ON THE GENERALIZATION OF A QUASI-NEWTONIAN METHOD

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1. Introduction

In 1972 Bass has suggested [1] a variable metric quasi-Newtonian method for minimizing convex functions, which does not need any line-search optimization step. However, the convergency properties of the above-mentioned method are not yet known. A suitable variable metric quasi-Newtonian method is planned to be extended for minimizing convex functionals defined over an infinite dimensional, but separable Hilbert space. The most suitable method for this aim seems to be a new family of variable metric quasi-Newtonian methods, which we have developed from the ideas of Bass. Let us point out, that there is a — though unimportant — mistake in the paper by Bass [1] but for our version it is vital to be improved.

Let us outline here first the essence of Bass method, with some important supplements. The third chapter will give a suggestion for a new family of quasi-Newtonian methods, formulated so as to include both the finite and the infinite dimensional case. The convergency properties of the family will be presented in the last chapter. The proofs will only be sketchy.

2. Conjugate directions and inversion of linear operators

In what follows, A and B denote symmetric (Hermitian) positive definite operators, mapping some separable Hilbert space onto itself. $\{d_i\}$ and $\{v_j\}$ are two linearly independent and complete sets, assisting the Gram–Schmidt method to build the conjugate directions of A and B^{-1} respectively, $\{a_k\}$ and $\{c_l\}$. Now, if $\{a_k\}$ is a complete set of A-conjugate directions, i.e. if

$$a_i^T A a_j = \delta_{ij}\alpha_j, \qquad \alpha_j = a_j^T A a_j \tag{1}$$

$\{c_l\}$ is a complete set of B^{-1}-conjugate directions, i.e. if

$$c_i^T B^{-1} c_j = \delta_{ij}\gamma_j, \qquad \gamma_j = c_j^T B^{-1} c_j \tag{2}$$

is valid for all i and j, then also

$$A^{-1} = \Sigma \alpha_k^{-1} a_k a_k^T \tag{3}$$

and

$$B = \Sigma \gamma_l^{-1} c_l c_l^T \tag{4}$$

In fact, multiplying (3) by Aa_j, (1) leads to:

$$\Sigma \alpha_k^{-1} a_k a_k^T A a_j = \Sigma \alpha_k^{-1} a_k \delta_{kj} \alpha_j = a_j,$$

and in the same way, multiplying (4) by $B^{-1}c_j$, we get c_j for all j.

Now, relationships (3) and (4) permit to recursively get the sum (3), starting from the operator 0, provided each of the a_k lie in the subspace spanned by $d_1, d_2, \ldots, d_k$, and have the form

$$a_k = d_k + \sum_{i<k} \alpha_{ik} d. \tag{5}$$

In a more or less similar way, starting from the operator B and ending with the operator 0 yields (4), provided each of the c lie in the subspace spanned by $Bv_1, Bv_2, \ldots, Bv_l$, and have the form

$$c_l = Bv_l + \sum_{j<l} \beta_{jl} B v_j \tag{6}$$

(This is exactly the mistake by Bass, namely that c is lying in the subspace $v_1, v_2, \ldots, v_l$. In fact, however, Bass needs only (3) in his method.)

Theorem 1. The recursions

$$A_0 = 0; \quad a_1 = d_1; \quad a_{i+1} = d_{i+1} - A_i A d_{i+1};$$
$$A_{i+1} = A_i + \frac{a_{i+1} a_{i+1}^T}{a_{i+1}^T A a_{i+1}} \tag{7}$$

and

$$B_1 = B; \quad c_1 = B_1 v_1; \quad c_{i+1} = B_{i+1} v_{i+1};$$
$$B_{i+1} = B_i - \frac{c_i c_i^T}{v_i^T c_i} \tag{8}$$

satisfy the above mentioned conditions.

Sketch of the proof by induction. First, a_{i+1} is lying in the subspace spanned by $d_1, d_2, \ldots, d_i, d_{i+1}$, and has the form (5), if this fact holds for $j=1, 2, \ldots, i$, because then A_i is a projections operator into the subspace spanned by $a_1, a_2, \ldots, a_i$. Second, a_{i+1} is an A-conjugate direction with respect to $a_1, a_2, \ldots, a_i$, if this fact holds for $j=1, 2, \ldots, i$, namely from (7)

$$a_{i+1}^T A a_j = a_j^T A a_{i+1} = a_j^T A d_{i+1} - a_j^T A A_i A d_{i+1} =$$
$$= a_j^T A d_{i+1} - \sum_{k \leq i} a_j^T A \frac{a_k a_k^T}{a_k^T A a_k} A d_{i+1} =$$
$$= a_j^T A d_{i+1} - \frac{(a_j^T A a_j) a_j^T}{a_j^T A a_j} A d_{i+1} = 0.$$

Third, c_{i+1} is lying in the subspace spanned by $Bv_1, Bv_2, \ldots, Bv_{i+1}$, and has the form (6), if this fact holds for $j=1, 2, \ldots, i$, because, by the inductive definition in (8), B_{i+1} has the form

$$B_{i+1} = B - \frac{c_1 c_1^T}{v_1^T c_1} - \ldots - \frac{c_i c_i^T}{v_i^T c_i} = B - P_i \tag{9}$$

where the projections operator P_i projects onto the subspace spanned by $c_1, c_2, \ldots, c_i$, hence, according to our hypothesis, into the subspace of $Bv_1, Bv_2, \ldots, Bv_i$, and therefore

$$c_{i+1} = B_{i+1} v_{i+1} = Bv_{i+1} - P_i v_{i+1} \tag{10}$$

has the form (6). Fourth, c_{i+1} is B^{-1} — conjugate with respect to $c_1, c_2, \ldots, c_i$, if this fact holds for $j=1, 2, \ldots, i$, because from (8) and (9):

$$c_{i+1}^T B^{-1} c_j = c_j^T B^{-1} c_{i+1} = c_j^T B^{-1} B_{i+1} v_{i+1} = c_j^T B^{-1} B v_{i+1} - c_j^T B^{-1} P_i v_{i+1} =$$

$$= c_j^T v_{i+1} - c_j^T B^{-1} \sum_{k \leq i} \frac{c_k c_k^T}{c_k^T v_k} v_{i+1} =$$

$$= c_j^T v_{i+1} - \frac{c_j^T B^{-1} c_j c_j^T}{c_j^T v_j} v_{i+1} = c_j^T v_{i+1} - c_j^T v_{i+1} = 0$$

if $c_j^T B^{-1} c_j = c_j^T v_j$ holds. However, from (8):

$$c_j^T B^{-1} c_j = v_j^T B_j B^{-1} c_j = v_j^T (B - P_{j-1}) B^{-1} c_j = v_j^T c_j - v_j^T P_{j-1} B^{-1} c_j =$$

$$= v_j^T c_j - v_j^T \sum_{k<j} \frac{c_k c_k^T}{v_k^T c_k} B^{-1} c_j = v_j^T c_j - 0 = v_j^T c_j, \qquad \text{q. e. d.}$$

3. Suggestion on the new family

Calculating the change of the gradient for a given discharge is known to yield some information on the local second derivative operator, at least on its mapping with respect to the direction considered. All of the single—and double—rank variable metric quasi-Newtonian methods apply just the above information for updating the approximate local Newtonian operator. Except Bass's method, however, all the others are using only once the above mentioned information in each step via the optimizer namely that provided automatically by the applied method. According to [1], information may be collected in several directions on the second derivative before updating it, independent of whether information is collected by stopping at a step and sweeping over the space or proceeding continuously towards the optimizer. Bass suggest to use (3) and (4) for updating both the local Newtonian operator, and the local second derivative operator. Namely the projections operators, derived from (3), have to be added to the new Newtonian operator, whereas the projections operators, derived from (4), have to be substracted from the former one. In the Bass version however the second half of this procedure is unimportant, because in each cycle as many information is collected in independent directions, as the basic space has dimensions, resulting in each cycle in the full sum of (3). This is why the mistake of Bass with respect to the subspace of the c's is unimportant. This suggestion is remarkable, because in other known methods the search directions are not freely chosen and therefore passing to all possible directions at an infinite frequency on the Newtonian operator cannot be taken as granted. Therefore in general there is no guarantee that the approximation of the Newtonian operator will converge to the right value. Bass's suggestions are felt to lead to the required convergency, but this is still to be demonstrated. Also the most important in Bass's suggestion is to be free to chosen the search direction, while the other aspect—namely to collect in each cycle as many information as there are dimensions before updating the Newtonian operator—is not essential. Collecting however in each cycle less information than the number of dimensions imposes to cleverly update the approximation of the Newtonian operator aptly combining (3) and (4). Thereby it becomes unimportant whether the basic space is finite dimensional, or not.

This method can be used to minimize convex functionals with sufficiently smooth second derivative, defined over a separable Hilbert space, and, the next Chapter will show it to improve the convergency over that by any other method.

Our suggestion is the following: in each cycle information is collected on the Newtonian operator in at least two independent directions of which at least one is chosen so that in n steps at least $n+1$ independent directions are tested.

One of the at least two directions per cycle will be imposed by the method itself, so as for any other variable metric quasi-Newtonian method (namely for this step the information results without extra calculations).

The other directions will be rotated, to infinitely often sweep the full space. It means that in the case of infinite dimensions at least two directions are needed per cycle i.e. at least three directions together with that imposed to by the very method. One possible choice of this two independent rotating directions in the infinite dimensional space is by providing the space a basis —maybe an orthonormal one—. In one of the two directions the basis is swept over once systematically, and while this direction takes the basic vectors with subscripts 2^n+1 and 2^{n+1} the other takes the basic vectors with subscripts 1 to 2^n.

To update the Newtonian operator, (3) and (4) are systematically used; (3) in the chosen directions, and with the local new Newtonian operator, to be approximated, where A in (3) denotes the unknown local second derivative operator. Also (4) is used, where B means the former approximation of the Newtonian operator to be updated. The most essential idea in our suggestion is how to choose directions v_i in (4). Our proposition is the following: for N directions per cycle, let

$$v_i = Aa_i \qquad \text{for} \qquad i \leqq N \tag{11}$$

and

$$v_i = Aa_i + \sum_{k \leqq N} \vartheta_{ik} Aa_k \qquad \text{for} \qquad i > N \tag{12}$$

where the coefficients ϑ_{ik} are chosen so as to assure the orthogonality of v_i to c_j $(i>k; j=1, 2, \ldots, k)$. Remark that (12) has only a theoretical importance and therefore no coefficients ϑ_{ik} are effectively needed. To be more precise, $Ad_i=\Delta g_i$ replaces (3), where d_i is the ith direction in the cycle considered, and Δg_i is the change of first derivative upon changing the independent variable by d_i. To be sure that the computed directional difference of the first derivative is close enough to the differential of the first derivative, $\|d_i\|=\mathcal{O}(h)$ must hold, where h denotes an upper bound of the instantaneous distance of the basic point (the temporary approximation by the minimizer) from the minimizer.

Our updating formula is the following:

$$B=B_{(\text{new})}=A_N+B_N \tag{13}$$

where B_N is generated by (4) from $B=B_{(\text{old})}$.

A family results, with each member defined by N ($N\geqq 2$ in the finite, and $N\geqq 3$ in the infinite dimensional case), and by the choice of the rotating directions in the cycles.

4. Local convergency investigations

Let us assume now that the functional f to be minimized is convex, and its second derivative uniformly satisfies a Lipschitz condition. Therefore for the global convergency investigations the ideas by Broyden, Dennis and Moré (Chapter 3 in [2]) can be made use of. However, to use these general ideas the estimates of the norms of $[A^{-1}-(A_N+B_N)]$ are needed. Three types of norms will be applied. First the ordinary operator norm, denoted by $\|.\|$, second, a transformed vector norm, and the corresponding operator norm, defined by a symmetric (Hermitian) positive definite auxiliary operator H, denoted by $\|.\|_H$, and defined for vector v by $\|v\|_H=\|H^{1/2}v\|$, and for the operator V by

$$\|V\|_H=\sup_v \frac{\|Vv\|_H}{\|v\|_H},$$

and third, a Frobenius-type norm, defined by a symmetric (Hermitian) auxiliary operator S and by an even number k, denoted by $\|.\|^F_{S,k}$ and defined for the operator V by

$$||V||^F_{S,k} = (\Sigma\, |\lambda_i|^k)^{1/k}$$

where λ_i is the ith eigenvalue of SVS.

First some estimation are needed for $(A_k + B_k)Aa_j$, where A is a symmetric positive definite operator, $\{d_i\}$ is a complete basis of the separable Hilbert space considered, a_j and A_k are the sequences generated by (3), $B = (A + \varepsilon AR)^{-1}$ is an approximation of A^{-1}, good enough (i.e. ε is sufficiently small), itself symmetric and positive definite (i.e. R is symmetric too), v_l is the sequence generated by (11) and (12), finally, B_k denotes the sequence of operators, generated by (4), with the directions v_l.

Theorem 2.

$$A_N A a_i = a_i \quad \text{if} \quad N \geqq i, \qquad A_N A a_i = 0, \quad \text{if} \quad N < i; \tag{14}$$

$$B_N v_i = 0, \quad \text{if} \quad N \geqq i; \qquad B_N v_i = B v_i, \quad \text{if} \quad N < i; \tag{15}$$

$$||[A^{-1} - (A_N + B_N)]v_i||_{A^2} = ||(A^{-1} - B)v_i||_{A^2} \tag{16}$$

are valid.

Sketch of the proof: Relationships (14) are trivial consequences of (3), because the projections operators in A_k and in A^{-1} have the form

$$(a_n a_n^T)/(a_n^T A a_n);$$

and the a_n are A conjugate to a_m, for $n \neq m$. The first group in (15) is also a trivial consequence of (4), because the projections operators in B and in B_k have the form $(c_n c_n^T)/(v_n^T c_n)$, and the c_n are B^{-1} conjugate directions, hence

$$\frac{c_n c_n^T}{v_n^T c_n} v_j = \frac{c_n c_n^T}{v_n^T c_n} B^{-1} B v_j = \frac{c_n c_n^T}{v_n^T c_n} B^{-1} [c_j + \sum_{l<j} \gamma_{jl} c_l] = 0$$

$$\text{if } n > j.$$

Now, from (4), the first operator in B_k is $\dfrac{c_{k+1} c_{k+1}^T}{v_{k+1}^T c_{k+1}}$, and therefore $B_k v_j = 0$ holds for $k \geqq j$.The second group in (15) is a trivial consequence of (12), because B and B_k differ only in projections operators of the form $(c_n c_n^T)/(v_n^T c_n)$ where $n \leqq k$, and from (12) they are all orthogonal to v_j, if $j > N$. Also (16) is trivial for $i \leqq N$, because for these indices $v_i = A a_i$, and therefore $A^{-1} v_i = A_N v_i$ and $B_N v_i = 0$, i.e. $[A^{-1} - (A_N + B_N)]v_i = 0$ for these subscripts. The only crucial point is the estimation $[A^{-1}-(A_N+B_N)]v_i=[A^{-1}-(A_N+B)]v_i$ for $i>N$, because $A_N v_i \neq 0$ from (12). To this aim will be computed all the quantities of interest in the order of ε. Assume:

$$B=[A(I+\varepsilon R)]^{-1} \sim (I-\varepsilon R)A^{-1} \tag{17}$$

To assure the orthogonality of v_i to c_j, coefficients ϑ_{ij} will be chosen to hold $v_i^T B v_j = v_j^T B v_i = 0$ for $i \leqq N$ and $j>N$, because from Theorem 1, c_j is lying in the subspace spanned by $Bv_1, Bv_2, \ldots, Bv_j$. Hence

$$a_i^T A(I-\varepsilon R)A^{-1}[Aa_j+\sum_{l \leqq N} \vartheta_{jl} A a_l] \sim 0 \tag{18}$$

This equation — by inductive demonstration, of $\vartheta_{jl}=\vartheta(\varepsilon)$ — yields the estimation

$$\vartheta_{ji} \sim -\varepsilon \frac{a_i^T A R a_j}{a_i^T A a_i}. \tag{19}$$

From (19) and (3):

$$\begin{aligned} A_N v_j &\cong \sum_{l \leqq N} \frac{a_l a_l^T}{a_l^T A a_l} \left[Aa_j - \varepsilon \sum_{i \leqq N} Aa_i \frac{a_i^T A R a_j}{a_i^T A a_i} \right] = \\ &= -\varepsilon \sum_{l \leqq N} a_l \frac{a_l^T A R a_j}{a_l^T A a_l} = A^{-1} v_j - a_j, \end{aligned} \tag{20}$$

whereas from (15)

$$\begin{aligned} Bv_j = B_N v_j &\cong (I-\varepsilon R)A^{-1} v_j = A^{-1} v_j - \varepsilon R A^{-1} v_j \cong \\ &\cong a_j + A_N v_j - \varepsilon R a_j = A^{-1} v_j - \varepsilon R a_j. \end{aligned} \tag{21}$$

therefore

$$(A^{-1}-B)v_j \simeq \varepsilon R a_j \tag{22}$$

whereas

$$[A^{-1}-(A_N+B_N)]v_j \simeq A_N v_j + \varepsilon R a_j. \tag{23}$$

But according to the third relationship (20) $A_N v_j$ is the projection of $-\varepsilon R a_j$ onto the subspace spanned by vectors $a_n = A^{-1} v_n$ ($n=1, 2, \ldots, N$). (In fact, the projection of a vector w onto this subspace is given — considering the A — conjugateness of the a_n — in the form $w = \sum_{i \leqq N} w_i a_i$, with

$$\mathrm{w}_i = \frac{\mathrm{w}^T A a_i}{\mathrm{a}_i^T \mathrm{A} \mathrm{a}_i}; \quad \text{if} \quad w = -\varepsilon R a_j, \quad \text{then}$$

$$w_i = -\varepsilon \frac{a_j^T R A a_i}{a_i^T A a_i} = -\varepsilon \frac{a_i^T A R a_j^T}{a_i^T A a_i} \tag{24}$$

and this the same sequence of coefficients as that in (20), q.e.d.)

Now (23), (20) and (24) show that (16) is valid, because all the maps of v_j with $j > N$ are diminished by its projection onto a subspace, substituting the operator $[A^{-1}-(A_N+B_N)]$ for $[A^{-1}-B]$. Of course this fact does not generally assure that $\|[A^{-1}-(A_N+B_N)]v_j\| \leqq \|[A^{-1}-B]v_j\|$ is valid, because there is no information about the signs of $a_i^T a_j$ for $i \leqq N$ and $j > N$, but (17) is valid, because a_i is A orthogonal to a_j for these subscripts.

Theorem 3

$$\|A^{-1}-(A_N+B_N)\|_A \leqq \|A^{-1}-B\|_A \tag{25}$$

$$\|A^{-1}-(A_N+B_N)\|^F_{A^{1/2},k} \leqq q(\varepsilon; N; k)\|A^{-1}-B\|^F_{A^{1/2},k} \tag{26}$$

where $q(\varepsilon; N, k) < 1$ if ε is small enough, (and if the sequence of the eigenvalues of AR tend to 0 fast enough in the infinite dimensional case).

Sketch of proof: (25) is a trivial consequence of (16), because the mapping of the complete basis $\{v_l\}$ by $[A^{-1}-(A_N+B_N)]$ or by $[A^{-1}-B]$ is divided in two A conjugate subspaces, and in the second of them (spanned by a_j with $j > N$) they coincide, whereas in the first the mappings by the first operator vanish, whereas by the second operator not.

The proof of (26) needs great many calculations; here the case $k=2$, will be insisted on giving only the details of the relationship between

$$\|A^{-1}-(A_1+B_1)\|^F_{A^{1/2},2} \quad \text{and} \quad \|A^{-1}-B\|^F_{A^{1/2},2}.$$

The ideas to be applied on the relationship between

$$\|A^{-1}-(A_{i+1}+B_{i+1})\|^F_{A^{1/2},2} \qquad \text{and} \qquad \|A^{-1}-(A_i+B_i)\|^F_{A^{1/2},2}$$

are the same, and give a similar inequality, and they together confirm (26). The basic idea is that the norms to be considered are closely related to the trace of the square of the operator — this is well known in the finite dimensional case, but it is easy to extend to the infinite dimensional case, where the trace can be defined by infinite dimensional determinants, as limits of its finite slices.

To this aim some estimates in the order of ε are needed. Using short notations:

$$\alpha_1 = a_1^T A a_1; \qquad \rho_1 = a_1^T A R a_1$$

we get

$$c_1 = BAa_1 \sim (I-\varepsilon R)a_1 \tag{27}$$

and

$$v_1^T c_1 = a_1^T A a_1 - \varepsilon a_1^T A R a_1 = \alpha_1 - \varepsilon\rho_1 = \alpha_1\left(1-\varepsilon\frac{\rho_1}{\alpha_1}\right) \tag{28}$$

hence

$$\frac{1}{v_1^T c_1} \simeq \frac{1}{\alpha_1}\left(1+\varepsilon\frac{\rho_1}{\alpha_1}\right), \tag{29}$$

and

$$\begin{aligned}\frac{a_1a_1^T}{\alpha_1}-\frac{c_1c_1^T}{\rho_1} &= \frac{a_1a_1^T}{\alpha_1}-\frac{(I-\varepsilon R)a_1a_1^T(I-\varepsilon R)}{\rho_1}\sim\\ &\sim \frac{a_1a_1^T}{\alpha_1}-\frac{a_1a_1^T-\varepsilon Ra_1a_1^T-\varepsilon a_1a_1^TR}{\alpha_1}\left(1+\varepsilon\frac{\rho_1}{\alpha_1}\right)\sim\\ &\sim -\varepsilon\frac{\rho_1}{\alpha_1}\frac{a_1a_1^T}{\alpha_1}+\varepsilon\frac{Ra_1a_1^T+a_1a_1^TR}{\alpha_1}.\end{aligned} \tag{30}$$

With the above we get

$$\{\|A^{-1}-B\|^{F}_{\sqrt{A}:2}\}^2=\left\{\|A^{-1}-(A_1+B_1)-\varepsilon\frac{\rho_1}{\alpha_1^2}a_1a_1^T+\right.$$

$$\left.+\varepsilon\frac{1}{\alpha_1}(Ra_1a_1^T+a_1a_1^TR)\|^{F}_{\sqrt{A}:2}\right\}^2=\{\|A^{-1}-(A_1+B_1)\|^{F}_{\sqrt{A}:2}\}^2+$$

$$+tr\left\{\left[-\varepsilon\frac{\rho_1}{\alpha_1^2}\sqrt{A}a_1a_1^T\sqrt{A}+\varepsilon\frac{1}{\alpha_1}\sqrt{A}Ra_1a_1^T\sqrt{A}+\varepsilon\frac{1}{\alpha_1}\sqrt{A}a_1a_1^TR\sqrt{A}\right]^2\right\}+$$

$$+2tr\left\{\sqrt{A}\left\{[(A^{-1}-A_1)-B_1]\cdot\left[-\varepsilon\frac{\rho_1}{\alpha_1^2}a_1a_1^T\sqrt{A}+\right.\right.\right.$$

$$\left.\left.\left.+\varepsilon\frac{1}{\alpha_1}Ra_1a_1^T\sqrt{A}+\varepsilon\frac{1}{\alpha_1}a_1a_1^TR\sqrt{A}\right]\right\}\right\}\cong$$

$$\cong\{\|A^{-1}-(A_1+B_1)\|^{F}_{\sqrt{A}:2}\}^2+\left(\varepsilon^2\frac{\rho_1^2}{\alpha_1^4}\alpha_1^2+\varepsilon^2\frac{1}{\alpha_1^2}\rho_1^2+\right.$$

$$+\varepsilon^2\frac{1}{\alpha_1^2}\rho_1^2-2\varepsilon^2\frac{\rho_1}{\alpha_1^3}\rho_1\alpha_1-2\varepsilon^2\frac{\rho_1}{\alpha_1^3}\alpha_1\rho_1+$$

$$\left.+2\varepsilon^2\frac{1}{\alpha_1^2}\alpha_1a_1^TRARa_1\right)+2\left\{0+0+0+tr\left\{\sqrt{A}\left(\varepsilon RA^{-1}+\right.\right.\right.$$

$$\left.+\varepsilon\frac{\beta_1}{\alpha_1^2}a_1a_1^T-\frac{\varepsilon}{\alpha_1}Ra_1a_1^T-\frac{\varepsilon}{\alpha_1}a_1a_1^TR\right)\left(\frac{\varepsilon}{\alpha_1}a_1a_1^TR\sqrt{A}\right)=$$

$$=\{\|A^{-1}-(A_1+B_1)\|^{F}_{\sqrt{A}:2}\}^2-\varepsilon^2\frac{\rho_1^2}{\alpha_1^2}+2\varepsilon^2\frac{1}{\alpha_1}a_1^TRARa_1+$$

$$+2\varepsilon^2\frac{1}{\alpha_1}a_1^TRARA^{-1}a_1+2\varepsilon^2\frac{\rho_1}{\alpha_1^3}\rho_1\alpha_1-2\varepsilon^2\frac{1}{\alpha_1^2}\alpha_1a_1^TRARa_1-$$

$$\left.-2\varepsilon^2\frac{1}{\alpha_1^2}\alpha_1a_1^TRARa_1\right\}=\left\{\|A^{-1}-(A_1+B_1)\|^{F}_{\sqrt{A}:2}\right\}^2+\varepsilon^2\frac{\rho_1^2}{\alpha_1^2},$$

that is the square of the Frobenius-type norm of $(A^{-1}-B)$ exceeds that of $[A^{-1}-(A_1+B_1)]$ by a term proportional to the square of the norms of the former.

Using the above yields a similar additive term in all steps which proves our assertion.

Results from using the general ideas of Broyden, Dennis and Moré

Theorem 4

If our assertions are valid, then the sequence $(A_N + B_N)$ tends toward the right value of the local Newtonian operator, hence the sequence for approximating the minimizer is Q-superlinearly convergent.

If our space is finite dimensional, than $(A_N + B_N)$ tends in geometrical order toward the local Newtonian operator. In the Frobenius-type norm also in the infinite dimensional case a geometrical convergence rate toward the local Newton operator, results if our first approximation was in this sense good enough. In the L-dimensional case the rate of convergence is of order τ where τ is the positive root of the equation

$$\tau^{L/N} - \tau^{(L/N)-1} - 1 = 0,$$

considering stepwise instead of cycle wise convergence-rate.

Summary

Extension of one suitable variable metric quasi-Newtonian method for minimizing functionals defined over an infinite dimensional but separable Hilbert space is attempted. The most suitable to this aim seems to be a new family of quasi-Newtonian methods, developed from the ideas of Bass.

Recalling the essentials of Bass's method with some important supplements, a new family of quasi-Newtonian methods is suggested formulated so as to included both the finite and infinite case. Finally the convergency properties of the family are presented. The proofs are only sketchy.

References

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