$$

where $X(t)$ is the vector of the state variables, $Y(t)$ that of the output signals and $U(t)$ of the input signals. A, B, C, D are the parameter matrices, of the system.

The continuous model described by Eq. (1) is converted into discrete model (Fig. 1) converting the input signals collected in vector $U(t)$ into impulse series $U^*(t)$ by means of sampler switches M_1, and in the simplest case $B=I$ transmitting them to the input of the continuous system via a holding circuit of zero order T, as signals $U_T(t)$, varying stepwise in time. The switches M_2 working synchronously with M_1, convert the continuous signals $X(t)$ into impulse series $X^*(t)$ and $U^*(t)$. Signals $X^*(t)$ and $U^*(t)$ are related by the difference equations:

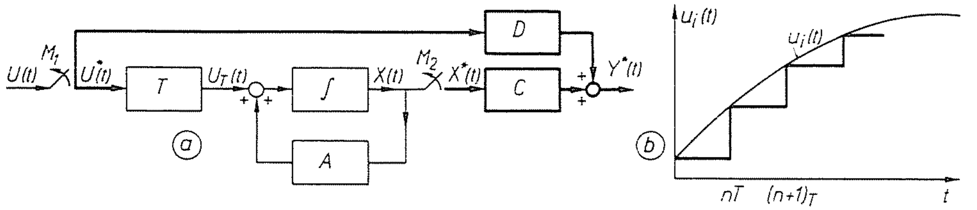

Fig. 1

$$X\{(n+1)T\}=A^*X(nT)+B^*U(nT) \tag{2a}$$

$$Y^*(nT)=C^*X(nT)+D^*U(nT) \tag{2b}$$

Here

$$A^*=e^{AT}; \quad B^*=A^{-1}(e^{AT}-I)B \tag{3}$$

where I is the unit matrix. These equations are the recurrent algorithms of the numerical solution of the state differential equation with the condition corresponding to Fig. 1b, i.e. the signal $u_i(nT)$ in the input of the system is constant between the moments nT and $(n+1)T$. The real signal $u_i(t)$ is substituted by the stepwise function $u_{iT}(t)$, equal to $u_i(t)$ at the sampling points and constant between two sampling points. Starting from the initial data the $X(T), X(2T)\ldots X(nT)$ and $Y(T)\ldots Y(nT)$ values are determined step by step. They equal to the sampled values of the vectors $X(t)$ and $Y(t)$ of a system excited by the stepwise curve U_T, so they can be regarded as the best numerical solution of the state differential equation for a given input signal U_T. This is possible in the linear case, since the analytical solution of the state differential equation is known.

Let us approximate coefficients A^* and B^* by the Taylor series A^* and B^*. From Eq. (3):

$$A^*=I+AT+\frac{A^2T^2}{2}+\frac{A^3T^3}{6}+\frac{A^4T^4}{24}+\ldots \tag{4a}$$

$$B^* = BT+\frac{ABT^2}{2}+\frac{A^2BT^3}{6}+\frac{A^3BT^4}{24}+\ldots \tag{4b}$$

Depending on the number of terms taken into consideration different A^* and B^* values are obtained. Substituting them into Eq. (2) different numerical solving algorithms result. Taking the first two terms of the Taylor series of the function exp (AT) into account (finishing the series in Eqs (4) with the terms linear in T) yields the well-known Euler formula.

$$X[(n+1)T]=(I+AT)X(nT)+BTU(nT) \tag{5}$$

The consideration of the second-order terms results in the generalized trapezoid formula, the consideration of the 4th order terms results in the Runge-Kutta procedure of the 4th order. Hence the algorithms with the exact coefficients A^* and B^* in Eq. (2), can be regarded as a Runge-Kutta procedure of "infinite order".

Since in the linear case the results of all the procedures can be produced in closed formula, using the z-transformation, the errors of each procedure can be analysed very simply. The z-transformed form of Eqs (2) under initial condition $X(0)=0$:

$$zX^*(z)=A^*X^*(z)+B^*U^*(z) \tag{6a}$$

$$Y^*(z)=C^*X^*(z)+D^*U^*(z) \tag{6b}$$

Let us examine continuous systems of one output ($U=u$) and one input ($Y=y$) signals, consisting of inertia elements of the 1st order (or of integral or proportional elements being the marginal cases of the formers). Their continuous and discrete state equations with zero initial conditions (the holding included) can be given in the s and z domain by equations:

$$x_1(s)=\frac{u(s)}{1+sT_i} \quad \text{or} \quad x_1^*(z)=\frac{b_1^*u^*(z)}{z-a_1^*} \tag{7}$$

$$\vdots \qquad\qquad \vdots$$

$$x_n(s)=\frac{u(s)}{1+sT_n} \qquad x_n^*(z)=\frac{b_n^*u^*(z)}{z-a_n^*} \tag{8}$$

$$w(s)=\frac{y(s)}{u(s)}=\frac{c_1}{1+sT_1}+\ldots+\frac{c_n}{1+sT_n} \qquad w^*(z)=c_1\frac{b_1^*}{z-a_1^*}+\ldots+\frac{c_nb_n^*}{z-a_n^*}$$

$T_1\ldots T_n$ are the time constants of the terms. The expressions are valid for the complex values of $1/T_k$ as well, hence the state variable form of systems containing oscillating elements are also included. Eqs (8) are the partial fractional forms of the transfer and of the pulse transfer functions:

$$w(s)=K\frac{(1+s\tau_1)(1+s\tau_2)\ldots}{(1+sT_1)(1+sT_2)\ldots} \quad \text{i.e.} \quad w^*(z)=K^*\frac{(z-\sigma_1)(z-\sigma_2)\ldots}{(z-a_1^*)(z-a_2^*)\ldots}, \tag{9}$$

Parameters τ and σ depend on parameters c and b or c^* and b^*.

The correct values of a_k^* and b_k^* corresponding to Eq. (3) are:

$$a_k^* = e^{-T/T_k}; \qquad b_k^* = 1 - e^{-T/T_k} \tag{10}$$

Substituting these values into $w^*(z)$, the pulse transfer function of the system completed with holding results, which can be regarded as analytical expression of numerical integration using the Runge-Kutta formula of the infinite order:

$$w_\infty^*(z) = \frac{c_1}{z - e^{-T/T_1}} + \ldots + \frac{c_n}{z - e^{-T/T_n}} \tag{11}$$

Substituting a_k^* and b_k^* by the first few members of the Taylor series of Eqs (10) yields the approximation of the function $w^*(z)$, which is the closed form of the finite order Runge-Kutta algorithm. This approximation can be obtained by replacing $\exp(-T/T_k)$ in Eq. (11) by the first few members of its Taylor series.

The closed expression of the Euler formulu directly results from using the continuous form without factoring it into partial fractions. The z-transform of Eq. (5) is:

$$\frac{z-1}{T} X^*(z) = AX^*(z) + BU^*(z) \tag{12}$$

Comparing this to the continuous state equation (2a), it is seen, that substituting the Laplace transforms of the variables in the state equation of the continuous system by the z-transforms of the sampled signals and substituting the variable s by the expression

$$s \sim \frac{z-1}{T}, \tag{13}$$

the approximate state equation of the sampled system results. Thus the approximate pulse transfer function is directly obtained in this case from the transfer function $w(s)$ of the continuous system using substitution (13). If the transfer function consists of serial terms $w(s) = w_1(s)w_2(s)$, the pulse transfer function $w^*(z) = w_1^*(z)w_2^*(z)$ can be determined by term by term conversion. This is equivalent to the insertion of a sampler switch and a holding circuit between the members (Fig. 2). Each pair of switches-holding circuits inserts into the system an additional time delay of about $T/2$ to, T, provided its effect is not compensated by the next zeros. In the cases of more members or significant T values the phase characteristics may be strongly distorted by this effect.

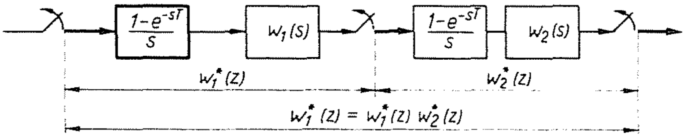

Fig. 2

The angular error can be reduced by substituting the variable s by the bilinear expression

$$s \sim \frac{2}{T} \frac{z-1}{z+1}, \tag{14}$$

leading to the numerical algorithm:

$$\frac{X(n+1)T - X(nT)}{T} = A\frac{X(n+1)T + X(nT)}{2} + B\frac{U(n+1)T + U(nT)}{2}. \tag{15}$$

The higher- order approximation of a^* and b^* cancels the possibility of direct substitution of s variable by z.

Table 1 shows the pulse transfer function of a first order system with one time constant and with the transfer function

$$w(s) = \frac{1}{1+sT_1}$$

in the following cases:

a) The accurate pulse transfer function:

$$w_\infty^*(z) = \frac{1-e^{-T/T_1}}{z-e^{-T/T_1}}. \tag{16}$$

b) Pulse transfer function $w_4(z)$ corresponding to the Runge-Kutta algorithm of the 4th order. The term $\exp(-T/T_1)$ in Eq. (16) is substituted by the first five terms from its Taylor series.

Table 1

$w^*(z)$	T/T_1					
	0,1	0,4	1	2	3	4
$w^*_\infty(z)$	$\frac{0.095}{z-0.905}$	$\frac{0.33}{z-0.67}$	$\frac{0.632}{z-0.370}$	$\frac{0.865}{z-0.135}$	$\frac{0.950}{z-0.05}$	$\frac{0.982}{z-0.018}$
$w^*_4(z)$	$\frac{0.095}{z-0.905}$	$\frac{0.33}{z-0.67}$	$\frac{0.625}{z-0.375}$	$\frac{0.666}{z-0.333}$	$\frac{-0.375}{z-1.375}$	$\frac{-4}{z-5}$
$\tilde{w}^*_1(z)$	$\frac{0.048(z+1)}{z-0.905}$	$\frac{0.167(z+1)}{z-0.667}$	$\frac{0.33(z+1)}{z-0.333}$	$\frac{0.5(z+1)}{z}$	$\frac{0.6(z+1)}{z+0.2}$	$\frac{0.667(z+1)}{z+0.333}$
$w^*_1(z)$	$\frac{0.1}{z-0.9}$	$\frac{0.4}{z-0.6}$	$\frac{1}{z}$	$\frac{2}{z+1}$	$\frac{3}{z+2}$	$\frac{4}{z+3}$

c) Using the bilinear algorithm of Eq. (14):

$$\tilde{w}^*_1(z) = \frac{z+1}{z - \dfrac{2-T/T_1}{2+T/T_1}}. \tag{17}$$

d) Pulse transfer function corresponding to the Euler formula, using the approximation of the first order in T:

$$w^*_1(z) = \frac{T/T_1}{z-(1-T/T_1)}. \tag{18}$$

Increasing T/T_1 the approximate formulas gradually deviate from the theoretically correct z-transformed form shown in Table 1. The Runge-Kutta formula of the 4th order is almost correct up to $T/T_1=1$, while the Euler formula gives the same correctness in the case $T/T_1<0.1$. In the range $T/T_1>1$ the approximations are ever less of use. For $T/T_1=3$ even the pole of the 4th-order expression is out of the unit circle, i.e. the stable system is shown to be unstable by the model, because of the numerical instability. Although the bilinear formula will not be unstable, but the pulse transfer function is absolutely inaccurate..

Table 2

$w(z)$	$T/T_1=0.1$	
	$T/T_2=0.4$	$T/T_2=3$
$w_\infty^*(z)$	$0.017\dfrac{z+0.847}{(z-0.905)(z-0.670)}$	$0.0657\dfrac{z+0.3768}{(z-0.905)(z-0.05)}$
$w_4^*(z)$	$0.017\dfrac{z+0.847}{(z-0.905)(z-0.670)}$	$0.111\dfrac{z-1.32}{(z-0.905)(z-1.375)}$
$\bar{w}_1^*(z)$	$0.008\dfrac{(z+1)^2}{(z-0.905)(z-0.67)}$	$0.0288\dfrac{(z+1)^2}{(z-0.905)(z+0.2)}$
$w_1^*(z)$	$0.04\dfrac{1}{(z-0.9)(z-0.6)}$	$\dfrac{0.3}{(z-0.9)(z+2)}$

Table 2 shows the comparison referring to the 2nd order system:

$$w(s) = \frac{1}{(1+sT_1)(1+sT_2)}. \tag{19}$$

The accuracy of the approximations may be characterized by angular and absolute value errors of the frequency responses referred to $w_\infty^*(j\omega)$.

The good approximation is important in the frequency band up to the cut-off frequency, for the control circuit is effective in this region.

There are two ways of the computer analysis of a continuous control system:

a) Discrete modelling or programming the differential equation of the closed loop. In this case, disregarding the errors caused by the sampling of the input signals, the result given by the exact z-transform algorithm is equal to the sampled values of the accurate output signal independent of the sampling intervals. Deviation results only from errors of the algorithms of finite order.

b) The simulation programms the feedback of the discrete model of the open loop. This is the only way in most real cases, namely simulation is usually needed when the analytical handling of the closed loop is impossible or difficult. The samplers and holders inserted in the control loop, (the discretization of the system), cause additive dead time, altering the phase caracteristic of the loop even making it unstable under adverse conditions.

For this reason the output signal can only be restored with a given error even by the algorithm of infinite order. The error of the algorithms of finite order is added to this.

These errors can be neglected when the sampling interval is significantly—e.g. by one order of magnitude—smaller than the smallest of the time constants of the system.

The ratio of the greatest and of the smallest time constants can be of the order of 10^3—10^4 in control loops, then the sampling interval of this kind results in multistep computation, increasing the running time, especially in smaller computers or in calculators. Since the transmission frequency range is determined by the cut-off frequency rather than by the smallest time constant, it is sufficient to choose a ratio $1:5-1:10$ between cut-off frequency and "corner frequency of sampling" $1/T$. In this case, however, some of the time constants might be smaller than the sampling interval T, causing poor convergency or instability of numerical algorithms as seen from the tables. This can be avoided by replacing the members with a time constant smaller than T causing phase shift of a few degrees, in the small frequency range either by pure proportional members or by proportional members with dead time T. Thus the poles causing numerical instability are avoided. The procedure can be used, when the pulse transfer functions of neglected and the remained elements can be realized separately. (The order of their numerator in z is not higher than the order of their denominator.)

Example

Let the transfer function of the open loop be (Fig. 3):

$$w_x(s) = \frac{0.1}{s(1+0.25s)} = 0.1\left[\frac{1}{s} - \frac{1}{(s+4)}\right] \tag{20}$$

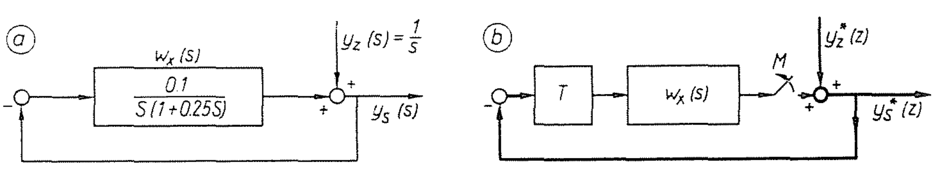

Fig. 3

Let the input signal be the step function $y_z(s) = 1/s$. The accurate value of the output signal in the frequency domain is:

$$y_s(s) = \frac{y_z(s)}{1+w_x(s)} = \frac{s+4}{(s+0.1)(s+3.9)} \tag{21}$$

and in the time domain:

$$y_s(t) = 1.027e^{-0.1t} - 0.027e^{-3.9t} \tag{22}$$

Let us examine $y_s(t)$ resulting from the digital simulation of the open loop.

The cut-off frequency of the open loop is $\omega_c = 0.1$. Let $1/T$ be one order greater, then $T = 1$.

The z-transform of the unit step function is:

$$y_z^*(z) = \frac{z}{z-1} \tag{23}$$

a) The pulse transfer function of the open loop is:

$$w_{x\infty}^*(z) = 0.1\left[\frac{1}{z-1} - \frac{1}{4}\frac{1-e^{-4}}{z-e^{-4}}\right] = 0.07546\frac{z+0.3010}{(z-1)(z-0.0183)} \tag{24}$$

The z-transform of the output signal is:

$$y_{s\infty}^*(z) = \frac{y_z^*(z)}{1+w_x^*(z)} = z\frac{z-0.0183}{(z-0.8971)(z-0.0457)} = \\ = \frac{1.032z}{z-0.8971} - \frac{0.032z}{z-0.0457} \tag{25}$$

This means two pulse sequences decreasing exponentially in the time domain, which are sampled values of the time function.

$$y_{s\infty}(t) = 1.032e^{-0.109t} - 0.032e^{-3.085t} \tag{26}$$

The difference from the true value (Eq. (22)) is due to the replacement of the continuous model of the open loop by a discrete model.

b) The effect of the use of the Runge-Kutta algorithm of the 4th order is the same as replacing the factor e^{-4} by its Taylor series of the 4th order in the pulse transfer function.

$$e^{-4} \sim 1-4+16/2-64/6+256/24=5\,. \tag{27}$$

Thus

$$w_{x4}^*(z)=0.1\left[\frac{1}{z-1}-\frac{1-5}{4(z-5)}\right]=0.2\,\frac{z-3}{(z-1)(z-5)} \tag{28}$$

$$y_s^*(z)=\frac{z(z-5)}{(z-0.8975)(z-4.9)}=\frac{1.024z}{z-0.8975}-\frac{0.0243z}{z-4.90} \tag{29}$$

The first term of the result is a pulse series decreasing exponentially with time constant $T_{14}=9.2$, almost equal to the first term of (22), but the second term is a pulse series increasing exponentially having the time constant $T_{24}=0.63$, making the result useless. Thus the integration using the Runge-Kutta algorithm of the 4th order isn't convergent. The reason is, that one of the time constants of the modelled system is significantly smaller (to a ratio 1:4) than the sampling interval.

c)The second term in the transfer function (20) causes a slight phase shift in the frequency band $\omega<0.1$. Approximately it can be replaced by a pure proportional term, thus:

$$w_x(s) \sim \frac{0.1}{s} \tag{30}$$

It can be modelled by a realizable pulse transfer function:

$$w_x^*(z)=\frac{0.1}{z-1} \tag{31}$$

$$y_{s4}^*(z)=\frac{z}{z-0.9} \tag{32}$$

This is the sampled form of the time function

$$y_{s4}(t)=e^{-0.105t} \tag{33}$$

It differs from Eq. (22) in the absence of the quickly decreasing second term causing only an insignificant error, regarding its small amplitude.

d) Replacing in Eq. (20) the small time constant by dead time T:

$$w_x(s) = \frac{0.1e^{-s}}{s} \tag{34}$$

$$w^*(z) = \frac{0.1z^{-1}}{z-1} = \frac{0.1}{z(z-1)} \tag{35}$$

$$y_{s4}^*(z) = \frac{1.145z}{z-0.8872} - \frac{0.145}{z-0.1127} \tag{36}$$

$$y_{s4}(t) = 1.145e^{-0.12t} - 0.145e^{-2.183t} \tag{37}$$

In this case the approximation c) is the more advantageous. d) would suit better for a neglected time constant nearer to T.

Summary

The problem of the convergence of digital modelling of control systems is dealt with. The cases where the one-step algorithms of constant interval lead to satisfactory solution are determined in a simple and clear way, based on the linear control circuits of constant parameters.

Reference

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