CALCULATION OF QUASI-TEM WAVES BY SERIES EXPANSION IN POWERS OF FREQUENCY

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Introduction

Quasi-TEM (qTEM) waves in closed waveguides with inhomogeneous dielectric are treated in this paper. qTEM waves are known to be encountered in shielded microstrips, coaxial cables, multicore lines, i.e. in waveguides which contain further conductors in addition to that bounding the wave field. This is why the cross section of the dielectric is a multiply connected region. Both the dielectric and the conductors are supposed to be lossless. The investigation is restricted to the case where neither the geometry nor the properties of the media (permittivity and permeability) vary in the direction of the wave propagation, and the latters are constant but different in each region of the cross section.

The functions describing the propagation coefficient and the field strengths will be obtained by series expansion in powers of frequency. A similar method has been presented in [1] for calculating the quasi-TM and quasi-TE modes, where the series expansion was performed about the cut-off frequency. The cut-off frequency of qTEM waves is zero. This explains to a certain extent why the method used for calculating the qTM and qTE waves cannot be applied to qTEM modes, and so another procedure had to be developed.

If the frequency dependence of the propagation coefficient (the dispersion characteristic) is determined by a usual method, then the complicated waveguide eigenvalue problem has to be solved for many distinct values of frequency. The method presented in this paper has the advantage that only a much simpler eigenvalue problem has to be solved, namely the eigenvalues of a quadratic matrix of as many rows and columns as there are conductors inside the wave field have to be determined. The coefficients of the power series in frequency can be calculated by solving simple boundary value problems.

Differential equations and boundary conditions of the problem

Let the z-axis of the co-ordinate system be parallel to the direction of wave propagation, and let the unit vector in this direction be denoted by $\mathbf{k}$. The complex value of the electric and magnetic field strength in the m-th region of the dielectric cross section A_m (see Fig. 1) will be written as:

$$\mathbf{E}_m = (\mathbf{e}_{Tm} + \mathbf{e}_{zm}) \exp(-pz) \tag{1}$$

$$\mathbf{H}_m = (\mathbf{h}_{Tm} + \mathbf{h}_{zm}) \exp(-pz), \tag{2}$$

where $p = \mathrm{j}\beta$ denotes the propagation coefficient. Vectors $\mathbf{e}_{Tm}$ and $\mathbf{h}_{Tm}$ are perpendicular to the z-axis, vectors $\mathbf{e}_{zm}$ and $\mathbf{h}_{zm}$ are parallel to it, and all the four vectors depend only on the two cross sectional co-ordinates.

Publications on waveguides (e.g. [2]) usually express the other components of the field strengths by means of functions e_{zm} and h_{zm}, writing the

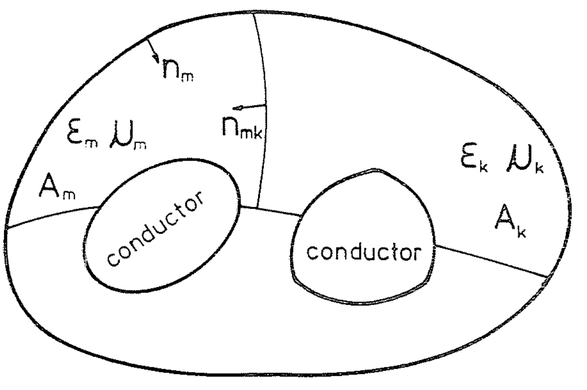

Fig. 1

differential equations to be solved in terms of these functions. Instead of this, now vector $\mathbf{e}_{Tm}$ is used as basis of calculation. This choice is advantageous because as against the boundary conditions for functions e_{zm} and h_{zm} those for vectors $\mathbf{e}_{Tm}$ do not contain the frequency and the unknown propagation coefficient. It is to be noted that vectors $\mathbf{h}_{Tm}$ may replace vectors $\mathbf{e}_{Tm}$ as basis of the calculation. The other components of the field strengths can be expressed in terms of vector $\mathbf{e}_{Tm}$ as:

$$\mathbf{e}_{zm} = \frac{\mathbf{k}}{p} \operatorname{div} \mathbf{e}_{Tm} \tag{3}$$

$$\mathbf{h}_{zm} = -\frac{1}{j\omega\mu_m} \operatorname{curl} \mathbf{e}_{Tm} \tag{4}$$

$$\begin{aligned} \mathbf{h}_{Tm} &= \frac{\mathbf{k}}{j\omega\mu_m} \times \left(p\mathbf{e}_{Tm} + \frac{1}{p} \operatorname{grad} \operatorname{div} \mathbf{e}_{Tm}\right) = \\ &= \frac{\mathbf{k}}{p} \times \left(j\omega\varepsilon_m \mathbf{e}_{Tm} + \frac{1}{j\omega\mu_m} \operatorname{curl} \operatorname{curl} \mathbf{e}_{Tm}\right). \end{aligned} \tag{5}$$

For convenience's sake, instead of frequency ω, the dimensionless quantity $w = \omega L/c$ is used for the direct parameter of expansion where c denotes the light velocity in vacuum, and L is an arbitrary constant of length dimension, which is suitably chosen so as to equal some characteristic geometry of the waveguide. So the power series of the square of the propagation coefficient and of vectors $\mathbf{e}_{Tm}$ are sought in the following form:

$$p^2 = L^{-2} \sum_{i=1}^{\infty} a_i w^{2i} \tag{6}$$

$$\mathbf{e}_{Tm} = \sum_{i=0}^{\infty} \mathbf{e}_{mi} w^{2i}. \tag{7}$$

Obviously [1], functions $\mathbf{e}_{mi}$ satisfy equation

$$\Delta \mathbf{e}_{m0} = 0 \tag{8}$$

$$\Delta \mathbf{e}_{mi} = -L^{-2}\left(\varepsilon_{rm}\mu_{rm}\mathbf{e}_{m,i-1} + \sum_{j=1}^{i} a_j \mathbf{e}_{m,i-j}\right) \tag{9}$$

$$i = 1, 2, \ldots,$$

where ε_{rm} and μ_{rm} denote the relative permittivity and permeability, resp., in the region A_m. In the calculation functions $\mathbf{e}_{m0}$ are first determined. Knowing these, functions $\mathbf{e}_{mi}$ will be determined consecutively by a recursion procedure based upon Eq. (9). In the recursion procedure first the divergence and curl of vectors $\mathbf{e}_{mi}$ are possibly determined, yielding, in turn, the vectors $\mathbf{e}_{mi}$ themselves. To this aim, let us introduce the following notations:

$$u_{mi} = \operatorname{div} \mathbf{e}_{mi} \tag{10}$$

$$v_{mi}\mathbf{k} = \operatorname{curl} \mathbf{e}_{mi}. \tag{11}$$

For qTEM modes $u_{m0}=0$ and $v_{m0}=0$ [1]. The equation for determining the other functions u_{mi} results by forming divergence of both sides of Eq. (9) and taking $u_{m0}=0$ into consideration:

$$\Delta u_{mi} = -L^{-2}\left(\varepsilon_{rm}\mu_{rm}u_{m,i-1} + \sum_{j=1}^{i-1} a_j u_{m,i-j}\right). \tag{12}$$

In knowledge of functions u_{mi} functions v_{mi} will be obtained from:

$$\operatorname{grad} v_{mi} = \mathbf{k} \times \left[\operatorname{grad} u_{mi} + L^{-2}\left(\varepsilon_{rm}\mu_{rm}\mathbf{e}_{m,i-1} + \sum_{j=1}^{i} a_j \mathbf{e}_{m,i-j}\right)\right], \tag{13}$$

also derived from (9).

In addition to the foregoing equations also the knowledge of the boundary conditions derived in [1] is needed for determining functions u_{mi}, v_{mi} and $\mathbf{e}_{mi}$. The following boundary conditions must be satisfied along the conductor outlines:

$$u_{mi} = 0 \tag{14}$$

$$\mathbf{n}_m \times \mathbf{e}_{mi} = 0 \tag{15}$$

where $\mathbf{n}_m$ denotes the unit vector normal to the outline and pointing inside the region A_m (see Fig. 1). Along the outline separating regions A_m and A_k of the dielectric the following boundary conditions have to be satisfied:

$$u_{mi}=u_{ki} \tag{16}$$

$$\frac{1}{\mu_m}\left(\frac{\partial u_{mi}}{\partial n_{mk}}+L^{-2}\sum_{j=1}^{i} a_j \mathbf{n}_{mk}\mathbf{e}_{m,i-j}\right)=\frac{1}{\mu_k}\left(\frac{\partial u_{ki}}{\partial n_{mk}}+L^{-2}\sum_{j=1}^{i} a_j \mathbf{n}_{mk}\mathbf{e}_{k,i-j}\right) \tag{17}$$

$$\frac{1}{\mu_m}v_{mi}=\frac{1}{\mu_k}v_{ki} \tag{18}$$

$$\mathbf{n}_{mk}\times\mathbf{e}_{mi}=\mathbf{n}_{mk}\times\mathbf{e}_{ki} \tag{19}$$

$$\varepsilon_m\mathbf{n}_{mk}\mathbf{e}_{mi}=\varepsilon_k\mathbf{n}_{mk}\mathbf{e}_{ki}, \tag{20}$$

where $\mathbf{n}_{mk}$ denotes the unit vector normal to the outline and pointing inside the region A_m.

Solution valid at low frequencies

At low frequencies it is sufficient to consider the first terms of series (6) and (7), i.e. the electric field can be approximated by vectors $\mathbf{e}_{m0}$, and the propagation coefficient can be approximated by the form $p\approx \mathrm{j}\sqrt{-\mathrm{a}_1}\,\omega/c$. Vectors $\mathbf{e}_{m0}$ determine an electrostatic field, what follovs also from equations div $\mathbf{e}_{m0}=0$ and curl $\mathbf{e}_{m0}=0$. Let us introduce N potential funtions φ_{mp} in order to describe the vector $\mathbf{e}_{m0}$, N being the number of the conductors inside the wave field. (In Fig. 1 $N=2$.) Let functions φ_{mp} satisfy Laplace's equation

$$\Delta\varphi_{mp}=0, \qquad p=1, 2, \dots N \tag{21}$$

and let along the outline of the p-th conductor

$$\varphi_{mp}=1 \tag{22}$$

and along the outlines of the other conductors (including the one bounding the wave field)

$$\varphi_{mp}=0, \tag{23}$$

but of course only the outlines bounding in fact the region A_m have to be considered. Along the outline separating regions A_m and A_k the following boundary conditions have to be satisfied:

$$\varphi_{mp}=\varphi_{kp} \tag{24}$$

$$\varepsilon_m \frac{\partial \varphi_{mp}}{\partial n_{mk}} = \varepsilon_k \frac{\partial \varphi_{kp}}{\partial n_{mk}} \tag{25}$$

The vector $\mathbf{e}_{m0}$ can be written in terms of functions φ_{mp} as:

$$\mathbf{e}_{m0} = -\sum_{p=1}^{N} \Phi_{0p} \text{ grad } \varphi_{mp} \tag{26}$$

where constants Φ_{0p} are undefined for the moment.

Now let us determine the function u_{mi} satisfying Laplace's equation according to (12), as well as boundary conditions (14), (16) and (17). Let us introduce N functions ψ_{mp} so that the solution satisfying the boundary conditions might be written in a perspicuous form. Let them satisfy Laplace's equation

$$\Delta\psi_{mp} = 0, \qquad p = 1, 2, \dots N \tag{27}$$

and let along the outline of the p-th conductor

$$\psi_{mp} = 1 \tag{28}$$

and along the outlines of the other conductors

$$\psi_{mp} = 0. \tag{29}$$

Along the outline separating regions A_m and A_k of the dielectric the following boundary conditions must be satisfied:

$$\psi_{mp} = \psi_{kp} \tag{30}$$

$$\frac{1}{\mu_m} \frac{\partial \psi_{mp}}{\partial n_{mk}} = \frac{1}{\mu_k} \frac{\partial \psi_{kp}}{\partial n_{mk}}. \tag{31}$$

The function u_{m1} can be given in terms of functions φ_{mp} and ψ_{mp} as:

$$u_{m1} = a_1 L^{-2} \sum_{p=1}^{N} \Phi_{0p}(\varphi_{mp} - \psi_{mp}), \tag{32}$$

where beside coefficients Φ_{0p} also the coefficient a_1 awaits to be determined.

If the first term of the series in ω^2 of function $\mathbf{h}_{Tm}$ is determined by using relationship (5) and the foregoing expression of u_{m1}, the following approximation valid at low frequencies is got for the transversal magnetic field:

$$\mathbf{h}_{m0}=\frac{\sqrt{-a_1 c}}{\mu_m L^2}\sum_{p=1}^{N}\Phi_{0p}\text{curl }\mathbf{k}\psi_{mp}, \tag{33}$$

pointing to the fact that functions ψ_{mp} can be regarded as functions that give magnetic vector potentials parallel to the z-axis and belonging to the magnetic fields of several excitations.

Substituting the expression (32) of u_{m1} into Eq. (13) yields the relationship

$$\text{grad } v_{m1}=-L^{-2}\mathbf{k}\times\sum_{p=1}^{N}\Phi_{0p}(a_1 \text{ grad } \psi_{mp}+\varepsilon_{rm}\mu_{rm} \text{ grad } \varphi_{mp}). \tag{34}$$

Existence of functions v_{m1} satisfying (34) and boundary conditions (18) requires the sum of $1/\mu_{rm}$ times the integrals of the vector in the right-hand side of (34) with respect to sections in regions A_m of an arbitrary closed curve defined in the dielectric cross section to be zero. If the closed curve encloses no conductor, then this condition is satisfied by definitions of functions ψ_{mp} and φ_{mp}. So it suffices to prescribe the fulfilment of the condition along the closed-curve outlines of N conductors:

$$\sum_m \int_{l_{qm}}\left[\mathbf{k}\times\sum_{p=1}^{N}\Phi_{0p}\left(\frac{a_1}{\mu_{rm}}\text{grad } \psi_{mp}+\varepsilon_{rm} \text{ grad } \varphi_{mp}\right)\right]\mathrm{d}\mathbf{l}=0 \tag{35}$$

$$q=1, 2, \ldots N.$$

Here l_{qm} denotes the common part of the outlines of the q-th conductor and region A_m, if it exists. Let the direction of the curve l_{qm} around the conductor co-ordinated to the z-axis according to the right-hand rule.

The N equations (35) will be assembled into one matrix equation. To this aim the vector $\bar{\Phi}_0=[\Phi_{0p}]$ with N components is introduced and matrices $\bar{\bar{C}}=[c_{qp}]$ *and* $\bar{\bar{F}}=[f_{qp}]$ are defined with elements:

$$c_{qp}=\sum_m\int_{l_{qm}}(\varepsilon_{rm} \text{ grad } \varphi_{mp}\times\mathbf{k})\mathrm{d}\mathbf{l} \tag{36}$$

$$f_{qp}=\sum_m\int_{l_{qm}}\left(\frac{1}{\mu_{rm}}\text{grad } \psi_{mp}\times\mathbf{k}\right)\mathrm{d}\mathbf{l}. \tag{37}$$

$\varepsilon_0\bar{\bar{C}}$ and $\mu_0\bar{\bar{F}}^{-1}$ are seen to be the so-called capacitance and inductance matrices, resp., hence $\bar{\bar{C}}$ and $\bar{\bar{F}}$ are symmetric.

With the notations above Eqs. (35) are rewrited into the matrix equation:

$$-\bar{\bar{F}}^{-1}\bar{\bar{C}}\bar{\Phi}_0=a_1\bar{\Phi}_0. \tag{38}$$

Accordingly coefficient a_1 and vector $\bar{\Phi}_0$ can be determined as eigenvalue and right-hand eigenvector of matrix $-\bar{\bar{F}}^{-1}\bar{\bar{C}}$. If the dielectric is homogeneous, $\psi_p=\varphi_p$ and so $\bar{\bar{C}}=\varepsilon_r\mu_r\bar{\bar{F}}$. Then the value $a_1=-\varepsilon_r\mu_r$ is an eigenvalue of multiplicity N, and so $\bar{\Phi}_0$ may be an arbitrary vector. Obviously in such a case all the other coefficients a_i are zero and the propagation coefficient is $p=\mathrm{j}\omega\sqrt{\varepsilon\mu}$. Apart from this case, no multiple eigenvalue can occur but for special combinations of parameter values. This is why only modes belonging to simple eigenvalues will be treated, although the presented method can be generalized without any theoretical difficulty for modes belonging to multiple eigenvalues. If the eigenvalue is simple, vector $\bar{\Phi}_0$ is defined uniquely by Eq. (38) up to a constant multiplier. This multiplier can be chosen arbitrarily unless the power transferred by the waveguide is given.

By determining coefficient a_1 and vector $\bar{\Phi}_0$, function $\mathbf{e}_{m0}$, u_{m1} and grad v_{m1} can be regarded as known. Functions v_{m1} up to a constant can be determined from the latter by integration and considering boundary condition (18). This constant can be computed taking the imperative to satisfy the equation

$$\sum_m \int_{A_m} v_{m1}\,\mathrm{d}A=0 \tag{39}$$

in consequence of boundary conditions (15) and (19) and Stokes' theorem into consideration.

The approximation $p\approx \mathrm{j}\omega\sqrt{-a_1/c}$ valid at low frequencies may result from the theory of transmission lines if the waveguide is regarded to be a system of transmission lines with N conductors, where the conductor bounding the wave field acts as the common return conductor. In this system the impedance matrix per unit length is $\bar{\bar{Z}}=\mathrm{j}\omega\mu_0\bar{\bar{F}}^{-1}$ and the admittance matrix per unit length is $\bar{\bar{Y}}=\mathrm{j}\omega\varepsilon_0\bar{\bar{C}}$. Just the foregoing approximation arises from this according to the theory of transmission lines [3]. If the dielectric is homogeneous, this result is exact, but if it is inhomogeneous, the approximation needs to be corrected for increasing frequencies. Further terms of the series of p^2 give this correction. The result by the theory of transmission lines is not exact because this theory neglects the component of the displacement current density in the direction of propagation. This neglect is no more acceptable at high frequencies except the ideal lines with a homogeneous dielectric, with no component of the electric field strength and so of the displacement current density in the direction of propagation. A similar problem, the comparison of the approximation by the theory of lines with the exact result by the theory of fields has already been treated by VÁGÓ in the case of systems of lossy transmission lines with conductors of circular cross section and earth [3].

Calculation of the coefficients of the power series

A recursion procedure will be presented in the following, suiting to determine successively the coefficients of the power series of the field and the propagation coefficient, starting from the approximation valid at low frequencies. At the beginning of the i-th step of the recursion procedure coefficients $a_1, a_2, \dots a_i$, functions $\mathbf{e}_{m0}, \mathbf{e}_{m1}, \dots \mathbf{e}_{m,i-1}$ with their divergence of course and functions u_{mi} and v_{mi} are known. It was already shown how to determine the coefficient a_1 and functions $\mathbf{e}_{m0}$, u_{m1} and v_{m1} necessary to begin the procedure.

Knowing functions $u_{mi} = \operatorname{div} \mathbf{e}_{mi}$ and $v_{mi} = \mathbf{k} \operatorname{curl} \mathbf{e}_{mi}$ the vectors $\mathbf{e}_{mi}$ can be determined, the expressions of which contain N unknown constants. Thereafter functions $u_{m,i+1}$ and grad $v_{m,i+1}$ can be calculated, in the expressions of which, besides the N unknown constants, also the coefficient a_{i+1} is missing. These N constants and coefficient a_{i+1} can be determined by means of consideration analogous to those made in connection with functions grad v_{m1} in order to determine coefficient a_1 and vector $\bar{\Phi}_0$. The course of this calculation will be detailed in the following.

Vectors $\mathbf{e}_{mi}$ will result from functions u_{mi} and v_{mi} by means of boundary conditions (15), (19) and (20). If $\mathbf{e}^*_{mi}$ denotes a particular solution of this problem, the general solution can be given as:

$$\mathbf{e}_{mi} = \mathbf{e}^*_{mi} - \sum_{p=1}^{N} \Phi_{ip} \operatorname{grad} \varphi_{mp}. \tag{40}$$

Coefficients Φ_{ip} have to be determined later. Particular solution $\mathbf{e}^*_{mi}$ can be calculated suitably from two potential functions:

$$\mathbf{e}^*_{mi} = \operatorname{grad} \lambda_{mi} + \mathbf{k} \times \operatorname{grad} \nu_{mi}. \tag{41}$$

Functions λ_{mi} and ν_{mi} have to satisfy Poisson's equations

$$\Delta \lambda_{mi} = u_{mi} \tag{42}$$

$$\Delta \nu_{mi} = v_{mi} \tag{43}$$

and the following boundary conditions. Along the outlines of the conductors

$$\frac{\partial v_{mi}}{\partial n_m}=0, \tag{44}$$

along the conductor bounding the wave field

$$\lambda_{mi}=0, \tag{45}$$

along the outline of the p-th conductor

$$\lambda_{mi}=\Lambda_p \qquad p=1,2,\ldots N, \tag{46}$$

where Λ_p denotes an arbitrary constant. Along the outlines separating regions A_m and A_k of the dielectric:

$$\lambda_{mi}=\lambda_{ki} \tag{47}$$

$$\varepsilon_m\frac{\partial\lambda_{mi}}{\partial n_{mk}}=\varepsilon_k\frac{\partial\lambda_{ki}}{\partial n_{mk}} \tag{48}$$

$$\varepsilon_m v_{mi}=\varepsilon_k v_{ki} \tag{49}$$

$$\frac{\partial v_{mi}}{\partial n_{mk}}=\frac{\partial v_{ki}}{\partial n_{mk}}. \tag{50}$$

Therefrom functions v_{mi} can be determined up to an additive constant, but this constant does not influence functions e^*_{mi}. Functions λ_{mi} depend upon the N indeterminate constants Λ_p. Any fixation of these means the choice of the particular solution e^*_{mi}.

After the particular solution e^*_{mi}, functions $u_{m,i+1}$ have to be determined. They are written in the form

$$u_{m,i+1}=s_{mi}+L^{-2}\sum_{p=1}^{N}(a_1\Phi_{ip}+a_{i+1}\Phi_{0p})(\varphi_{mp}-\psi_{mp}), \tag{51}$$

where the meaning of coefficients Φ_{ip} is given by relationship (40). In consequence of relationship (12), functions s_{mi} have to satisfy the equation:

$$\Delta s_{mi}=-L^{-2}(\varepsilon_{rm}\mu_{rm}u_{mi}+\sum_{j=1}^{i}a_j u_{m,i-j+1}). \tag{52}$$

Functions s_{mi} have to satisfy the boundary condition

$$s_{mi}=0 \tag{53}$$

along the outline of the conductor and the boundary conditions

$$s_{mi} = s_{ki} \tag{54}$$

$$\frac{1}{\mu_m}\left[\frac{\partial s_{mi}}{\partial n_{mk}} + \frac{\mathbf{n}_{mk}}{L^2}\left(a_1 \mathbf{e}^*_{mi} + \sum_{j=2}^{i} a_j \mathbf{e}_{m,i-j+1}\right)\right] = \\ = \frac{1}{\mu_k}\left[\frac{\partial s_{ki}}{\partial n_{mk}} + \frac{\mathbf{n}_{mk}}{L^2}\left(a_1 \mathbf{e}^*_{ki} + \sum_{i=2}^{i} a_j \mathbf{e}_{k,i-j+1}\right)\right] \tag{55}$$

along the outline separating regions A_m and A_k of the dielectric so that function $u_{m,i+1}$ fulfils boundary conditions (14), (16) and (17). By means of Eq. (52) and boundary conditions (53) to (55), functions s_{mi} can be determined uniquely.

Using the form of function $u_{m,i+1}$ given in (51) the following expression arises from (13) for the gradient of function $v_{m,i+1}$:

$$\operatorname{grad} v_{m,i+1} = \mathbf{k} \times \left\{\operatorname{grad} s_{mi} + L^{-2}\left[(\varepsilon_{rm}\mu_{rm} + a_1)\mathbf{e}^*_{mi} + \sum_{j=2}^{i} a_j \mathbf{e}_{m,i-j+1} - \right.\right. \\ \left.\left. - \sum_{p=1}^{N}\left(\Phi_{ip}(\varepsilon_{rm}\mu_{rm} \operatorname{grad} \varphi_{mp} + a_1 \operatorname{grad} \psi_{mp}) + a_{i+1}\Phi_{0p} \operatorname{grad} \psi_{mp}\right)\right]\right\}. \tag{56}$$

Multiplying the vector in the right-hand side of Eq. (56) by $1/\mu_{rm}$ and integrating along the parts in several regions A_m of an arbitrary closed curve given in the cross section of the dielectric, the sum of these integrals has to equal zero so that functions $v_{m,i+1}$ satisfy the foregoing equation and boundary condition (18). If the closed curve encloses no conductor, then from Eq. (52), the definition of functions φ_{mp} and ψ_{mp} and the Gauss' theorem, this condition appears to be satisfied. So it suffices to prescribe the condition to be met along the outlines of N conductors as closed curves. The resulting N equations can be assembled into one matrix equation. Therefore quantities Φ_{ip} are assembled into a vector $\bar{\Phi}_i$ and vectors $\bar{E}_i$, $\bar{D}_i$, $\bar{M}_i$ and $\bar{S}_i$ are introduced, the components of which are defined by the following relationships:

$$E_{ip} = \sum_m \int_{l_{pm}} \frac{1}{\mu_{rm}} (\mathbf{k} \times \mathbf{e}_{mi})\, d\mathbf{l} \tag{57}$$

$$D_{ip} = \sum_m \int_{l_{pm}} \varepsilon_{rm}(\mathbf{k} \times \mathbf{e}^*_{mi})\, d\mathbf{l} \tag{58}$$

$$M_{ip}=\sum_m \int_{l_{rm}} \frac{1}{\mu_{rm}}(\mathbf{k}\times\mathbf{e}^*_{mi})\,\mathrm{d}\mathbf{l} \tag{59}$$

$$S_{ip}=\sum_m \int_{l_{rm}} \frac{1}{\mu_{rm}}(\mathbf{k}\times \operatorname{grad} s_{mi})\,\mathrm{d}\mathbf{l}. \tag{60}$$

Using them and the previously introduced notations, the N equations are assembled — after some transformations — into the following matrix equation:

$$(\bar{\bar{F}}^{-1}\bar{\bar{C}}+a_1\bar{\bar{1}})\bar{\Phi}_i=-\bar{\bar{F}}^{-1}\left(L^2\bar{S}_i+\bar{D}_i+a_1\bar{M}_i+\sum_{j=2}^{i}a_j\bar{E}_{i-j+1}\right)-a_{i+1}\bar{\Phi}_0. \tag{61}$$

The matrix which is the multiplier of vector $\bar{\Phi}_i$ in the left-hand side of this equation is singular. So, in order to fulfil the equation, the vector on the right-hand side must be orthogonal to the proper left-hand eigenvector of matrix $\bar{\bar{F}}^{-1}\bar{\bar{C}}$. Denoting this vector by $\bar{A}^+$ the following expression arises for the coefficient a_{i+1}

$$a_{i+1}=-\frac{\bar{A}^+\bar{\bar{F}}^{-1}\left(a_1\bar{M}_i+\sum_{j=2}^{i}a_j\bar{E}_{i-j+1}+\bar{D}_i+L^2\bar{S}_i\right)}{\bar{A}^+\bar{\Phi}_0}. \tag{62}$$

With this value of coefficient a_{i+1} the vector $\bar{\Phi}_i$ can be determined from Eq. (61). Function $\mathbf{e}_{mi}$ arises by introducing this into relationship (40). So further terms of series (6) and (7) giving the dispersion characteristic and the transversal electric field strength, resp., are got.

Functions $u_{m,i+1}$ and $v_{m,i+1}$ are needed for the determination of the following terms of the series. The former is given by relationship (51) in knowledge of coefficient a_{i+1} and vector $\bar{\Phi}_i$. Thereafter functions grad $v_{m,i+1}$ are got by using (13). Functions $v_{m,i+1}$ are determined up to a constant from these by integrating and considering boundary condition (18). The constant can be computed by taking into account that boundary conditions (15) and (19) impose to satisfy:

$$\sum_m \int_{A_m} v_{m,i+1}\,\mathrm{d}A=0. \tag{63}$$

The i-th step of the recursion procedure ends by determining this constant.

The course of the calculation is seen in the flow chart in *Fig. 2.* The quantities already known before beginning the i-th recursion step are seen in the first row and results of the i-th step in the last one. The flow chart shows the relationships and the sequence of determining the latters and the intermediate quantities of the calculation.

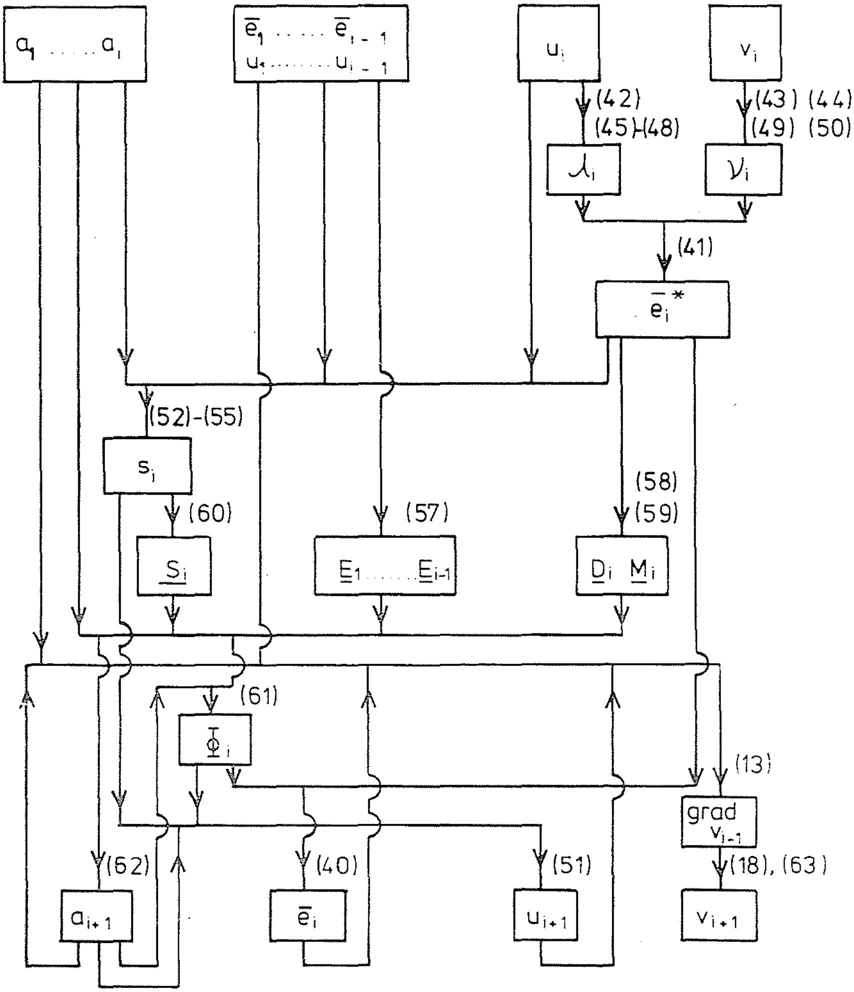

Fig. 2

Calculation of the qTEM mode in coaxial cable with an inhomogeneous dielectric

Let us examine the coaxial cable with inhomogeneous dielectric in *Fig. 3*. Because of the simplicity of the arrangement it is possible to adopt usual methods, and the procedure presented in the paper also suits analytic methods, permitting a kind of comparison.

Conventional, lengthy calculations yields the dispersion equation:

$$\begin{aligned}&\varepsilon_2 k_1[J_0(k_1R_1)N_0(k_1R)-N_0(k_1R_1)J_0(k_1R)]\times\\&\times[J_0(k_2R_2)N_1(k_2R)-N_0(k_2R_2)J_1(k_2R)]=\\&=\varepsilon_1 k_2[J_0(k_2R_2)N_0(k_2R)-N_0(k_2R_2)J_0(k_2R)]\times\\&\times[J_0(k_1R_1)N_1(k_1R)-N_0(k_1R_1)J_1(k_1R)],\end{aligned}$$

where the costumary notations of Bessel's and Neumann's functions are used introducing notations

$$k_1=\sqrt{p^2+\omega^2\varepsilon_1\mu_0}\qquad k_2=\sqrt{p^2+\omega^2\varepsilon_2\mu_0}.$$

This equation has at least one, but at most a finite number of negative real solutions in p^2. The least of these roots (the highest one in absolute value) belongs to the qTEM mode. For this mode one of k_1 and k_2 is real, the other is purely imaginary. If $\varepsilon_1>\varepsilon_2$, k_1 is real and k_2 is imaginary. In such a case

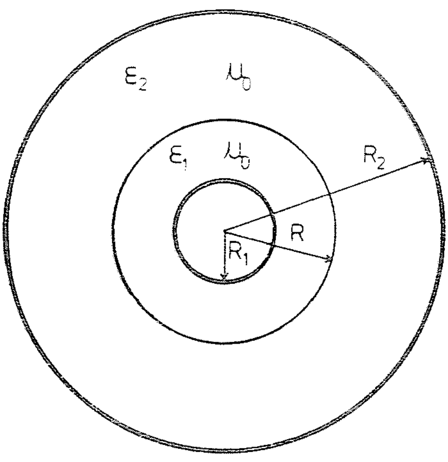

Fig. 3

Neumann's functions $N_0(k_2R)$, $N_1(k_2R)$ and $N_0(k_2R_2)$ are suitably substituted by Hankel's functions $H_0(k_2R)$, $H_1(k_2R)$ and $H_0(k_2R_2)$ in the dispersion equation. The determination of the propagation coefficient from this equation requires a very lengthy calculation. In the following the presented method is applied for calculating the propagation coefficient and the electric field strength.

The norming parameter of the series expansion is chosen to $L=R_2$. As there is only one inner conductor, the subscript refering to it is omitted. The problem is treated in cylindrical co-ordinates. Functions φ_m and ψ_m defined by Eqs (21) and (27) and the boundary conditions are got in the following form:

$$\varphi_1=\frac{\varepsilon_2 \ln\frac{r}{R}+\varepsilon_1 \ln\frac{R}{R_2}}{\varepsilon_2 \ln\frac{R_1}{R}+\varepsilon_1 \ln\frac{R}{R_2}} \qquad R_2\leqq r\leqq R$$

$$\varphi_2=\frac{\varepsilon_1 \ln\frac{r}{R_2}}{\varepsilon_2 \ln\frac{R_1}{R}+\varepsilon_1 \ln\frac{R}{R_2}} \qquad R\leqq r\leqq R_2$$

$$\psi_{1,2}=\frac{\ln\frac{r}{R_2}}{\ln\frac{R_1}{R_2}} \qquad R_1\leqq r\leqq R_2.$$

Matrices $\bar{\bar{C}}$ and $\bar{\bar{F}}$ defined by relationships (36) and (37) are scalars, their values are

$$C=\frac{2\pi\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r2} \ln\frac{R}{R_1}+\varepsilon_{r1} \ln\frac{R_2}{R}} \qquad F=\frac{2\pi}{\ln\frac{R_2}{R_1}}.$$

Vectors $\mathbf{e}_{mi}$ are all radial, this is why only their radial components are given, denoted simply by e_{mi}. By means of Eq. (26) e_{m0} is given as:

$$e_{m0}=\frac{\Phi_0}{\varepsilon_m\left(\frac{1}{\varepsilon_1}\ln\frac{R}{R_1}+\frac{1}{\varepsilon_2}\ln\frac{R_2}{R}\right)r} \qquad m=1,2,$$

where Φ_0 may be chosen arbitrarily. By means of relationship (38) the coefficient a_1 is got as:

$$a_1 = -\frac{\varepsilon_{r1}\varepsilon_{r2} \ln\frac{R_2}{R_1}}{\varepsilon_{r2} \ln\frac{R}{R_1} + \varepsilon_{r1} \ln\frac{R_2}{R}}$$

By means of (32), functions u_{m1} are written as:

$$u_{m1} = \frac{\Phi_0 \varepsilon_{r1}\varepsilon_{r2}(\varepsilon_{r1} - \varepsilon_{r2}) \ln\frac{R}{R'_m} \ln\frac{r}{R_m}}{R_2^2\left(\varepsilon_{r2} \ln\frac{R}{R_1} + \varepsilon_{r1} \ln\frac{R_2}{R}\right)^2} \qquad m = 1, 2,$$

where $R'_m = R_2$ if $m = 1$ and $R'_m = R_1$ if $m = 2$. Of course, according to (34):

$$v_{m1} = 0.$$

Now the recursion procedure can be begun, its i-th step being the following. Functions u_{mi} are written as:

$$u_{mi} = \sum_{j=0}^{i-1} \left(A_{mij} \ln\frac{r}{R_m} + B_{mij}\right)\left(\frac{r}{R_m}\right)^{2j},$$

where the coeficients A_{mij} and B_{mij} are already known, and

$$v_{mi} = 0.$$

As there is only one inner conductor, vectors $\bar{\Phi}_i$ have a single arbitrary component. Let us choose it zero and so $e_{mi} = e^*_{mi}$. Functions e_{mi} are written as:

$$e_{mi} = \sum_{j=0}^{i-1} \left(C_{mij} \ln\frac{r}{R_m} + D_{mij}\right)\left(\frac{r}{R_m}\right)^{2j} r + E_{mi}\frac{R_m^2}{r},$$

where

$$C_{mij} = \frac{A_{mij}}{2j+2} \qquad D_{mij} = \frac{1}{2j+2}\left(B_{mij} - \frac{A_{mij}}{2j+2}\right).$$

One of the coefficients E_{1i} and E_{2i} can be chosen arbitrarily, the other can be determined by means of the equation

$$\varepsilon_1 e_{1i}|_{r=R} = \varepsilon_2 e_{2i}|_{r=R},$$

which is got from boundary condition (20).

Functions s_{mi} satisfying Eq. (52) are written as:

$$s_{mi}=F_{mi}\ln\frac{r}{R_m}+B_{m,i+1,0}+\sum_{j=1}^{i}\left(A_{m,i+1,j}\ln\frac{r}{R_m}+B_{m,i+1,j}\right)\left(\frac{r}{R_m}\right)^{2j},$$

where

$$A_{m,i+1,j}=-\frac{1}{4j^2}\left(\varepsilon_{rm}A_{m,i,j-1}-\sum_{k=1}^{i-j+1}a_kB_{m,i-k+1,j-1}\right)\left(\frac{R_m}{R_2}\right)^2$$

$$B_{m,i+1,j}=-\left[\frac{A_{m,i+1,j}}{j}+\frac{1}{4j^2}\left(\varepsilon_{rm}B_{m,i,j-1}-\sum_{k=1}^{i-j+1}a_kB_{m,i-k+1,j-1}\right)\right]\left(\frac{R_m}{R_2}\right)^2$$

$$j=1,2,\dots i$$

$$B_{m,i+1,0}=-\sum_{j=1}^{i}B_{m,i+1,j}.$$

Coefficients F_{1i} and F_{2i} can be determined from the system of linear equations

$$s_{1i}|_{r=R}=s_{2i}|_{r=R}$$

$$\left[\frac{ds_{1i}}{dr}+\frac{1}{R_2^2}\sum_{j=1}^{i}a_je_{1,i-j+1}\right]_{r=R}=\left[\frac{ds_{2i}}{dr}+\frac{1}{R_2^2}\sum_{j=1}^{i}a_je_{2,i-j+1}\right]_{r=R},$$

resulting from boundary conditions (54) and (55). Now formula (62) can be evaluated, from which

$$a_{i+1}=-\frac{\ln\frac{R_2}{R_1}}{\Phi_0}\Bigg\{\varepsilon_{r1}E_{1i}R_1^2+F_{1i}R_2^2+\sum_{j=1}^{i}\Bigg[\varepsilon_{r1}D_{1,i,j-1}R_1^2+$$
$$+(A_{1,j+1,j}+2jB_{1,i+1,j})R_2^2+a_j\left(E_{1,i-j+1}+\sum_{k=0}^{i-j}D_{1,i-j+1,k}\right)R_1^2\Bigg]\Bigg\}$$

results. In knowledge of the coefficient a_{i+1}, functions $u_{m,i+1}$ can be determined, where all coefficients but $A_{m,i+1,0}$ have already been calculated. The latter ones can be calculated from the relationship

$$A_{m,i+1,0}=F_{mi}+\frac{a_{i+1}\Phi_0(\varepsilon_{r1}-\varepsilon_{r2})\ln\frac{R'_m}{R}}{R_1^2\left(\varepsilon_{r2}\ln\frac{R}{R_1}+\varepsilon_{r1}\ln\frac{R_2}{R}\right)\ln\frac{R_2}{R_1}}$$

with R'_m defined previously. After this the procedure continues with the next recursion step.

As a numerical example, the first twelve coefficients a_i are given for an arrangement where $R_2=3R_1$, $R=2R_1$ and $\varepsilon_1=10\varepsilon_2$:

$$\begin{aligned} a_1 &= -2.3139\varepsilon_{r2} & a_2 &= -0.49333\varepsilon_{r2}^2 \\ a_3 &= -0.17911\varepsilon_{r2}^3 & a_4 &= -0.052132\varepsilon_{r2}^4 \\ a_5 &= -0.0092141\varepsilon_{r2}^5 & a_6 &= 0.0013254\varepsilon_{r2}^6 \\ a_7 &= 0.0020566\varepsilon_{r2}^7 & a_8 &= 0.0010230\varepsilon_{r2}^8 \\ a_9 &= 2.8913\cdot 10^{-4}\varepsilon_{r2}^9 & a_{10} &= 4.6075\cdot 10^{-6}\varepsilon_{r2}^{10} \\ a_{11} &= -4.8687\cdot 10^{-5}\varepsilon_{r2}^{11} & a_{12} &= -3.2111\cdot 10^{-5}\varepsilon_{r2}^{12}. \end{aligned}$$

In *Fig. 4* functions $\beta c_2/\omega$ computed from several different Taylor's polynomials approximating the function $p^2(\omega^2)$ have been plotted in continuous lines. Here c_2 denotes the light velocity in a medium of permittivity ε_2 and n the number of terms in Taylor's polynomials. The exact function $\beta c_2/\omega$ computed from the

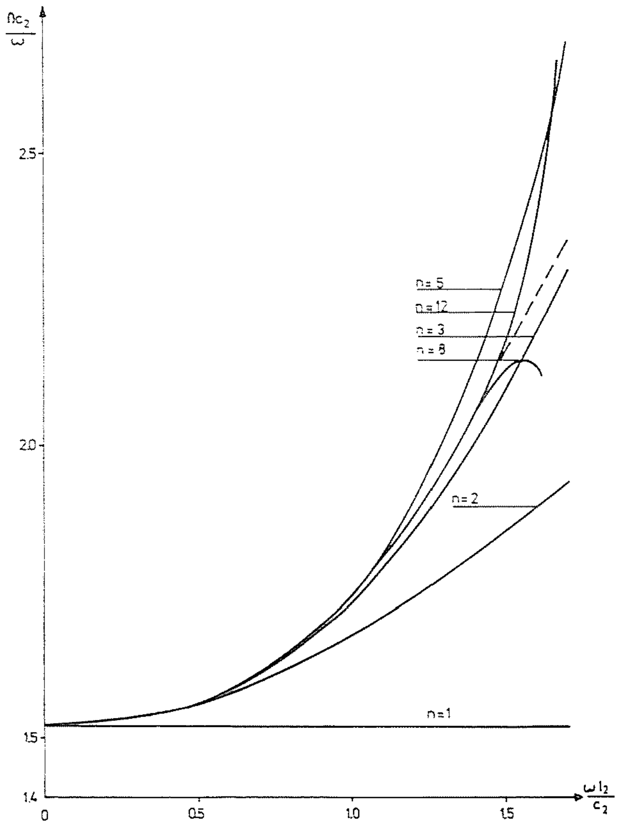

Fig. 4

dispersion equation is seen in dotted lines for values of $\omega R_2/c_2$ higher than 1.4. Below this value it practically equals the approximation with twelve terms. Taylor's series is seen to converge approximately up to the value $\omega R_2/c_2 = 1.5$. The problem of convergence has been treated in [1].

Summary

The dispersion function and the transversal electric field of quasi-TEM modes in waveguides with inhomogeneous dielectric have been determined by means of Taylor's series expansion about zero frequency. A recursion procedure has been given to determine the coefficients of these frequency power series. Against the usual method, in this procedure no complicated eigenvalues problem occurs, only the eigenvalues of a matrix of a size given by the number of the inner conductors have to be found, and then boundary value problems involving no eigenvalue problem have to be solved.

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