THE LEAST ACTION AND THE VARIATIONAL CALCULUS IN ELECTRODYNAMICS

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Introduction

Solution by the variational calculus of electromagnetic fields can be derived as the stationary function of a suitable functional constructed from the potential function. The stationary function yields a solution satisfying Maxwell's equations. For the case of static and stationary electrical fields and electromagnetic wave phenomena of sinusoidal excitation suitable functionals are found in the literature [7, 8, 9, 10, 12] sometimes referring to the physical meaning of the functional.

It will be shown that the functionals for different cases derive from the same principle, the principle of least action, or in special cases, Hamilton's principle. Thus the variational principles in electrodynamics can uniformly be discussed and the functional for the case of arbitrary excitation can also be formulated. The formulation of the functional and the proof that the stationary function of the functional solves Maxwell's equations, are deduced from the theory of relativity using four-dimensional vectors. The mathematical formalism necessary for relativistic discussion and solution of the Maxwell equations is given in the Appendix. More detailed discussion is found in [13].

It is also possible to apply the functional formulated on the principle of least action to different cases of electrodynamics, such as electrostatics, magnetostatics, stationary electric and magnetic fields, quasistationary electromagnetic fields and electromagnetic wave phenomena.

The principle of least action

A mathematical formalism common in several fields of physics derives the laws of certain phenomena from the principle of least action or its special form, the Hamilton principle. According to this theory, phenomena proceed in a way that the action-integral (or functional) formulated by the extensive parameters has extremal value. The necessary condition for this is the zero value of the first variation of the action integral. The first variation can only vanish if the

Euler—Lagrange set of differential equations corresponding to the action integral is satisfied. This means the satisfaction of the differential equations characterizing the phenomenon by means of its extensive parameters. So the action integral considered as the functional of the extensive variables is stationarized by the function satisfying Maxwell's equations.

The action integral of the joint set of the electromagnetic field and the moving particles has three parts: the first and the second depend on the properties of the field and the particles; the third depends on the interaction of the field and the particles. Since convective current is not discussed, the part depending on the properties of the particles only is missing. So the action integral is:

$$I_a = -j/c \int_{\Omega} \left[A^+ S + \frac{1}{4\mu} \, Tr(\mathbf{F}^2) \right] d\Omega \tag{1}$$

where I_a is the action, S is the current vector of four dimensions, $\mathbf{F}$ is the tensor of the electromagnetic field intensities, c is light velocity and j is the imaginary unit, mark $+$ denotes the transpose of a matrix, and $Tr(\mathbf{F}^2)$ means the trace of tensor $\mathbf{F}^2$. The integration has to be performed over the four-dimensional region, where the elementary domain is:

$$d\Omega = dx_1 \, dx_2 \, dx_3 \, dx_4 \, . \tag{2}$$

Introducing the four-dimensional vector potential,

$$\mathbf{F} = \operatorname{rot} A \, , \tag{3}$$

the action integral as a function of the vector potential can be written in the following form [1, 13]:

$$I_a = -j/c \int_{\Omega} \left[A^+ S + \frac{1}{4\mu} \, Tr(\operatorname{rot}^2 A) \right] d\Omega \, . \tag{4}$$

If the current density is not known, equation

$$S = \sigma \mathbf{F} \boldsymbol{u} = \sigma \operatorname{rot} A \, \boldsymbol{u} \tag{5}$$

may be used. Here σ is the conductivity of the media, $\boldsymbol{u}$ is the four-dimensional vector of the velocities (see Appendix). Substitution of (5) into (4) yields the functional for this case. However, the first part of the action integral should be divided by two, since otherwise the interaction of the field and the particles would be considered twice. So the action integral is:

$$I_a = -j/c \int_\Omega \left[\frac{1}{2} \sigma A^+ \operatorname{rot} A\, u + \frac{1}{4\mu} (Tr(\operatorname{rot}^2 A)) \right] d\Omega . \tag{6}$$

Performing the scalar product operation, the action integrals (1) and (6) may be written in the following three-dimensional form:

$$I_a = \int_0^t \int_V \left[\bar{\mathbf{A}}\bar{\mathbf{J}} - \rho\varphi + \frac{1}{2} (\bar{\mathbf{E}}\bar{\mathbf{D}} - \bar{\mathbf{H}}\bar{\mathbf{B}}) \right] dV d\tau , \tag{7}$$

and

$$I_a = \int_0^t \int_V \frac{1}{2} [\sigma \bar{\mathbf{A}}\bar{\mathbf{E}} + (\bar{\mathbf{E}}\bar{\mathbf{D}} - \bar{\mathbf{H}}\bar{\mathbf{B}})]\, dV d\tau . \tag{8}$$

Using the formulas:

$$\bar{\mathbf{B}} = \operatorname{rot} \bar{\mathbf{A}} , \tag{9}$$

$$\bar{\mathbf{E}} = -\operatorname{grad} \varphi - \frac{\partial \bar{\mathbf{A}}}{\partial t} , \tag{10}$$

we obtain:

$$I_a = \int_0^t \int_V \left[\bar{\mathbf{A}}\bar{\mathbf{J}} - \rho\varphi + \frac{\varepsilon}{2} \left(-\operatorname{grad} \varphi - \frac{\partial \bar{\mathbf{A}}}{\partial \tau} \right)^2 - \frac{1}{2\mu} \operatorname{rot}^2 \bar{\mathbf{A}} \right] dV d\tau , \tag{11}$$

and

$$I_a = \int_0^t \int_V \frac{1}{2} \left[\sigma \bar{\mathbf{A}} \left(-\operatorname{grad} \varphi - \frac{\partial \bar{\mathbf{A}}}{\partial \tau} \right) + \varepsilon \left(-\operatorname{grad} \varphi - \frac{\partial \bar{\mathbf{A}}}{\partial \tau} \right)^2 - \frac{1}{\mu} \operatorname{rot}^2 \bar{\mathbf{A}} \right] dV d\tau . \tag{12}$$

The action integral is seen to be a time integral of an energy-like function:

$$I_a = \int_0^t W(\tau)\, d\tau . \tag{13}$$

The energy-like function is:

$$W(\tau)=\int_v L dV, \tag{14}$$

where L is the Lagrange function of the field. The Lagrange function is the integrand of integrals (11) and (12). Also $W(\tau)$ is seen to be the difference of the electric and the magnetic energies.

In the following, the stationary function of the action integral as the functional of the four-dimensional vector potential will be proven to satisfy the Maxwell equations.

The Euler—Lagrange differential equation of the action integral

As it is known, if the first variation of a functional equals zero, the Euler—Lagrange differential equation of the functional is satisfied. The Euler—Lagrange differential equation of functionals (11) and (6) can be written on the basis of the following formula:

$$\frac{\partial L}{\partial A_i}-\sum_{j=1}^{4}\frac{\partial}{\partial x_j}\frac{\partial L}{\partial\left(\frac{\partial A_i}{\partial x_j}\right)}=0 \qquad i=1, 2, 3, 4. \tag{15}$$

The Lagrangian functions are, according to (1) and (6):

$$L=-j/c\left[A\,S+\frac{1}{4\mu}(Tr(\mathrm{rot}^2 A))\right], \tag{16}$$

$$L=-j/c\left[\frac{1}{2}\sigma A^{+}\,\mathrm{rot}\,A\,u+\frac{1}{4\mu}(Tr(\mathrm{rot}^2 A))\right], \tag{17}$$

respectively.

It is to be proved that the Euler—Lagrange differential equations of functionals (6) and (1) are the differential equations for the four-dimensional vector potential derived from Maxwell's equations. In order to obtain this formula, (15) has to be reformulated by means of four-dimensional vectors:

$$\frac{\partial L}{\partial A}-\frac{\partial L}{\partial(A\nabla^{+})}\nabla=0 \tag{18}$$

(∇ is the four-dimensional differential operator, and laws of derivatives on matrices and column vectors are found in the Appendix.) On application to Lagrange function (16):

$$\frac{\partial L}{\partial A} = -j/cS\,, \tag{19}$$

and

$$\frac{\partial L}{\partial(A\nabla^+)} = -\frac{j}{c}\frac{1}{\mu}(\nabla A^+ - A\nabla^+) = -\frac{j}{c\mu}\operatorname{rot} A\,. \tag{20}$$

Thus

$$S + \frac{1}{\mu}\operatorname{div}\operatorname{rot} A = 0\,. \tag{21}$$

Using the identity:

$$\operatorname{div}\operatorname{rot} A = \operatorname{grad}\operatorname{div} A - \Box A\,, \tag{22}$$

we get:

$$\Box A = -\mu S\,, \tag{23}$$

and

$$\operatorname{div} A = 0\,. \tag{24}$$

(Mark $\Box$ denotes the four-dimensional Laplace operator (see Appendix).) Eq. (23) is the differential equation for the vector potential derived from Maxwell's equations, and Eq. (24) is the Lorentz condition.

If current density is not known, Lagrange function (17) is to be used. In that case the equations are:

$$\frac{\partial L}{\partial A} = -\frac{j}{2c}\,\sigma \operatorname{rot} A\, u\,, \tag{25}$$

$$\frac{\partial L}{\partial(A\nabla^+)} = -\frac{j}{c}\left[-\frac{1}{2}\sigma A\nabla^+ u + \frac{1}{\mu}(\nabla A^+ - A\nabla^+)\right]. \tag{26}$$

So the differential equations are:

$$\Box A + \mu\sigma A\nabla^+ u = 0\,, \tag{27}$$

and

$$A^+\nabla = \sigma\mu A^+ u\,. \tag{28}$$

Eq. (27) is the differential equation for the vector potential derived from Maxwell's equations, and Eq. (28) is the Lorentz condition.

Action integrals of electrodynamics

In the following, the three-dimensional action integrals (11) and (12) will be discussed in several cases of electrodynamics.

1. Electrostatics

In the case of static electric field there is no magnetic field and the field intensities are independent of time. So the minimum of action integral means the minimum of energy function (14). Using the equations:

$$\bar{\mathbf{H}}=\bar{\mathbf{0}}, \qquad \bar{\mathbf{J}}=\bar{\mathbf{0}}, \qquad \frac{\partial}{\partial t}=0, \quad \text{and}$$

$$\bar{\mathbf{E}}=-\operatorname{grad}\varphi\,, \tag{29}$$

the energy-like functional of the electric field to be minimised is:

$$W=\frac{1}{2}\int_V \varepsilon\operatorname{grad}^2\varphi dV-\int_V \rho\varphi dV, \tag{30}$$

where φ is the scalar potential.

2. Magnetostatics

A duality is known to exist between electric and magnetic fields [7, 13]. So the equations of electric field are valid for magnetic field, too. Using the equation

$$\bar{\mathbf{H}}=-\operatorname{grad}\varphi_m\,, \tag{31}$$

(where φ_m is the magnetic scalar potential) and $\bar{\mathbf{E}}=\bar{\mathbf{0}}$, $\frac{\partial}{\partial t}=0$, the magnetic field energy-like functional to be extremized is:

$$\mathrm{W}_m=-\int_V \frac{1}{2}\mu\operatorname{grad}^2\varphi_m dV. \tag{32}$$

3. *Stationary electric field*

Stationary electric fields can be discussed on the basis of the analogy to static field. Therefore $\rho=0$, and replacing ε by σ, functional (30) is of the following form:

$$W=\frac{1}{2}\int_V \sigma\,\mathrm{grad}^2\varphi\,dV. \tag{33}$$

4. *Stationary magnetic field*

Since the electric field is omitted, $\varphi=0$ and $\frac{\partial}{\partial t}=0$. But the interaction of electric and magnetic fields has to be taken into account. So the functional is:

$$W=-\frac{1}{2}\int_V \frac{1}{\mu}\mathrm{rot}^2\,\bar{\mathrm{A}}\,dV+\int_V \bar{\mathrm{A}}\bar{\mathrm{J}}\,dV. \tag{34}$$

5. *Quasi-stationary electromagnetic field*

In the case of static and stationary electromagnetic field the scalar and vector potentials are independent, so they can be determined independently. (Except stationary magnetic fields when the current density is not known. In that case the electric field can be determined at first, and in the knowledge of the electric field, the magnetic field can be calculated.) Time-varying electric and magnetic fields are not independent of each other, so it is necessary to know both the scalar and the vector potential for the solution. It is sufficient to know the vector potential for the solution in the case of quasistationary field or wave phenomenon in an ideal isolator. Both the electric and the magnetic field can be derived from the vector potential, since the scalar potential may be termed by Lorentz condition (28):

$$\varphi=-\frac{1}{\mu\sigma}\,\mathrm{div}\bar{\mathrm{A}}\,. \tag{35}$$

This means, that if $\rho=0$, φ is arbitrary, so φ can be zero. Thus the action integral is:

$$I_a = \int_0^t \int_V \left[-\frac{1}{2} \sigma \bar{\mathrm{A}} + \frac{\partial \bar{\mathrm{A}}}{\partial \tau} - \frac{1}{2\mu} \mathrm{rot}^2\, \bar{\mathrm{A}} \right] dV d\tau\,. \tag{36}$$

If the assumptions of the problem cannot be met in this way, using formula (35), the action integral is:

$$I_a = \int_0^t \int_V \left[\frac{1}{2} \sigma \bar{\mathrm{A}} + \left(\frac{1}{\mu\sigma} \,\mathrm{grad\,div}\,\bar{\mathrm{A}} - \frac{\partial \bar{\mathrm{A}}}{\partial \mathrm{t}} \right) - \frac{1}{2\mu} \mathrm{rot}^2\, \bar{\mathrm{A}} \right] dV d\tau\,. \tag{37}$$

6. *Electromagnetic wave phenomena*

In the case of electromagnetic waves in an ideal isolator, the action integral can be derived from Eq. (12), substituting $\varphi = 0$ and $\sigma = 0$ into the equation:

$$I_a = \int_0^t \int_V \left[\frac{\varepsilon}{2} \left(\frac{\partial \bar{\mathrm{A}}}{\partial \tau} \right)^2 - \frac{1}{2\mu} \mathrm{rot}^2\, \bar{\mathrm{A}} \right] dV d\tau\,. \tag{38}$$

The most general form of the action integral has to be used in the case where the assumption $\varphi = 0$ contradicts the conditions of the problem or the isolator is not ideal. In this case the vector potential $\bar{\mathrm{A}}$ and the scalar potential φ cannot be determined independently. Therefore, in the case of not ideal isolator, the action integral is:

$$I_a = \int_0^t \int_V \left[-\frac{\sigma}{2} \bar{\mathrm{A}} \frac{\partial \bar{\mathrm{A}}}{\partial \tau} - \frac{1}{2} \sigma \bar{\mathrm{A}}\, \mathrm{grad}\, \varphi + \frac{\varepsilon}{2} \left(\frac{\partial \bar{\mathrm{A}}}{\partial \tau} \right)^2 + \right.$$

$$\left. + \varepsilon \frac{\partial \bar{\mathrm{A}}}{\partial \tau} \mathrm{grad}\, \varphi + \frac{\varepsilon}{2} \mathrm{grad}^2\, \varphi - \frac{1}{2\mu} \mathrm{rot}^2\, \bar{\mathrm{A}} \right] dV d\tau\,,$$

and the Lorentz condition

$$\mathrm{div}\, \bar{\mathrm{A}} = -\mu \sigma \varphi - \mu \varepsilon \frac{\partial \varphi}{\partial \tau} \tag{40}$$

is valid.

Appendix

In the following the formulation and the solution of Maxwell equations are discussed by means of four-dimensional mathematical formalism. First of all, the three-dimensional form of Maxwell equations are considered:

$$\operatorname{rot}\bar{\mathbf{H}}=\bar{\mathbf{J}}+\frac{\partial\bar{\mathbf{D}}}{\partial t} \tag{F.1}$$

$$\operatorname{rot}\bar{\mathbf{E}}=-\frac{\partial\bar{\mathbf{B}}}{\partial t} \tag{F.2}$$

$$\operatorname{div}\bar{\mathbf{D}}=\rho \tag{F.3}$$

$$\operatorname{div}\bar{\mathbf{B}}=0 \tag{F.4}$$

$$\bar{\mathbf{D}}=\varepsilon\bar{\mathbf{E}}\ ;\quad \bar{\mathbf{B}}=\mu\bar{\mathbf{H}}\ ;\quad \bar{\mathbf{J}}=\sigma\bar{\mathbf{E}}\ ;\quad \varepsilon=\varepsilon_0\varepsilon_r\ ;\quad \mu=\mu_0\mu_r \tag{F.5}$$

(the inserted electric field intensity $\bar{\mathbf{E}}_i$ being zero.)

The four-dimensional form of the equations is:

$$\operatorname{div}\mathbb{F}=\mu S \tag{F.6}$$

$$\operatorname{div}\mathbb{P}=0 \tag{F.7}$$

$$S=\sigma\mathbb{F}u \tag{F.8}$$

$$\mathbb{P}=-j\sqrt{\frac{\varepsilon}{\mu}}\,\mathbb{F}^*, \tag{F.9}$$

where

$$\mathbb{F}=\begin{bmatrix} 0 & B_z & -B_y & -j/cE_x \\ -B_z & 0 & B_x & -j/cE_y \\ B_y & -B_x & 0 & -j/cE_z \\ j/cE_x & j/cE_y & j/cE_z & 0 \end{bmatrix} \tag{F.10}$$

$$S=\begin{bmatrix} \bar{\mathbf{J}} \\ jc\rho \end{bmatrix}, \tag{F.11}$$

$$u=\begin{bmatrix} \kappa\bar{\mathbf{v}} \\ j\kappa c \end{bmatrix}; \qquad \kappa=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}; \tag{F.12}$$

$\bar{\mathbf{v}}$ is the velocity vector of charges, c is light velocity.

Let us introduce the four-dimensional vector potential:

$$A=\begin{bmatrix} \bar{\mathbf{A}} \\ j/c\varphi \end{bmatrix}, \tag{F.13}$$

where $\bar{\mathbf{A}}$ is the three-dimensional vector potential and φ is the scalar potential.

Introducing

$$\mathbb{F}=\text{rot}\, A\,, \tag{F.14}$$

the four-dimensional differential equation

$$\Box A=-\mu S\,, \tag{F.15}$$

and the Lorentz condition

$$\text{div}\, A=0 \tag{F.16}$$

is derived.

If current density is not known, using Eq. (F.8) we get:

$$\Box A-\mu\sigma A\,\nabla^{+}u=0\,, \tag{F.17}$$

$$A^{+}\nabla=\sigma\mu A^{+}u\,. \tag{F.18}$$

The most important definitions and operations relating to four-dimensional vectors and matrices are summarized in the following.

In an alternating or antimetric matrix, the elements symmetrically arranged with respect to the main diagonal differ only by sign:

$$V_{ik}=-V_{ki}\,.$$

Hence,

$$V_{kk}=0\,.$$

A matrix $\mathbb{V}^*$ can be assigned to an alternating tensor $\mathbb{V}$ with elements:

$$V^*_{kl}=V_{mn}\,.$$

Here k, l, m, n represent the even permutations of the numbers 1, 2, 3, 4. $\mathbf{V}^*$ constructed in this way is the dual matrix of matrix $\mathbf{V}$.

Thus if

$$\mathbf{V} = \begin{bmatrix} 0 & V_{12} & V_{13} & V_{14} \\ -V_{12} & 0 & V_{23} & V_{24} \\ -V_{13} & -V_{23} & 0 & V_{34} \\ -V_{14} & -V_{24} & -V_{34} & 0 \end{bmatrix}$$

then the dual matrix is:

$$\mathbf{V}^* = \begin{bmatrix} 0 & V_{34} & -V_{24} & V_{23} \\ -V_{34} & 0 & V_{14} & -V_{13} \\ V_{24} & -V_{14} & 0 & V_{12} \\ -V_{23} & V_{13} & -V_{12} & 0 \end{bmatrix}$$

The four-dimensional Hamilton's operator is given by

$$v = \begin{bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \\ \dfrac{\partial}{\partial X_4} \end{bmatrix}$$

The four dimensional Laplace operator is:

$$\square = \nabla^{+} \nabla = \frac{\partial}{\partial X_1^2} + \frac{\partial}{\partial X_2^2} + \frac{\partial}{\partial X_3^2} + \frac{\partial}{\partial X_4^2}$$

(∇^{+} denotes the transpose of ∇).

The operations on the operators are controlled by the formal rules of matrix algebra:

$$\operatorname{grad} S=\nabla S \qquad (S \text{ is scalar})$$

$$\operatorname{div} V=\nabla^{+} V=V^{+}\nabla$$

$$\operatorname{rot} V=\nabla V^{+}-V\nabla^{+}$$

$$\operatorname{grad} V=V\nabla^{+}$$

$$\operatorname{div} \mathbf{T}=\mathbf{T}\nabla=[\nabla^{+}\mathbf{T}^{+}]^{+}$$

$$\operatorname{div}\operatorname{grad} S=\nabla^{+}\nabla S=\Box S$$

$$\operatorname{div}\operatorname{grad} V=V\nabla^{+}\nabla=V\Box$$

$$\operatorname{div}\operatorname{rot} V=[\nabla V^{+}-V\nabla^{+}]\nabla=\nabla V^{+}\nabla-V\nabla^{+}\nabla=\operatorname{grad}\operatorname{div} V-V\Box .$$

$$\operatorname{rot}\mathbb{V}=-\operatorname{div} V^{*}=-\mathbb{V}\nabla \qquad (\text{if } \mathbb{V} \text{ is an alternating matrix}).$$

The derivative of a scalar by a column matrix is:

$$\frac{\partial S}{\partial X}=\begin{bmatrix} \dfrac{\partial S}{\partial X_1} \\ \dfrac{\partial S}{\partial X_2} \\ \dfrac{\partial S}{\partial X_3} \\ \dfrac{\partial S}{\partial X_4} \end{bmatrix}, \quad \text{where} \quad X=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

The derivative of a scalar by a matrix is:

$$\frac{\partial S}{\partial \mathbb{X}}=\begin{bmatrix} \dfrac{S}{\partial X_{11}} & \dfrac{\partial S}{\partial X_{12}} & \cdots & \dfrac{\partial S}{\partial X_{14}} \\ \dfrac{\partial S}{\partial X_{21}} & \cdots & \cdots & \dfrac{\partial S}{\partial X_{24}} \\ \vdots & & & \\ \dfrac{\partial S}{\partial X_{41}} & \cdots & \cdots & \dfrac{\partial S}{\partial X_{44}} \end{bmatrix}$$

where

$$\mathbb{X}=\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{14} \\ X_{21} & \cdots & \cdots & X_{24} \\ \vdots & & & \\ X_{41} & \cdots & \cdots & X_{44} \end{bmatrix}$$

Summary

Functionals used in variational calculus of electrodynamics by means of the least action principle are formulated. The four dimensional mathematical formalism of the relativistic electrodynamics is used for discussion. Functionals for electrostatics, magnetostatics, stationary electric and magnetic fields, quasistationary electromagnetic fields and electromagnetic wave phenomena are derived from the general principle of least action.

References

1. Landau, L. D.—Lifsic, E. M.: Theoretical Physics (In Hungarian) Vol. II. Tankönyvkiadó, Budapest, 1976.
2. Gombás, P.—Kisdi, D.: Introduction to Theoretical Physics. (In Hungarian) Akadémiai Kiadó, Budapest, 1971.
3. Fényes, I.: Modern fizikai kisenciklopédia (In Hungarian). Gondolat Könyvkiadó, Budapest, 1971.
4. Kompaneyets, A. S.: Theoretical physics. MIR PUBLISHERS, Moscow, 1965.
5. Panofsky, W.—Phillips, M.: Classical electricity and magnetism. Addison--Wesley Publishing Company, Inc. 1955.

6. MOON, P.—SPENCER, D. E.: Foundation of electrodynamics. Boston Technical Publishers, Inc. 1965.
7. FODOR, GY.—VÁGÓ, I.: Electricity (In Hungarian) Booklet 12. Tankönyvkiadó, Budapest, 1975.
8. KANTOROVICS, L. V.—KRILOV, W. J.: Priblishenie Metodi Vyshewo Analiso (In Russian). Moscow, 1962.
9. MIKHLIN, S. G.—SMOLITSKIY, K. L.: Approximate methods for solution of differential and integral equations. New York, Elsevier, 1967.
10. MIKHLIN, S. G.: Variational methods in mathematical physics. New York, Macmillan, 1964.
11. FRANK, R.—MISES, PH.: A mechanika és fizika differenciál- és integrálegyenletei (In Hungarian) Vol. I, II. Műszaki Könyvkiadó, Budapest, 1966.
12. BÁRDI, I.—BÍRÓ, O.—GYIMESI, M.: Calculation of static and stationary fields in inhomogeneous, anisotropic media by the variational method. Periodica Polytechnica, Electrical Engineering, Vol. 20. No. 2. Budapest, 1976.
13. BÁRDI, I.: Solution of Maxwell's equations by relativistic electric and magnetic potentials. Periodica Polytechnica, Electrical Engineering Vol. 19. No. 3. Budapest, 1975.
14. BÁRDI, I.: Contribution to the variational methods for solving static and stationary electromagnetic field problems. Periodica Polytechnica, Electrical Engineering Vol. 21. No. 2. Budapest, 1977.

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