A SIMPLE VARIATION OF KRONS'METHOD OF SOLVING NETWORKS BASED ON THE THEOREM OF LINEAR VECTOR SPACES

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In the field of electric power systems, the most extensively used method of solving networks is in which the mesh, the node and the cut-set equations are derived by a simple matrix conversion of the equations of the Ohm theorem and a necessary number of Kirchhoff's current and voltage equations written on the basis of certain considerations of the graph theorem. The matrix model is advantageous from the point of view of numerical solution and also because this linear algebraic technique is widely used in electrical engineering. On the other hand, it conceals the fundamental background of solving networks, or in other words, it directs the development of solving networks only towards improving or speeding-up numerical calculation.

Strictly speaking, the method described above is a variation of G. Kron's method of solving networks, completed with some notions of the graph theorem. After the "classicals" the most creative analysis of electrical networks have undoubtedly been mad by G. KRON [6], [7]. Like WEYL [12], he, too, interprets current as a contravariant and voltage as a covariant vector. Their dimension numbers are always equal to the number of the branches of a circuit. This is natural for a circuit torn into branches, the so called "primitive network" and it is also valid for the so-called "orthogonal network" constructed from loops and open paths because the sum of independent loops and open paths is equal to the number of branches (see later). Thus the matrices connecting such networks termed by KRON "connection matrices" are always quadratic ones, which makes it possible to recognize that some of the network quantities (voltage, current) are transformed as tensors of valence 1 while others as tensors of valence 2. The tensorial interpretation makes it clear that the network quantities are invariant; for example, the invariance of power; or the primitive branch currents and the set of loop currents and open currents are different forms of the *same* current distributions. This is how this interpretation is — deliberately — connected with the theory of relativity. The solution — the calculation of unknown currents and voltages — becomes simple in the "orthogonal network" because Kirchhoff's laws can simply be formulated there, and thus the quantities have to be transformed into the "orthogonal network". A quarter of a century later G. Kron developed a new method for the solution of the network problem through tearing [7], called by him diacoptics. However, it is quite difficult to understand diacoptics in the systems used nowadays for solving network problems.

The method of G. Kron was not sufficiently understood and could not spread because he did not sufficiently elucidate the notions used, and did not give the extact proof of his results and, in addition, his style is rather diffuse. From among his followers and from those who have tried to explain his work, we should like to mention H. H. Happ [4] who discusses Kron's method of solving network problems in a clearer, systematized way, mainly to make diacoptics perspicuous, in which he has put also new elements. J. Paul Roth [8] laid down the algebraic topological fundations of Kron's work, which was later used by Kron himself. Unfortunately, this did not make the method easier to understand for it requires that the reader should be familiar not only with linear vector spaces and tensors but also with some notions of algebraic topology. P. Slepian [11] — apparently independently of Kron and Roth — also uses algebraic topological methods and makes a construction in which the currents and the voltages are in the same vector space; he renews the so-called III. and IV. Kirchhoff laws as methods of solution, which are not very suitable for practical calculations. A. Klos [5], too, wants to put the currents and the voltages into the same vector space, using simple linear algebraic methods, but in his analysis there are steps which cannot be interpreted in physics: he takes the difference between the original and the transformed current distribution (and the voltage distribution) vectors and derives further equations from it. His aim could consistently be accomplished if he confined himself only to orthogonal transformation matrices which could be derived by the orthogonalisation of the transformation matrix which he made from the mesh matrix and the cut-set matrix. Only with this complication could the invariance of power be maintained in the case when voltage and current distributions are put into the same vector space.

Generally, it is not fortunate to put the voltage interpreted between two points, and the current interpreted in one point (in the case of concentrated elements) into a common vector space; it is more natural to separate them (except for the case of the n-pole in which current and voltage distribution appear in the same point, and it is only in this case that we can speak of the eigenvectors and eigenvalues of the impedance tensor, i.e. of the method of symmetrical components). We also have to mention Brammeller's book [1] in which the author discusses diacoptics with the application of matrices.

The aim of the present paper is to obtain a physically lucid and exact variant of diacoptics and Kron's method for solving network based exclusively on the theorem of linear vector spaces. In respect of the connections between linear vector spaces and tensors, our work is based mostly on Gelfand's work

[2] from the mathematical literature mentioned in the bibliography [2, 3, 9]. In this work, we only deal with the calculation of stationary current distributions. To make it easy to understand, we use the index calculus and matrix presentation as well.

Before coming to the point, we should like to give the linear algebraic background of solving the network problems mentioned above by a method used nowadays which works with n-dimensional vector spaces containing an ordered number of n. This method is supported by the following equations:

$$[A][I_r]=[0]; \qquad [B][U_r]=[0] \qquad \text{(Kirchhoff's equations)}$$

$$\begin{aligned}[U_r]+[U_g]&=[Z]([I_r]+[I_g])\\ [\mathrm{I}_r]+[I_g]&=[Y]([U_r]+[U_g])\end{aligned} \qquad \text{(Ohm theorem)}$$

$$[I_r]=[B]_t[I_h]; \qquad [U_r]=[A]_t[U_c] \qquad (t \text{ denotes transposing})$$

where $[U_r]$ and $[I_r]$ are the voltage and current column vectors formed from the *resultant* branch quantities, $[U_g]$ and $[I_g]$ are the voltage and current vectors, respectively, formed from the *generating* branch quantities; $[A]$ is the incidence matrix characteristic of the graph of the network, $[B]$ is the mesh-matrix, $[I_h]$ and $[U_c]$ are column vectors formed from the independent loop currents and nodal voltages respectively, $[Z]$ is the branch impedance matrix, $[Y]$ is the branch admittance matrix, while $[B]$ can be derived from $[A]$. In a network of e branches, v nodes and p subnetworks $[U_r]$, $[I_r]$, $[U_g]$ and $[I_g]$ contain e components, $[I_h]$ consists of $(e-v+p)$ components and the number of components of $[U_c]$ is $(v-p)$. $[A]$ matrix has e columns and $(v-p)$ rows, $[B]$ matrix has e columns and $(e-v+p)$ rows, while $[Z]$ and $[Y]$ have e columns and e rows.

Since the columns of $[B]_t$ represent unit intensity loop currents, and as such satisfy the nodal equations, therefore the range of $[B]_t$ falls within the zero subspace of $[A]$:

$$R([B]_t) \subseteqq N([A])$$

and

$$[A][B]_t=[0],$$

and from this follows

$$[B][A]_t=[0]$$

which shows that:

$$R([A]_t)\subseteqq N([B]),$$

i.e. the range of $[A]_t$ falls within the zero subspace of $[B]$. Since $R([B]_t)$ and $N([B])$, and similarly $R([A]_t)$ and $N([A])$ are orthogonal subspaces and the columns of $[B]_t$ and $[A]_t$ are linearly independent, therefore

$$\dim R([B]_t)=e-v+p, \qquad \dim N([B])=v-p$$

$$\dim R([A]_t)=v-p; \qquad \dim N([A])=e-v+p,$$

whereby

$$R([B]_t)=N([A]) \qquad N([B])=R([A]_t).$$

As can be seen, the Kirchhoff laws are satisfied if

$$[I_r]\subseteqq N([A]) \quad \text{and} \quad [U_r]\subseteqq N([B]),$$

and thus $[U_r]$ and $[I_r]$ are orthogonal to each other.

$$[\hat{I}_r]_t[U_r]=[0] \quad \text{(Tellegen's theorem)}$$

($\hat{}$ denotes conjugation).
Getting to $N([A])$ and $N([B])$ is ensured by formation from $[I_h]$ and $[U_c]$. Multiplying Ohm's theorem by $[B]$ and $[A]$, the following equations is obtained:

$$([B][Z][B]_t)[I_h]=[B][Z][I_r]=[B]([U_g]-[Z][I_g])$$

$$([A][Y][A]_t)[U_c]=[A][Y][U_r]=[A]([I_g]-[Y][U_g])$$

From the first equation, the unknown $[I_h]$ (mesh method) and from the second equation the unknown $[U_c]$ (node method) can be calculated.

In the following *Figure 1a*, the way of solution is shown by transformation between vector spaces, given by their dimensional numbers and denoted symbolically in accordance with the introduction.

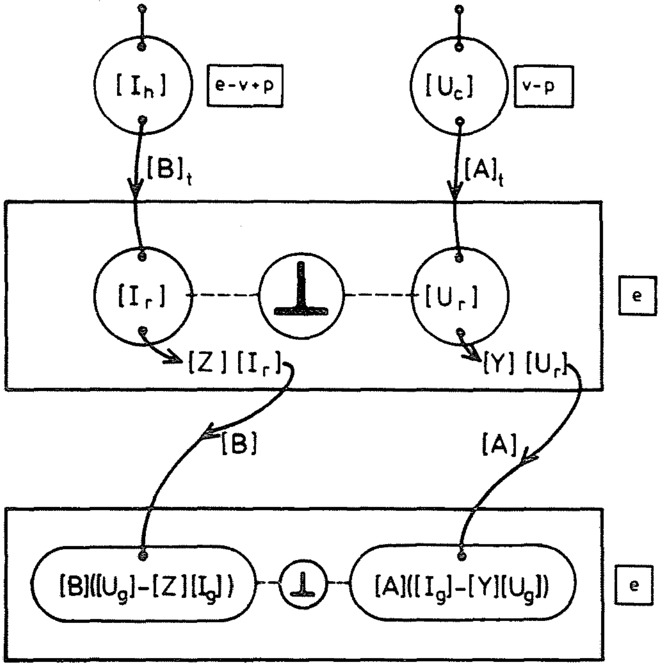

[I_h]
e-v+p
[U_c]
v-p
[B]_t
[A]_t
[I_r]
[U_r]
e
[Z] [I_r]
[Y] [U_r]
[B]
[A]
[B]([U_g]-[Z][I_g])
[A]([I_g]-[Y][U_g])
e

Fig. 1a

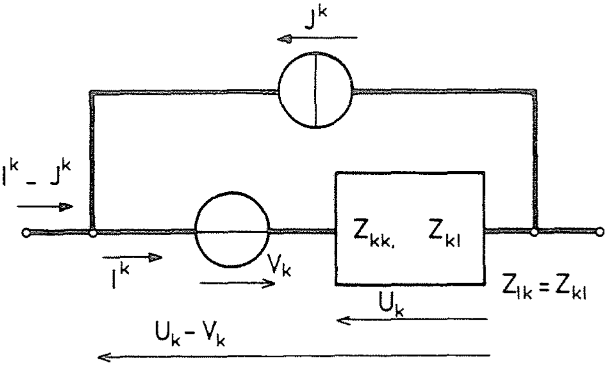

J^k
I^k - J^k
I^k
V_k
Z_kk, Z_kl
Z_lk = Z_kl
U_k
U_k - V_k

Fig. 1b

1. The vector space of current and voltage distribution; primitive bases

Consider an "*e*"-branch, stationary, invariant network energized from stationary, sinusoidal sources. The structure of the k-th branch of the network is shown in *Fig. 1b.*

The network does not contain a controlled generator. (However, controlled elements can also be considered with proper types of tensors).

(In the Figure, the reference directions of voltage rise are shown.) J^k and V_k are the active currents and voltage, respectively, of the ideal sources, I^k and U_k are the passive currents and voltage, respectively, developed on the self- and mutual impedances. A branch of a given network does not necessarily contain a current source, voltage source or mutual impedance; these are included in the general branch arrangement shown in Fig. 1b with substituting elements of zero value. The basic task of network computation is to determine the passive voltages and currents in the case of fixed impedances, when the active voltages and currents are known.

It seems that the active currents and voltages can cover the whole complex range in the case of any value of impedance, since Kirchhoff's laws are satisfied by the development of passive currents and voltages. But it is not so! If the self-impedances in all the branches running into one node are infinite (there is a disconnection), then, in accordance with the node law, one of the active currents cannot be arbitrary, since the passive currents in these branches are zero. Similarly, if all the self- and mutual impedances in a loop are equal to zero, then the passive voltage is also equal to zero, so one of the active voltages cannot be arbitrary in accordance with the loop law. The passive currents and voltages cannot either run the whole range of complex numbers in the case of any impedance system, since the current of a torn branch or the voltage of a branch having zero self- and mutual impedance can only be zero. Since there is an infinite number of impedance systems at which the active or passive voltages and currents can have any value, the largest range of variation of voltages and currents must be considered to describe connections of universal validity; exceptional cases will be derived from the extraordinary values of the parameters of the mathematical model.

Let the active and passive current distributions be characterized by column matrices $[J]$ and $[I]$, respectively, of which the k-th row is just the current of the k-th branch, i.e. J^k and I^k. Interpreting the linear combination of the current distributions as the linear combination of these column matrices, we get the *linear vector space of current distributions.* (Since the linear combination derived can also be a current distribution.) Since, in accordance with the former analysis, the elements of the "*e*" row column matrix can have any complex value, the linear vector space derived is *e-dimensional*. We would like to stress that the active and passive current distributions are in the *same* linear vector space. The current distribution vectors — the elements of the linear vector space — will be denoted by bold-faced Roman letters. So, if the column matrix of $\mathbf{I}_1$ is $[I_1]$ and the column matrix of $\mathbf{I}_2$ is $[I_2]$, then $\lambda_1[I_1]+\lambda_2[I_2]$ is the column matrix of the vector $\lambda_1\mathbf{I}_1+\lambda_2\mathbf{I}_2$.

(In this paper the expression of "column matrix" is used instead of the usual "column vector" because the word "vector" is kept for the elements of linear vector spaces which — as will be seen later — can be given with different column matrices in different bases.)

It is known that a *base* of the linear vector space can be formed by just as many linearly independent vectors as the dimensional number of the vector space, and any of the elements of the vector space can be expressed by the linear combination of these vectors, and the coefficients of the linear combination are called the components of the vector in this base. Let $\mathbf{g}_k$ denote a current distribution vector in which the current of the k-th branch is of a unit and that of the other branches is zero; so the column matrix of this is just the k-th column of the "e"-dimensional unit matrix. As is known, the current distribution vectors $\mathbf{g}_1, \mathbf{g}_2, \ldots \mathbf{g}_e$ are linearly independent and as such they form a base of an "e"-dimensional current distribution vector space. This is called "primitive base" in accordance with Kron who called the network torn into branches a primitive network.

Since

$$[I]=\begin{bmatrix} I^1 \\ I^2 \\ \cdot \\ \cdot \\ \cdot \\ I^e \end{bmatrix} = I^1\begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + I^2\begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \ldots + I^e\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

therefore,

$$\mathbf{I}=I^1\mathbf{g}_1+I^2\mathbf{g}_2+\ldots+I^e\mathbf{g}_e=\sum_{k=1}^{e} I^k\mathbf{g}_k=I^k\mathbf{g}_k. \tag{1}$$

In the last step we used the *summational convention* i.e. similar lower and upper subscripts mean summing up to the dimensional number. (This abbreviation will also be used later). According to Equation (1), the components of the current distribution vector in the primitive base are just the branch currents or, in other words, the column matrix (discussed so far) is the *matrix of the current distribution vector in the primitive base.*

The linear vector space of (passive or active) voltage distributions, which does not coincide with the current vector space, can be built up in a similar way. This is also "e"-dimensional. It also has its primitive base the k-th vector of which is $\mathbf{g}^k$, a voltage distribution in which the voltage of the k-th branch is of a unit and that of the other branches is zero.

For the column matrices composed from the branch voltages, the following equations can be written:

$$[U]=\begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_e \end{bmatrix} = U_1 \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + U_2 \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \ldots + U_e \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

So, the voltage distribution vector is

$$\mathbf{U} = U_1\mathbf{g}^1 + U_2\mathbf{g}^2 + \ldots + U_e\mathbf{g}^e = U_k\mathbf{g}^k. \tag{2}$$

This equation expresses that in the primitive base the components of the voltage distribution vector are the branch voltages, and the column matrix composed from them is the matrix of the voltage distribution vector in the primitive base.

The resultant complex power generated on the passive elements of the network is

$$S = U_1\hat{I}^1 + U_2\hat{I}^2 + \ldots + U_e\hat{I}^e \equiv U_k\hat{I}^k \tag{3}$$

(where $\hat{}$ denotes the conjugated complex number). This can be considered to be a *scalar multiplication between the elements of the two vector space*, since, taking the equation

$$\mathbf{U}\mathbf{I} = U_k\hat{I}^k = [U]_t[\hat{I}] = [\hat{I}]_t[U] \tag{4}$$

as a definition, the well-known characteristics of the scalar product of hermitic symmetry can easily be derived:

$$(\lambda_1\mathbf{U}_1+\lambda_2\mathbf{U}_2)\mathbf{I}=\lambda_1(\mathbf{U}_1\mathbf{I})+\lambda_2(\mathbf{U}_2\mathbf{I}) \tag{5.a}$$

$$\mathbf{U}(\lambda_1\mathbf{I}_1+\lambda_2\mathbf{I}_2)=\hat{\lambda}_1(\mathbf{U}\mathbf{I}_1)+\hat{\lambda}_2(\mathbf{U}\mathbf{I}_2) \tag{5.b}$$

(subscript "t" denotes transponent).

The scalar product of the primitive base vectors of the current distribution and the primitive base vectors of the voltage distribution, according to equation (4) is:

$$\mathbf{g}^l\mathbf{g}_m=\delta^l_{\cdot m}=\begin{cases}1, & \text{if} \quad l=m\\ 0, & \text{if} \quad l\neq m\end{cases} \tag{6}$$

(where $\delta^l_{\cdot m}$ is the Kronecker symbol).

Equation (6) can also be explained physically. The current and voltage distributions in the primitive bases can generate power — which is just unity — if the unit current flows in the branch where the voltage is also unity. So, as such, the two primitive bases form a *bi-orthogonal* or *reciprocal* system.

In mathematics, the linear vector space called here the space of the current vectors is usually termed as *contravariant*, and the space of voltage vectors as *covariant*.

2. The transformation of the components of the current and voltage distributions

If a base different from the primitive base is considered, then the components of the current and voltage distribution vectors will be different from the branch current and the branch voltage. If $\mathbf{h}_{k'}$ is the k'-th vector of the new base of the current vector space and $\mathbf{h}^{k'}$ is the vector of the new base of the voltage vector space, then in accordance with equation (1) and equation (2) one can write:

$$\mathbf{I}=I^k\mathbf{g}_k=I^{k'}\mathbf{h}_{k'} \tag{7}$$

and

$$\mathbf{U}=U_k\mathbf{g}^k=U_{k'}\mathbf{h}^{k'} \tag{8}$$

where $I^{k'}$ and $U_{k'}$ are the components of I and U, respectively, in the new base.

We specify that the new bases, similarly to the primitive bases, are biorthogonal, so that similarly to equation (6):

$$[h^{l'}],[\hat{h}_{m'}]=\mathbf{h}^{l'}\mathbf{h}_{m'}=\delta^{l'}_{\cdot m'} \tag{9}$$

With this specification and from the second right-hand side of equation (7) and (8) and based on equations (5a) and (5b), the power can be written as:

$$\mathbf{U}\mathbf{I}=(U_{k'}\mathbf{h}^{k'})(I^{l'}\mathbf{h}_{l'})=U_{k'}\hat{I}^{l'}\delta^{k'}_{\cdot l'}$$

or

$$\mathbf{U}\mathbf{I}=U_{k'}\hat{I}^{k'} \tag{10}$$

Having biorthogonal bases, the calculation of complex power is as simple as in the primitive bases [see eqn. (4)]. If eqn. (7) is pre-multiplied by $\mathbf{h}^{l'}$ and then by $\mathbf{g}^{l'}$, eqn. (8) is post-multiplied by $\mathbf{h}_{l'}$ and then by $\mathbf{g}_{l'}$, and considering the biorthogonal relations [eqns. (6) and (9)], the following equations can be derived:

$$\mathbf{h}^{l'}\mathbf{I}=I^{k}(\mathbf{h}^{l'}\mathbf{g}_{k})=I^{k'}\delta^{l'}_{\cdot k'}\ ;$$

$$\mathbf{g}^{l}\mathbf{I}=I^{k}\delta^{l}_{\cdot k}=I^{k'}(\mathbf{g}^{l}\mathbf{h}_{k'})\ ;$$

$$\mathbf{U}\mathbf{h}_{l'}=U_{k}(\mathbf{g}^{k}\mathbf{h}_{l'})=U_{k'}\delta^{k'}_{\cdot l'}\ ;$$

$$\mathbf{U}\mathbf{g}_{l}=U_{k}\delta^{k}_{\cdot l}=U_{k'}(\mathbf{h}^{k'}\mathbf{g}_{l})\ ;$$

Introducing the following symbols

$$(\mathbf{h}^{k'}\mathbf{g}_{k})=t^{k'}_{\cdot k} \tag{11.a}$$

$$(\mathbf{g}^{k}\mathbf{h}_{k'})=t^{k}_{\cdot k'} \tag{11.b}$$

and using the meaning of $\delta^{l}_{\cdot k}$ and $\delta^{l'}_{\cdot k'}$ the previous equations can be rewritten as:

$$I^{k'} = t^{k'}_{\cdot k} I^k \tag{12.a}$$

$$I^k = t^k_{\cdot k'} I^k \tag{12.b}$$

$$U_{k'} = \hat{t}^k_{k'} U_k \tag{12.c}$$

$$U_k = \hat{t}^{k'}_{\cdot k} U_{k'} \tag{12.d}$$

Substituting Eq. (12.b) into the right side of Eq. (12.a):

$$I^{k'} = t^{k'}_{\cdot k}(t^k_{\cdot l'} I^{l'}) = (t^{k'}_{\cdot k} t^k_{\cdot l'}) I^{l'} \ ;$$

This identity is valid for every possible value of k' and l' from 1 to e, if

$$t^{k'}_{\cdot k} t^{k'}_{\cdot l'} = \delta^{k'}_{\cdot l'} \tag{13}$$

Eq. (12) give the transformation law between the primitive base and the new base. If $t^k_{\cdot k'}$ are considered to be the elements of a quadratic *transformation matrix* $[T]$, then according to eqn. (13), $t^{k'}_{\cdot k}$ are the elements of the matrix $[T]^{-1}$. Let the column matrices be composed from the components $I^{k'}$ and $U_{k'}$ be denoted by $[I']$ and $[U']$, respectively.

So the matrix form of the transformation equations (12) can be given by:

$$[I'] = [T]^{-1}[I] \tag{14.a}$$

$$[I] = [\mathrm{T}][\mathrm{I}'] \tag{14.b}$$

$$[U'] = [T]^{*}[U] \tag{14.c}$$

$$[U] = ([T]^{-1})^{*}[U'] \tag{14.d}$$

where * denotes the transposed conjugated, i.e. the adjunct value. (The transposed matrix should be taken because the row subscripts of matrix $[T]$ in the Eqs (12.c) and (12.d) are the subscripts of summing, since these coincide with the subscripts of the voltage components.) Only a non-singular matrix can be transformation matrix, which is a consequence of the fact that the new base vectors are linearly independent ones. Supposing that the vectors $\mathbf{h}^{k'}$ are linearly dependent, then, from Eq. (11.a), the rows of the matrix $t^{k'}_{\cdot k}$ are also dependent.

The calculation of powers in both of the bases can be written as:

$$S=[I]^*[U]=[\hat{I}]_t[U]=[\hat{I}']_t[U']=[I']^*[U']$$

which follows from Eqs (2), (4) and (10).

If $[T]$ is real, then $[T]^*$ and $([T]^{-1})^*$ are replaced by $[T]_t$ and $[T]_t^{-1}$, respectively.

3. The network form of Ohm's theorem

Exclude disconnection for the time being. With the self- and mutual impedances, the following equations can be written to express the relation between the passive voltages and currents of the branches:

$$U_1=Z_{11}I^1+Z_{12}I^2+\dots+Z_{1e}I^e$$

$$U_2=Z_{21}I^1+Z_{22}I^2+\dots+Z_{2e}I^e$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$U_e=Z_{e1}I^1+Z_{e2}I^2+\dots+Z_{ee}I^e$$

(at the current, there are superscripts and not exponents).

Using the summational convention, this can be rewritten in a short form as

$$U_k=Z_{kl}I^l. \tag{16.a}$$

Denoting the quadratic branch impedance matrix Z_{kl} with $[Z]$, eqn. (16.a) can be written in a matrix form:

$$[U]=[Z][I] \tag{16.b}$$

The l-th column of matrix $[Z]$ can also be defined by multiplying the matrix by the l-th column of the unit matrix and, taking Eq. (16.b) also into account, this means that if a current corresponding to the l-th primitive base vector is flowing through the passive branches, then the l-th column of $[Z]$ will be given by the branch voltages. So, if *a unit current is flowing through, but only* *through the l-th passive branch, then the elements of the l-th column of the branch impedance matrix will be given by the values of branch voltages.*

Eqs (16.a) and (16.b) are the network form of Ohm's theorem. They can be written in different bases, too. Using Eqs (12.c) (16.a), (12.b) and Eqs (14.c), (16.b), (14.b), one can write:

$$U_{k'} = \hat{t}^{k}_{\cdot k'} Z_{kl} t^{l}_{\cdot l'} I^{l'}$$

or

$$[U'] = [T]^{*} [Z] [T] [I']$$

Introducing the following symbols:

$$Z_{k'l'} = \hat{t}^{k}_{\cdot k'} Z_{kl} t^{l}_{\cdot l'} \tag{17.a}$$

$$[Z'] = [T]^{*} [Z] [T], \tag{17.b}$$

Ohm's theorem in the new base will be

$$U_{k'} = Z_{k'l'} I^{l'} \tag{18.a}$$

or

$$[U'] = [Z'] [I'] \tag{18.b}$$

Comparing these to eqns. (16.a) and (16.b), we can see that the form of Ohm's theorem is unchanged, but according to Eq. (17) the impedance matrix has to be transformed by a so-called adjunct transformation.

Actually Eqs (16) and (18) are the subscript form and the matrix form of the same tensor equation in the primitive and in another arbitrary base. The tensor equation is:

$$\mathbf{U} = \boldsymbol{Z} \mathbf{I}; \tag{19}$$

(tensors are denoted by bold-faced italics).

Z is the impedance tensor the component matrix of which in the primitive base is the branch impedance matrix $[Z]$ and its component matrix in another base is $[Z']$, which can be determined by the transformation law according to Eq. (17.b)

If the branch impedance matrix is a non-singular one — i.e. the impedance tensor is not degenerated, so its matrix is not singular in any of the base, — then from Eqs (16.a) or (16.b); (18.a) or (18.b) and (19), the current can be expressed:

$$I^m = Y^{mk} U_k \tag{20.a}$$

$$[I] = [Y][U] \tag{20.b}$$

$$I^{m'} = Y^{m'k'} U_{k'} \tag{21.a}$$

$$[I'] = [Y'][U'] \tag{21.b}$$

and

$$\mathbf{I} = Y\mathbf{U} \tag{22}$$

The admittance tensor Y is the inverse of the impedance tensor Z:

$$YZ = E \tag{23}$$

(where E is the unit tensor).

The matrix of the tensor Y in the primitive base is the *branch admittance matrix*,

$$Y^{mk} \quad \text{or} \quad [Y]$$

which is the inverse of the branch impedance matrix.

Therefore,

$$Y^{mk} Z_{kl} = \delta^m_{\cdot l} \tag{22.a}$$

or

$$[Y][Z] = [E] \tag{22.b}$$

($[E]$ is the unit matrix).

The matrix of the tensor Y in any arbitrary base satisfies the following equations:

$$Y^{m'k'} Z_{k'l'} = \delta^{m'}_{\cdot l'} \tag{23.a}$$

or

$$[Y'][Z'] = [E], \tag{23.b}$$

where $Z_{k'l'}$ or $[Z']$ is the matrix of the impedance tensor in the same base.

From Eqs (12.a) (20.a) and (12.d) or from Eqs (14.a), (20.b) and (14.d), we can write:

$$I^{m'} = t^{m'}_{\cdot m} Y^{mk} \hat{t}^{k'}_{\cdot k} U_{k'}$$

or

$$[I'] = [T]^{-1} [Y] ([T]^{-1})^* [U'],$$

If these are compared with Eqs (21.a) and (21.b), then the transformation law between admittance tensors in different bases can be derived:

$$Y^{m'k'} = t^{m'}_{\cdot m} Y^{mk} \hat{t}^{k'}_{\cdot k} \tag{24.a}$$

or

$$[Y'] = [T]^{-1} [Y] ([T]^{-1})^* \tag{24.b}$$

The k-th column of the branch admittance matrix $[Y]$ can also be written as a multiplication between this matrix and the k-th column of the unit matrix and, taking Eq. (20.b) also into account, this means that if the k-th voltage distribution of the primitive base of the voltage distribution vector space is on the passive branches, then the k-th column of $[Y]$ is given by the numerical values of the branch currents. So *if a unit voltage is connected to the k-th passive branch and the other passive branches are short-circuited, then the elements of the k-th column of the branch admittance matrix will be given by the branch currents* (numerically).

In the previous chapters it was supposed that both the impedance tensor and the admittance tensor exist. Since a tensor can be given by those vectors into which the vectors of a base transformed by the tensor, so using Eqs (19) and (22) it can be stated: the necessary and sufficient condition for the existence of the impedance tensor is that a base of the passive current distribution vector space generates *finite* passive voltage distributions, and the similar condition for the existence of the admittance tensor is that a base of the passive voltage distribution vector space generates *finite* passive current distributions. The simplest control can be carried out on the primitive base. It is easy to see e.g. that a network containing a branch with disconnection has no impedance tensor, while networks containing no such branches have impedance tensors. If the effective (real) part of the self-impedance of each branch is different from zero, then the network has an admittance tensor since the resistance limit the current to a finite value everywhere.

There are also networks which have neither an impedance tensor nor an admittance tensor because, e.g., there may be a disconnection in one of the branches and a short-circuit in another branch, which means that both the self- and mutual impedances are zero.

Disconnections can be avoided by leaving out the disconnected branches, but in this case, the current generators of these branches should be placed into the branches of another path between the nodes concerned to compensate for the current generator left off.

In this way of decreasing the "*e*"-dimensional number one can achieve a network having an impedance tensor.

Coming back to the interpretation of the branch impedance and the branch admittance matrices with the use of the primitive base, it can be seen that *the self-impedance elements in the principal diagonal of the branch impedance matrix are the open-circuit measuring impedances, the elements of mutual impedances are the open-circuit transfer impedances, the elements of the principal diagonal of the branch admittance matrix are the short-circuit measuring admittances, and the elements outside the principal diagonal are the short-circuit transfer admittances.*

The impedance tensor having components with two lower subscripts is a so-called (pure) *covariant tensor*,and the admittance tensor having components with two superscripts is a so-called (pure) *contravariant tensor*. Both types of tensor establish a homogeneous linear relation between two different vector spaces. (Current distribution and voltage distribution, or contravariant and covariant.)

4. Sub-spaces of the divergenceless current distributions and of the irrotational voltage distributions; the Kirchhoff laws

In the following chapters the passive (or active) current distributions satisfying the nodal law will be referred to as *divergenceless*, and the passive (or active) voltage distributions satisfying the loop law will be referred to as *irrotational* ones. The divergenceless current distributions form a sub-set of the current distribution vector space. It can easily be seen that the result of a linear combination of the divergenceless current distribution vectors are also divergenceless current vectors, so the aforementioned sub-set is a *sub-space*, which will be denoted by G. Similarly, the sub-set of the irrotational voltage distributions is the sub-space of the voltage distribution vector space, and will be denoted by M.

If the number of nodes of the "e" branch network considered is v, and it has a p number of sub-networks where the branches are in galvanic connections, but there is no galvanic connection between the sub-networks, then a $(v-p)$ number of independent nodal equations can be written which has to be satisfied by a divergenceless current distributions. This means a $(v-p)$ number of limitations, therefore the dimensional number of G will be:

$$\dim G = e-(v-p) = e-v+p. \tag{25}$$

Irrotational or potentional voltage distributions can simply be characterized by the potential differences between the nodes. Adopting a reference node in each sub-network, we shall have a $(v-p)$ number of potentials, so

$$\dim M = v-p.$$

At the current distributions in G, there are no currents flowing out of or into the nodes. The so-called *loop-current* flowing in a single closed loop is similar. Therefore G contains the *loop-currents* or any linear combination of them. If a loop-current having the same branch-current as the branch-current of $\mathbf{I}_G$ in the same branch is subtracted from an arbitrary current vector $\mathbf{I}_G$ of G, and then the same procedure is applied to the remainder, leaving the branches already nilled, all branches can be nilled in the end, otherwise $\mathbf{I}_G$ will not be divergenceless. It follows that all vectors of G can be interpreted as a linear combination of loop currents. Since loop currents with irrotational voltage distributions have no power — for the sum of voltages in a loop is zero, — therefore all $\mathbf{I}_G$ vectors of G are orthogonal to all $\mathbf{V}_M$ vectors of M. (Tellegen's theorem).

$$\mathbf{U}_M\mathbf{I}_G = 0. \tag{27.a}$$

Or we can simply say that G and M sub-spaces are orthogonal ones:

$$G \perp M. \tag{27.b}$$

Sub-sets not included in G, i.e. sub-sets of divergent current distributions are not sub-spaces, since their linear combination can result in divergenceless distribution.

On the other hand, outside of G, a $(v-p)$ number of linearly independent current vectors can be considered, (which form an "e"-dimensional base with the $(e-v+p)$ number of linearly independent vectors of sub-space G) which form sub-space H so its dimensional number is:

$$\dim H = v-p. \tag{28}$$

There are an infinite number of such sub-space H.

A sub-set not included in M, i.e. sub-sets of rotational voltage distributions are not sub-spaces either, (since a linear combination of the rotational voltage distributions may result in irrotational ones) a sub-space N, formed by $(e-v+p)$ linearly independent rotational voltage distributions may be considered, whose dimensional number is:

$$\dim N = e-v+p. \tag{29}$$

There are also an infinite number of such sub-spaces N. Choose N orthogonal to a previously selected H, and specify that:

$$H \perp N. \tag{30}$$

However, this is already enough to determine the sub-space N.

The sub-spaces derived this way and their characteristics are shown in *Fig. 2.* (The dimensional numbers are written under the symbols of sub-spaces.) All sub-spaces contain the vector $\mathbf{O}$.

It follows that all passive and active current vectors can be decomposed to components falling into sub-spaces G and H.

$$\mathbf{I} = \mathbf{I}_G + \mathbf{I}_H \tag{31.a}$$

$$\mathbf{J} = \mathbf{J}_G + \mathbf{J}_H \tag{31.b}$$

In a similar way, all passive and active voltage vectors can be decomposed to components falling into sub-spaces M and N:

$$\mathbf{U} = \mathbf{U}_M + \mathbf{U}_N \tag{32.a}$$

$$\mathbf{V} = \mathbf{V}_M + \mathbf{V}_N . \tag{32.b}$$

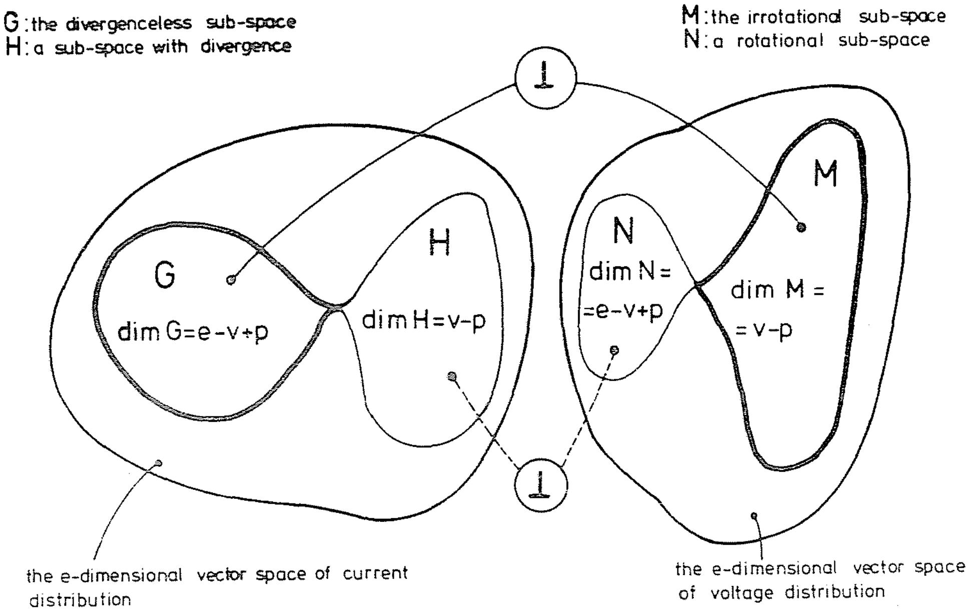


Fig. 2

From Fig. 1b it follows that the resultant of the passive and active currents in the primitive base is:

$$I^k - J^k$$

and the resultant of the passive and active voltages in the primitive base is:

$$U_k - V_k \, .$$

Therefore, the resultant current distribution vector and the voltage distribution vector are:

$$\mathbf{I}-\mathbf{J} \quad \text{and} \quad \mathbf{U}-\mathbf{V}, \qquad \text{respectively.}$$

These resultants *must* satisfy Kirchhoff's laws, which means that the vector

$$\mathbf{I}-\mathbf{J}=(\mathbf{I}_G-\mathbf{J}_G)+(\mathbf{I}_H-\mathbf{J}_H)$$

can only have components falling into the divergenceless sub-space G, and the vector

$$\mathbf{U}-\mathbf{V}=(\mathbf{U}_M-\mathbf{V}_M)+(\mathbf{U}_N-\mathbf{V}_N)$$

can only have components falling into the irrotational sub-space M. (Meanwhile we used Eqs (31) and (32).) So Kirchhoff's laws will have the following forms:

$$\mathbf{I}_H=\mathbf{J}_H \tag{33.a}$$

$$\mathbf{U}_N=\mathbf{V}_N \tag{33.b}$$

Substituting these into Eqs (31.a) and (32.a):

$$\mathbf{I}=\mathbf{I}_G+\mathbf{J}_H$$

and

$$\mathbf{U}=\mathbf{U}_M+\mathbf{V}_N$$

Then taking into account Ohm's theorem, according to Eqs (19) and (20) we have the following tensor equations:

$$\mathbf{U}_M+\mathbf{V}_N=Z\,(\mathbf{I}_G+\mathbf{J}_H) \tag{34.a}$$

or

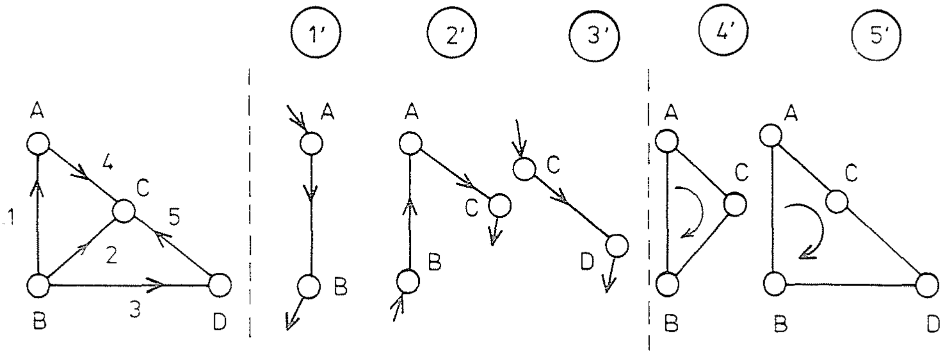
$$\mathbf{I}_G+\mathbf{J}_H=Y(\mathbf{U}_M+\mathbf{V}_N) \tag{34.b}$$

to determine the unknown $(e-V+p)$ dimensional, divergenceless $\mathbf{I}_G$, and the $(v-p)$ dimensional irrotational vectors $\mathbf{U}_M$.

Actual calculations can be carried out only in determined bases. The primitive base — though the matrices of $\mathbf{Y}$ and $\mathbf{Z}$ are simple in it — is not suitable because the sub-spaces cannot be separated. Therefore, bases are needed in which this separation can easily be done, but for this, on the other hand, the branch impedance and branch admittance matrices have to be transformed.

5. Kron's scheme of closed and open currents as a new base

Loop currents were called *closed* currents, and those currents which enter a node and then leave another node without forming a loop in the network considered were called *open* currents by G. KRON. Since all the vectors of sub-space G are linear combinations of loop currents, therefore the divergenceless sub-space G is *generated* by $(e-v+p)$ linearly independent loop currents of unit intensity. On the other hand, a $(v-p)$ number of unit intensity open currents linearly independent of the aforementioned vectors and of one another — forming no loops with one another — generate a divergent sub-space H. The common system of closed and open currents derived in this way span the whole current vector space, and so it is a base of that. If we also make the reciprocal base in the voltage distribution vector space, we obtain a new base in which the separation into sub-spaces G and H, and M and H is simple.



a *b* *c*

Fig. 3

As an example, consider the network shown in *Fig. 3a.* in which the number of sub-spaces $p=1$, the number of vertices (nodes) $v=4$ and the number of branches $e=5$.

In the Figure, also the reference directions of the passive currents of the branches are shown. Since $v-p=3$ and $e-v+p=2$, then 3 linearly independent open currents (of unit intensity) and 2 closed currents (of unit intensity) have to be assumed.

The open currents are shown in *Fig. 3.b*, while the loop currents in *Fig. 3.c.*

The column matrix of the vectors of the new base formed by the open and closed paths in the original primitive base are the following:

$$\begin{array}{cl}
\mathbf{h}_1, \quad \mathbf{h}_2, \quad \mathbf{h}_3, \quad \mathbf{h}_4, \quad \mathbf{h}_5, & \text{new base vectors} \\
\underbrace{\begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\-1\end{bmatrix}}_{H}\underbrace{\begin{bmatrix}1\\-1\\0\\1\\0\end{bmatrix}\begin{bmatrix}1\\0\\-1\\1\\-1\end{bmatrix}}_{G} & \begin{array}{c}\text{column matrices}\\ \text{as components in}\\ \text{the primitive base}\end{array} \\
 & \text{generated sub-spaces}
\end{array}$$

(The branch components of the base-current distributions are 1 or -1 depending on whether the direction of the current of the unit current distribution is similar or opposite, respectively to the orientation of the branch, or zero if it has no such a branch current.)

Since the components of the new base vectors in the new base are given by the columns of the unit matrix, therefore, in accordance with Eq. (12.b) or Eq. (14.b), the set of the previously written column matrices is just the transformation matrix $t^{k}_{\cdot k'} = [T]$

So,

$$t^{k}_{\cdot k'} = [T] = \begin{array}{c} \begin{array}{cc} \quad H \quad & \quad G \quad \\ (v-p) & (e-v+p) \end{array} \\ \left[\begin{array}{ccc:cc} -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{array}\right] \end{array}$$

$$t^{k'}_{\cdot k} = [T]^{-1} = \left[\begin{array}{ccccc} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ \hdashline 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right] \begin{array}{c} M \\ (v-p) \\ \\ N \\ (e-v+p) \end{array}$$

where the details of the calculation of inversion are neglected. If $[I]_t = [-2j; 3; 1+j; 0; -1]$ is the matrix of a current distribution vector in the primitive base, then its matrix in the current path base according to Eq. (12.a) or Eq. (14.a) is:

$$[I'] = [T]^{-1}[I] = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2j \\ 3 \\ 1+j \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2j \\ 4+j \\ 2+j \\ -3 \\ -1-j \end{bmatrix}$$

The first three are the opened path components, the components with divergence, and the following two are the closed path divergenceless components. If the matrix of a voltage distribution in the primitive base is $[U]_t = [-j;\, 1;\, -2+j;\, -3;\, 0]$, then in the new base, according to Eq. (12.c) or (14. c), it is:

$$[U']_t = [U]_t[T] = [-j;\, 1;\, -2+j;\, -3;\, 0] \begin{bmatrix} -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{bmatrix} =$$

$$= [j;\, -3-j;\, 0;\, -4-j;\, -1-2j]\,.$$

The first three are the irrotational, the following two the rotational components. The separation to sub-spaces can generally be denoted by the partitioning of matrices $[T]$ and $[T]^{-1}$. Let *subscripts a and b denote the opened path or the irrotational components, i.e. the components in the sub-spaces H or M and subscripts r and s denote the closed path, or the rotational components in sub-spaces G or N*. Then the required partitioning is:

$$t^{k}_{\cdot k'} = [T] = [\underbrace{t^{k}_{\cdot a}}_{(v-p)} \quad \underbrace{t^{k}_{\cdot r}}_{(e-v+p)}]\}(e) \tag{35.a}$$

and

$$t^{k'}_{\cdot k} = [T]^{-1} = \left[\begin{array}{c} t^{a}_{\cdot k} \\ \hline t^{r}_{\cdot k} \end{array}\right] \begin{array}{l} \}\ (v-p) \\ \}\ (e-v+p) \end{array} \tag{35.b}$$

Then according to Eqs (12.a) and (35.b)

$$I^{a} = t^{a}_{\cdot k} I^{k}; \qquad 1 \leqq a \leqq v-p \tag{36.a}$$

and

$$I^r = t^r_{\cdot k} I^k; \qquad e - v + p \leqq r \leqq e. \tag{36.b}$$

Moreover

$$[I] = \begin{bmatrix} I^a \\ \hdashline 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline I^r \end{bmatrix} \begin{matrix} \} \, (v-p) \\ \} \, (e-v+p) \end{matrix} \tag{36.c}$$

and, in accordance with Eqs (12.c) and (35.a),

$$U_a = U_k t^k_{\cdot a} \tag{3\hat{7}.a}$$

and

$$U_r = U_k t^k_{\cdot r} \tag{3\hat{7}.b}$$

and

$$[U'] = \begin{bmatrix} U_a \\ \hdashline 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline U_r \end{bmatrix} \begin{matrix} \} \, (v-p) \\ \} \, (e-v+p) \end{matrix} \tag{37.c}$$

If we want to determine the components in the primitive base from those in the current-path base and the summational convention is used also in the subspaces, then according to Eqs (12.b) and (35.a) and later Eqs (12.d) and (35.b)

$$I^k = t^k_{\cdot a} I^a + t^k_{\cdot r} I^r \tag{38}$$

and

$$U_k = U_a \hat{t}^a_{\cdot k} + U_r \hat{t}^r_{\cdot k} \tag{39}$$

(The symbols of conjugation can be omitted, since the transformational matrix is a real one.) Similar equations can be derived also for the active currents and voltages.

The form of Kirchhoff's I. law in the new base, in accordance with Eq. (33.a), is:

$$I^a = J^a \tag{40.a}$$

or using Eq. (36.a), too,

$$t^a_{\cdot k} I^k = t^a_{\cdot k} J^k \tag{40.b}$$

The form of Kirchhoff's II. law in the new base, in accordance with Eq. (33.b), is:

$$U_r = V_r \tag{41.a}$$

or using Eq. (37.b), too

$$U_k \hat{t}^k_{\cdot r} = V_k \hat{t}^k_{\cdot r} \tag{41.b}$$

Performing the transformations of Eqs (17) and (24) and partitioning the matrices the form of Ohm's theorem in the current-path base, in accordance with Eqs (18) and (21), will be

$$\left[\begin{array}{c} U_a \\ \hline U_r \end{array}\right] = \begin{array}{c} \begin{array}{cc} (v-p) & (e-v+p) \end{array} \\ \left[\begin{array}{c|c} Z_{ab} & Z_{as} \\ \hline Z_{rb} & Z_{rs} \end{array}\right] \end{array} \left[\begin{array}{c} I_b \\ \hline I^s \end{array}\right] \begin{array}{l} (v-p) \\ (e-v+p) \end{array} \tag{42.a}$$

or

$$\left[\begin{array}{c} I^a \\ \hline I^r \end{array}\right] = \left[\begin{array}{c|c} Y^{ab} & Y^{as} \\ \hline Y^{rb} & Y^{rs} \end{array}\right] \left[\begin{array}{c} U_b \\ \hline U_s \end{array}\right] \begin{array}{l} (v-p) \\ (e-v+p) \end{array} \tag{42.b}$$

Then the solution of the basic problem of network calculation is as follows:

Substituting Kirchhoff's laws (Eqs (40.a) and (41.a)) into the equation of Ohm's theorem (Eqs (42.a) and (42.b)), then parting the equations into two parts, we can write that:

$$\left.\begin{array}{l} U_a = Z_{ab} J^b + Z_{as} I^s \\ V_r = Z_{rb} J^b + Z_{rs} I^s \end{array}\right\} \tag{43.a}$$

and

$$\left.\begin{array}{l} J^a = Y^{ab} U_b + Y^{as} V_s \\ I^r = Y^{rb} U_b + Y^{rs} V_s \end{array}\right\} \tag{43.b}$$

The unknown *mesh-current* I^s can be calculated from the second equation of 43.a and the irrotational voltages U_b i.e. the *potentials* can be calculated from the first equation of 43.b provided that the active quantities J^k and V_k are known.

The algorithm of the mesh-current method:

I. Determination of the components with divergence of the active currents in accordance with Eq. (36.a)

$$J^b = t^b_{\cdot k} J^k$$

II. Calculation of the rotational components of active voltages in accordance with Eq. (37.b)

$$V_r = V_k \hat{t}^k_{\cdot r}$$

III. Transformation of the impedance matrix in accordance with Eq. (17)

$$Z_{k'l'} = \hat{t}^k_{\cdot k'} Z_{kl} t^l_{\cdot l'}\,; \quad [Z'] = [\hat{T}]_t [Z][T]$$

IV. Partitioning:

$$Z_{k'l'} = \begin{array}{cc} \begin{array}{cc} (v-p) & (e-v+p) \end{array} & \\ \left[\begin{array}{c|c} Z_{ab} & Z_{as} \\ \hline Z_{rb} & Z_{rs} \end{array}\right] & \begin{array}{l} (v-p) \\ (e-v+p) \end{array} \end{array}$$

V. Solution of the set of linear equations

$$Z_{rs} I^s = V_r - Z_{rb} J^b$$

derived from the second equation of Eq. (43.a) for I^s.

VI. Calculation of the potential voltages from the first equation of Eq. (43.a)

$$U_a = Z_{ab} J^b + Z_{as} I^s$$

VII. Calculation of the passive branch currents from Eq. (38) with the use of Eq. (40.a)

$$I^k = t^k_{\cdot b} J^b + t^k_{\cdot s} I^s$$

VIII. Calculation of the passive branch voltages from Eq. (39), using Eq. (41.a)

$$U_k = U_a \hat{t}^a_{\cdot k} + V_r \hat{t}^r_{\cdot k}$$

Algorithm of the potential method

(Only the steps different from those of the mesh current method are indicated)

III. Transformation of the admittance matrix in accordance with Eq. (24):

$$Y^{m'k'} = t^{m'}_{\cdot m} Y^{mk} \hat{t}^{k'}_{\cdot k}; \; [Y'] = [T]^{-1}[Y][\hat{T}]_t^{-1}$$

IV. Partitioning:

$$Y^{m'k'} = \begin{bmatrix} Y^{ab} & Y^{as} \\ Y^{rb} & Y^{rs} \end{bmatrix}$$

V. Solution of the linear equation system

$$Y^{ab} U_b = J^a - Y^{as} V_s$$

for U_b derived from the first equation of (43.b).

VI. Determination of the mesh-currents from the second equation of (43.b)

$$I^r = Y^{rb} U_b + Y^{rs} V_s$$

Cases at which there are *nodal current inputs* are not included in the arrangement of Fig. 1. (Of course, the sum of these inputs in a sub-network is zero). It can be seen that these are linear combinations of opened currents and, as such, their opened path components can easily be written. For example, if the current inputs at the nodes according to *Fig. 3a* are K^A, K^B, K^C, $K^D = -K^A - K^B - K^C$, then this current distribution, with the (unit) open currents indicated in *Fig. 3b*, can be written as follows:

$$K^A \mathbf{h}_{1'} + (K^B + K^A)\mathbf{h}_{2'} + (K^C + K^B + K^A)\mathbf{h}_{3'}$$

and this means that the opened path components are:

$$K^{1'} = K^A, K^{2'} = K^A + K^B, K^{3'} = K^A + K^B + K^C$$

Then the outer nodal current input have to be taken into account by adding K^a to the opened path components of the active currents, i.e.

$$J^a + K^a \text{ is written instead of } J^a$$

in V. and VI.

The following theorem is very important from the point of view of solvability of the basic problem of network calculation:

If among the elements of the symmetrical branchimpedance matrix only the self-impedances have non-zero real parts and these are *finite and positive*, then the matrix Z_{rs} is a non-singular one, and so it can be inverted. To prove this theorem, consider the divergenceless current distributions. In this case, the current distribution has no opened path components, i.e.

$$I^a = 0,\ I^{k'} = \begin{cases} 0, & \text{if} \quad k' = 1, 2, \ldots, v-p \\ I^r, & \text{if} \quad k' > v-p \end{cases}$$

and therefore the power is:

$$S = \hat{I}^{k'} Z_{k'l'} I^{l'} = \hat{I}^r Z_{rs} I^s$$

Because of the invariance of power

$$\hat{I}^{k'} Z_{k'l'} I^{l'} = \hat{I}^k Z_{kl} I^l$$

thus,

$$S = \hat{I}^r Z_{rs} I^s = \hat{I}^k Z_{kl} I^l$$

(k, l are subscripts in the primitive base).

On the other hand, according to the condition

$$Z_{kl} = R_{kk} + jX_{kl} \qquad (\text{and } X_{kl} = X_{lk}),$$

therefore

$$S = \left(\sum_{k=1}^{e} R_{kk} I^k \hat{I}^k \right) + j(\hat{I}^k X_{kl} I^l) = \left(\sum_{k=1}^{e} R_{kk} |I^k|^2 \right) + jQ$$

$$(Q \text{ is real because } \hat{I}^k X_{kl} I^l = \hat{I}^l X_{lk} I^k = \hat{I}^l X_{kl} I^k = I^k X_{kl} \hat{I}^l)$$

So $Re(S) > 0$ (except in the case of zero current distribution).

Therefore, the form of second degree $S = \hat{I}^r Z_{rs} I^s$ is never zero, except when $I^r = 0$, and so Z_{rs} *is not singular*. The non singularity of Y^{ab} can be proved in a similar way.

6. Diakoptics

The calculation of networks, the solution of the set of linear equations needs a great deal of calculations and operations requires high storage capacity. Its matrix is usually "sparse", i.e. it contains many zero elements. Diakoptics, the method worked out by G. KRON tears the network into sub-networks and then connecting the simplified subnetworks, the problem can be solved. In this way, there are a number of small compact quadratic matrices instead of a large "sparse" one. Obviously, the storage capacity requires is reduced and the calculation work is also much less, since the number of operations in solving a set of linear equations is proportional to the cube of the size, and so about $k\left(\frac{n}{k}\right)^3=\frac{n^3}{k^2}$ operations are required instead of n^3 operations. (n is the size of matrix of the set of linear equations before tearing, and k is the number of sub-networks.)

A general theorem of network calculation has an important role in diakoptics. The theorem can be easily understood on the basis of the previous chapters.

Conversion into equivalent loopless network

The conversion theorem is applicable to networks which are not inductively coupled with other networks.

The potential methods are based on the first equation of (43.b).

$$Y^{ab}U_b=J^a-Y^{as}V_s$$

J^a is the opened path (with divergence) component of the current generators, which also contains the converted values of the nodal current generators originally included, i.e.,

$$J^a=t^a_{\cdot k}I^k+K^a \tag{44.a}$$

V_s is the rotational component of voltage generators, i.e.,

$$V_s=V_kt^k_{\cdot s'} \tag{44.b}$$

U_b is the potential component of the passive voltage wanted. Also the irrotational component of the voltage generators will be needed:

$$V_b=V_kt^k_{\cdot b} \tag{44.c}$$

The meaning of $t^a_{\cdot k'}$ and $t^k_{\cdot s}$ is according to Eq. (35).

The equivalent conversion will be used for the coherent sub-network of a large network. In the sub-networks, torn out from the network, the other parts of the network will be considered by the help of nodal current generators which will be denoted with $\tilde{K}^i$, while $\tilde{K}^a$ will denote their divergence current components in the following. (It will be seen that there is no need of the value of $\tilde{K}^a$). J^a does not contain this $\tilde{K}^a$, therefore it should be added to the right side of the equation preceding Eq. (44.a):

$$Y^{ab}U_b = J^a - Y^{as}V_s + \tilde{K}^a$$

If the (existing) inverse of Y^{ab} is denoted with W_{ba}, i.e.,

$$(Y^{ab})^{-1} = W_{ba}, \tag{45}$$

then, from the previous equation, one can write:

$$U_b = W_{ba}(J^a - Y^{as}V_s + \tilde{K}^a). \tag{46}$$

Considering Eq. (46) to be Ohm's theorem in the primitive base, it can be stated that *outwardly*, i.e. from the point of view of the voltages between the nodes and the outer currents flowing into the nodes, *the network examined behaves as a loopless network, containing only opened paths at which the voltage generator of the opened paths is* V_b, *its current generator is*

$$J^b - Y^{bs}V_s,$$

its self-and mutual impedance is W_{ba}, *and its outer nodal-current is* $\tilde{K}^b$. The b-th branch of the substituting opened path network is shown in *Fig. 4*.

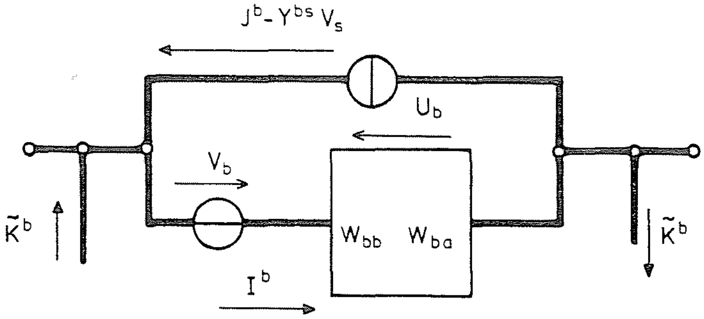


Fig. 4

The substitution discussed above means that not only the closed paths are omitted but also that *every opened path is replaced by a new one* and Eq. (46) expresses Ohm's theorem in the primitive base, concerning these new paths. To enlighten this, consider the network shown in *Fig. 5.a* and take the system of opened paths according to *Fig. 5.b*. The new network built up from these opened paths is shown in *Fig. 5.c*.

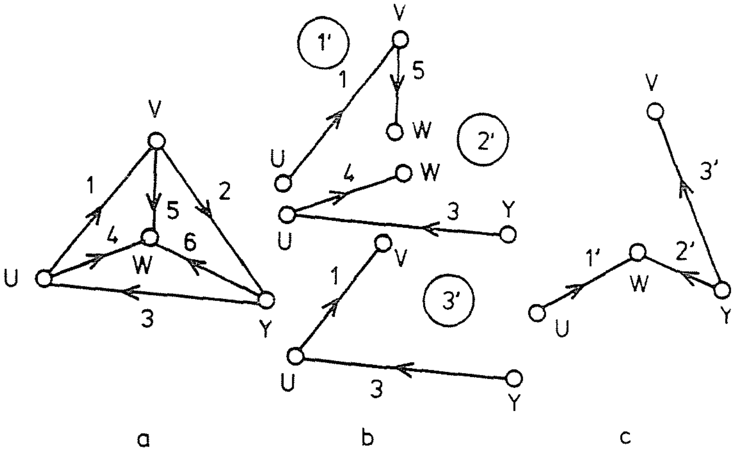


Fig. 5

Then consider a large connected network (*Fig. 6.a*) and tear it into sub-networks (*Fig. 6.e*) in such a way that after removing the total of the torn branches — the so-called inter-subnetwork (*Fig. 6.d*), — the sub-networks are still connected to each other without loops (*Fig. 6.c*). We are going to discuss in the following, how this can be carried out practically. If the sub-networks have no mutual impedances with one another and with the inter-subnetworks, then the passive current and voltage distribution of the whole network can be determined as follows:

a) The sub-networks are replaced by meshless network according to the teorem discussed above.

b) Connect the substituting networks of the sub-networks and the inter-subnetwork (consisting of the torn branches), so $\tilde{K}^a$ will be dropped out (that is why it was not needed to calculate its value); and since the sub-networks are connected to one another without loops, the substituting network contains only as many meshes as the number of the torn branches.

c) Determine the voltage and current distribution of the substituting network with the *mesh method.*

d) At the end, calculate the real currents of the sub-networks.

Disintergration into sub-networks can be done as follows: First determined nodes and their branch connections will be comprised into sub-networks, with carefully leaving out as few branches as possible. (*Fig. 6.a* and *6.b*). Then one more node, with the appropriate branches, will be connected to all the sub-networks except one (*Fig. 6.c*), and the common nodes obtained in this way will provide the loopless connection of the sub-networks. Namely, let

$$v_1, v_2, \ldots, v_m$$

the number of nodes of the separate sub-networks, then after establishing common nodes, the number of nodes of all the sub-networks, except say the *i*-th sub-network, will be increased by one. So the final number of nodes of the sub-networks are:

$$v_1+1,\ v_2+1,\ \ldots,\ v_i,\ \ldots,\ v_m+1.$$

After forming the sub-networks into loopless network, they have

$$v_1,\ v_2,\ \ldots,\ v_i-1,\ \ldots,\ v_m$$

branches, therefore the total number of the branches of the substituting sub-networks is:

$$\left(\sum_{k=1}^{m} v_k\right)-1=v-1$$

where v is the total number of nodes. So the branches solely of the substituting subnetworks form

$$[(v-1)-v]+1=0$$

number of loops.

The branches left out constitute the inter-subnetwork (*Fig. 6.d*) *Fig. 6.e* shows how the same disintegration can be obtained, using the original tearing technique. There is an important, special case in which the sub-networks are connected to one another in a single common node (on the 0-bar). In this case, this common node is assumed to belong to the sub-network without branch growth.

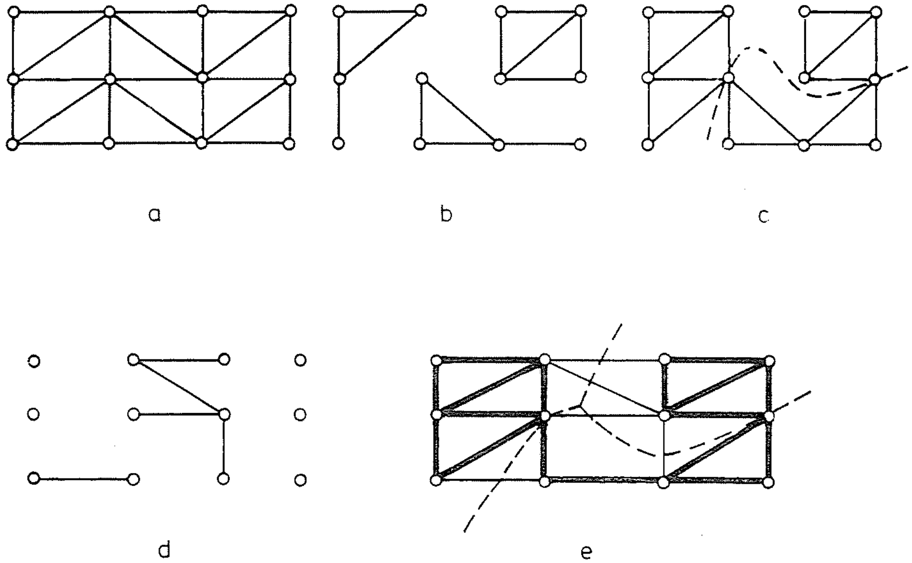


Fig. 6

This paper, did not discuss the calculation of the transformation matrices from the graph of the network. This problem — important from practical point of view — is discussed in detail in a paper, following the present one, of György Tevan, of the authors of the present paper, "Algorithm to determine the transformation matrix of Kron's method for calculating networks and for diakoptics."

Summary

According to this paper, Kron's method of network calculation is given a simple, clear linear algebraic foundation in such a way that the sets of stationary current and voltage distributions are regarded as linear vector spaces of dimension equal to the number of branches, and the scalar multiplication, resulting the power, is explained as an operation between these vector spaces. The base transformations of these two vector spaces are connected to each other by the realization of the invariance of power. The current vector space and the voltage vector space can be divided into a direct sum of the sourceless and a source subspace, and the irrotational and a rotational subspace, respectively. The base chosen according to this corresponds to Kron's "orthogonal network", comprising opened and closed current paths, and the base chosen according to the branches corresponds to the "primitive network". Finally, this paper shows Kron's diakoptics from the point of view of the previous chapters.

References

1. A. Brameller—M. N. John—M. H. Scott: Practical Diakoptics for Electrical Networks; (London) Chapman and Hall, (1969).
2. I. M. Gelfand: Lekcii po linejnoj algebre, Moszkva 1951.
3. P. R. Halmos: Finite — Dimensional Vector Spaces, D. Van Nostrand Company, Inc., Princeton, N. J. 1950.
4. H. H. Happ: Diakoptics and Networks; Academic Press 1971, New York—London
5. A. Klos: Algebraic Modal of Electrical Network; IEEE Transactions on Power Apparatus and Systems, vol. PAS-89, No. 2, February 1970.
6. G. Kron: Tensor Analysis of Networks; Wiley, 1939, New York.
7. G. Kron: Diakoptics; the Piecewise Solution of Large Scale Systems; MacDonald: London, 1963.
8. J. P. Routh: An Application of Algebraic Topology: Kron's Method of Tearing; Quarterly Of applied mathematics, Vol. XVII. April 1959. No. 1.
9. J. A. Schooten: Ricci—Calculus; Springer—Verlag, Berlin—Göttingen—Heidelberg, 1954.
10. S. Seshu and M. B. Reed: Linear Graphs and Electrical Networks. Addison—Werley, 1961. Reading Mass.
11. P. Slepian: Mathematical Foundations of Network Analysis; Springer 1968. Berlin—Heidelberg—New York.
12. H. Weyl: Repartition de corriente et uno red conductora, Revisto matematica, Hispano—Americana 5, 153—164 (1923).

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