DETERMINATION OF STATIC AND STATIONARY ELECTROMAGNETIC FIELDS BY USING VARIATION CALCULUS

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1. Introduction

The determinations of static and stationary electric or magnetic fields represent mathematically similar problems. During the following examinations the substance filling the field is supposed to be homogeneous, and it is supposed that there is no space charge in the case of a static field ($\rho = 0$), or no current in the case of stationary magnetic field ($\mathbf{J} = 0$) in the examined closed region. By introducing scalar potential $\Phi(x, y, z)$, the solution of Laplace equation

$$\Delta\Phi(x, y, z) = 0 \tag{1}$$

satisfying boundary conditions is required for the determination of field strengths $\mathbf{E}$ and $\mathbf{H}$, characterizing the electric and magnetic fields, respectively. In the case of planar problems the solution of the two-variable Laplace equation

$$\Delta\Phi(x, y) = 0 \tag{2}$$

under given boundary conditions is required. In the followings the solution of planar problems of Dirichlet or homogeneous Neumann type boundary conditions will only be discussed.

Variation calculus can be applied for the solution of partial differential equations of second order satisfying given boundary conditions [1]. In the closed planar region Ω the solution of Laplace equation (2) will be the scalar potential function $\Phi(x, y)$ minimizing the integral

$$I = \frac{1}{2} \iint_{\Omega} \operatorname{grad}^2 \Phi(x, y)\, d\Omega \tag{3}$$

and satisfying the Dirichlet type boundary conditions [2], [3].

Solving Eq. (2) by the variational method, the potential function obtained numerically can be written in analytic form. The following method permits the exact satisfaction of Dirichlet type boundary conditions by the solution. The satisfaction of homogeneous Neumann type boundary conditions is obtained as the natural boundary condition of integral (3). The method suits also in the case of concave and convex planar regions [6], if the boundary curves can be written analytically, at least by sections.

2. Satisfying Dirichlet type boundary conditions

In the case of a Dirichlet type boundary condition the value of the two-variable potential function $\Phi(x, y)$ satisfying Laplace equation (2) will be

$$\Phi(x, y)|_{\Gamma_i} = \Phi_i, \qquad i = 1, 2, \ldots m, \tag{4}$$

prescribed on curves Γ_i, $i = 1, 2, \ldots m$, bordering the closed planar region Ω (Fig. 1). Let us write potencial $\Phi(x, y)$ as the sum of two terms:

$$\Phi((x, y) = \Phi_\delta (x, y) + \Phi_\alpha (x, y), \tag{5}$$

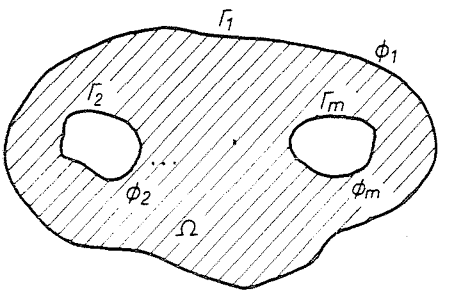

Fig. 1. The examined regions

where $\Phi_\delta(x, y)$ is an arbitrary function, continuous in the examined closed planar region, at least twice differentiable and satisfying the prescribed boundary condition (4) at the border of the planar region:

$$\Phi_\delta(x, y)|_{\Gamma_i} = \Phi_i, \qquad i = 1, 2, \ldots m, \tag{6}$$

$\Phi_\alpha(x, y)$ in turn has a zero value at the boundary curve Γ_i, $i = 1, 2, \ldots, m$ of the examined planar region Ω,

$$\Phi_\alpha(x, y)|_{\Gamma_i} = 0, \qquad i = 1, 2, \ldots, m. \tag{7}$$

Thus $\Phi(x, y)$ satisfies boundary condition (4). (7) is satisfied if it is of the form

$$\Phi_{\alpha}(x, y) = w_D(x, y)\,\Phi_{\Sigma}(x, y). \tag{8}$$

Here $w_D(x, y)$ is an arbitrary function positive and continuous in the examined closed region Ω, at least twice differentiable and having zero value at the boundary Γ_i, $i = 1, 2, \ldots, m$ of the region:

$$w_D(x, y)|_{\Gamma_i} = 0, \qquad i = 1, 2, \ldots, m. \tag{9}$$

$\Phi_{\Sigma}(x, y)$ is an approximative series of functions, consisting of n terms,

$$\Phi_{\Sigma}(x, y) = \sum_{k=1}^{n} a_k f_k(x, y) \tag{10}$$

where $f_k(x, y)$ is the k-th element of a complete system of functions defined in the planar region, and a_k is the unknown coefficient.

By using the Ritz's method [3], the coefficients can be determined from the following system of linear equations [1], [2], [3]:

$$\mathbf{A}\mathbf{a} + \mathbf{b} = \mathbf{0}. \tag{11}$$

Here A is a quadratic matrix of n-th order, the elements of which are

$$A_{kl} = \iint_{\Omega} \operatorname{grad}\,[w_D(x, y) f_k(x, y)] \cdot \operatorname{grad}\,[w_D(x, y) f_l(x, y)]\, d\Omega\,, \tag{12}$$

$\mathbf{b}$ is a column vector of n elements, with

$$b_k = \iint_{\Omega} \operatorname{grad}\,\Phi_{\delta}(x, y) \cdot \operatorname{grad}\,[w_D(x, y) f_k(x, y)]\, d\Omega\,. \tag{13}$$

$\mathbf{a}$ is the n-element column vector of unknown coefficients. System of equations (11) can be solved if $\Phi_{\delta}(x, y)$ and $w_D(x, y)$ are known. To construct these functions the method by Rwatchew [4], [5] for the description of planar regions has been applied.

3. Description of planar regions

Some open subregion of an infinite planar region is Ω^*, the boundary curve of the region Γ, and the union

$$\Omega = \Omega^* \cup \Gamma \tag{14}$$

is the closed planar region, which can be considered as the set of all points inside the region or on its boundary. The closed planar region Ω can be described by the following continuous and at least twice differentiable function:

$$w(x, y) \quad \begin{aligned} &>0, \quad \text{if} \quad (x, y) \in \Omega^*, \\ &=0, \quad \text{if} \quad (x, y) \in \Gamma, \\ &<0, \quad \text{if} \quad (x, y) \notin \Omega. \end{aligned} \tag{15}$$

Rwatchew composed the examined region as the union (unification) or intersection of subregions and as their complementary regions. Function $w(x, y)$ satisfying condition (15) is given analytically by using the functions describing the subregions as follows. Let function $w_1(x, y)$ describe region Ω_1 according to (15), and function $w_2(x, y)$ the region Ω_2. The union

$$\Omega = \Omega_1 \cup \Omega_2 \tag{16}$$

of planar region Ω_1 and Ω_2 (Fig. 2.a) is described by the function

$$\begin{aligned} w(x, y) &= w_1(x, y) \vee w_2(x, y) = \\ &= w_1(x, y) + w_2(x, y) + \sqrt{w_1^2(x, y) + w_2^2(x, y)}. \end{aligned} \tag{17}$$

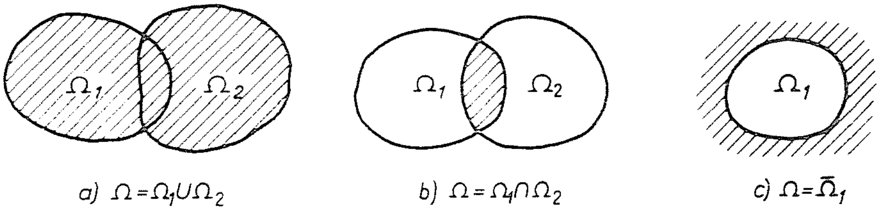

Fig. 2. Operations taken from the theory of sets

To the intersection

$$\Omega = \Omega_1 \cap \Omega_2 \tag{18}$$

of planar region Ω_1 and Ω_2 (Fig. 2.b) the function

$$\begin{aligned} w(x, y) &= w_1(x, y) \wedge w_2(x, y) = \\ &= w_1(x, y) + w_2(x, y) - \sqrt{w_1^2(x, y) + w_2^2} \end{aligned} \tag{19}$$

can be ordered. The complementary in relation to the complete infinite plane (Fig. 2.c)

$$\Omega = \bar{\Omega}_1 \tag{20}$$

of region Ω_1 is described by the function

$$w(x, y) = \neg\, w_1(x, y) = -w_1(x, y). \tag{21}$$

Functions (17), (19), and (21) describe regions (16), (18), and (20), respectively, in the way defined in (15). These functions are continuous and at least twice differentiable with the exception of the simultaneous satisfaction of $w_1(x, y) = 0$ and $w_2(x, y) = 0$. This case, however, may occur only at the boundary of the region, where the derivative of the function from inside the region can, in general, be interpreted.

The above considerations permit the construction of functions satisfying conditions prescribed by Eqs (6) and (9). $w_D(x, y)$ satisfies condition (9), if

$$w_D(x, y) = w(x, y), \tag{22}$$

where $w(x, y)$ is the function describing region Ω. The potential function of the form

$$\Phi_\delta(x, y) = \frac{\sum\limits_{i=1}^{m} \tau_i(x, y)\Phi_i}{\sum\limits_{i=1}^{m} \tau_i(x, y)} \tag{23}$$

satisfies boundary condition (6), if $\tau_i(x, y)$, $i = 1, 2, \ldots, m$ is a positive, continuous, at least twice differentiable function everywhere inside the examined region Ω. At the border of region Ω, at the i-th electrode, $i = 1, 2, \ldots, m$, it has a positive non-zero value, while on the other electrodes its value is zero,

$$\tau_i(x, y) \begin{cases} \neq 0, & \text{if} \quad (x, y) \in \Gamma_i, \\ = 0, & \text{if} \quad (x, y) \in \Gamma_j, \end{cases} \qquad \text{if} \quad \begin{array}{l} i = 1, 2, \ldots, m, \\ j = 1, 2, \ldots, m, j \neq i. \end{array} \tag{24}$$

For the construction of functions $\tau_i(x, y)$, $i = 1, 2, \ldots, m$ let us select regions Ω_i, $i = 1, 2, \ldots, m$ as follows. Let the curve Γ_i, $i = 1, 2, \ldots, m$ of the i-th electrode be part of the boundary of the region Ω_i, $i = 1, 2, \ldots, m$ including the examined region Ω. Subregions Ω_i, $i = 1, 2, \ldots, m$ are described by functions $w_i(x, y)$ as defined in (15). In this case

$$\begin{aligned} \tau_i(x, y) &= w_1(x, y) \wedge w_2(x, y) \wedge \ldots \wedge w_{i-1}(x, y) \wedge \\ &\wedge w_{i+1}(x, y) \wedge \ldots \wedge w_m(x, y) = \\ &= \bigwedge_{\substack{j=1 \\ j \neq i}}^{m} w_j(x, y), \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{25}$$

4. Example

To demonstrate the application of the described method, let us determine the magnetic field in the case of the arrangement shown in Fig. 3. On the curve

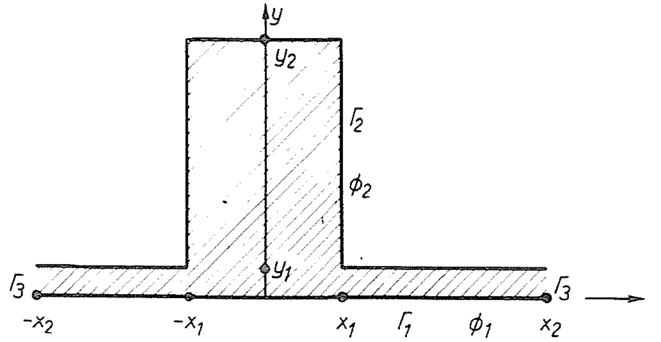

Fig. 3. Planar section of the examined arrangement

sections bordering the region, boundary conditions of Dirichlet and homogeneous Neumann type are prescribed.

$$\left.\begin{aligned}\Phi(x,y)|_{\Gamma_1}&=\Phi_1,\\ \Phi(x,y)|_{\Gamma_2}&=\Phi_2,\end{aligned}\right\} \tag{26}$$

$$\left.\frac{\partial\Phi(x,y)}{\partial n}\right|_{\Gamma_3}=0. \tag{27}$$

The examined region can be composed of subregions pertaining to the electrodes: the 1st eletrode $\Omega_1\,(y\geqq 0)$, the 2nd electrode $\Omega_{21}\,(y_2-y\geqq 0)$, $\Omega_{22}\,(y_1-y\geqq 0)$, $\Omega_{23}\,(x_1+x\geqq 0)$, $\Omega_{24}\,(x_1-x\geqq 0)$. The functions describing the subregions according to (15) are: $w_1(x,y)=y$; $w_{21}(x,y)=y_2-y$; $w_{22}(x,y)=y_1-y$; $w_{23}(x,y)=x_1+x$; $w_{24}(x,y)=x_1-x$. Function $w_D(x,y)$ satisfying condition (9) is found to be:

$$\begin{aligned}w_D(x,y)=((w_{23}(x,y)\wedge w_{24}(x,y))\vee w_{22}(x,y))\wedge\\ \wedge\, w_{21}(x,y)\wedge w_1(x,y).\end{aligned} \tag{28}$$

Function $\Phi_\delta(x,y)$ satisfying the Dirichlet type boundary condition (26) is, according to (23), the following:

$$\Phi_\delta(x,y)=\frac{\Phi_0\tau_2(x,y)}{\tau_1(x,y)+\tau_2(x,y)}, \tag{29}$$

where

$$\tau_1(x,y)=((w_{23}(x,y)\wedge w_{24}(x,y))\vee w_{22}(x,y))\wedge w_{21}(x,y) \tag{30}$$

and

$$\tau_2(x,y)=w_1(x,y), \tag{31}$$

in accordance with the arrangement shown in Fig. 4. According to (27) curve Γ_3 is a line of force, satisfied in consequence of the natural boundary condition of the variation method. Elements of the approximating series of functions (10)

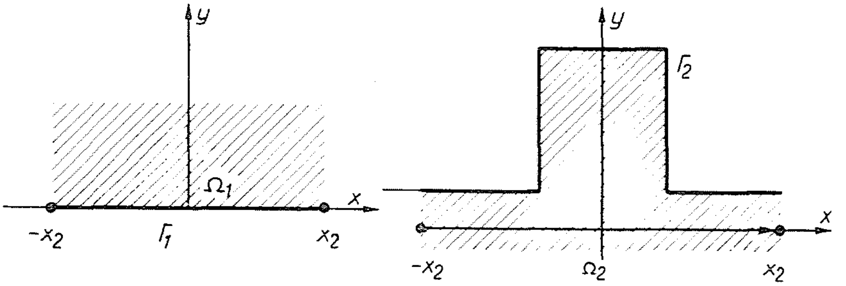

Fig. 4. Partial regions

were selected from the sphere of Tchebysheff polynomials. The k-th element of this is found to be

$$f_k(x, y) = T_i(x/x_2)\, T_j(y/y_2)\,, \qquad (32)$$

$$\text{if} \quad i = 0, 1, 2, \ldots, N_1\,, \quad j = 0, 1, 2, \ldots, N_2\,, \quad k = 1, 2, \ldots, (N_1+1)\cdot(N_2+1)\,,$$

where $T_i(x/x_2)$ denotes the i-th Tchebysheff polynomial depending on x, $T_j(y/y_2)$ the j-th Tchebysheff polynomial depending on y. In the case of normed quantities $x_1/y_2 = 0.1$; $x_2/y_2 = 0.24$; $y_1/y_2 = 0.04$; $\Phi_1 = 0$; $\Phi_2 = 1.0$ and in case of

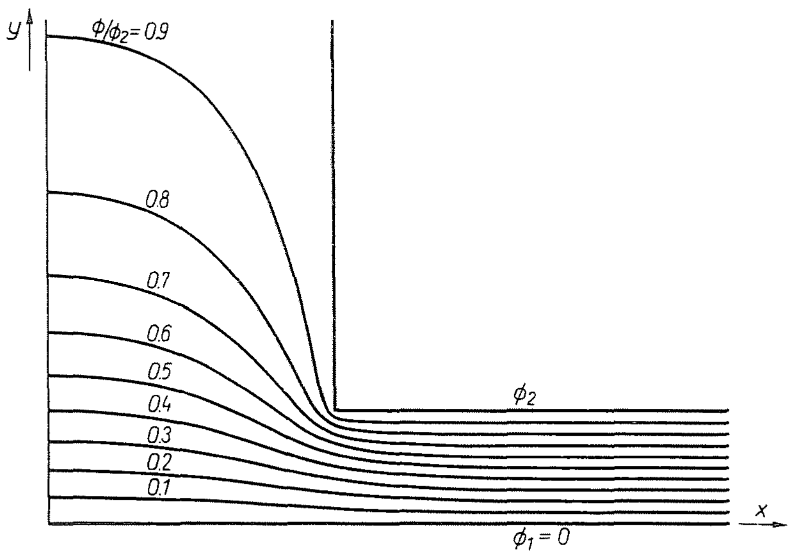

Fig. 5. Equipotential curves

$N_1 = 2$, $N_2 = 2$, utilizing the symmetry of the arrangement ($T_i(x/x_2)$ is an even function of x), approximating by $N = 6$ terms, the coefficients of the approximative functions are found from (11) to be as follows:

$$\mathbf{a} = \begin{bmatrix} 2.6659 \\ -1.3434 \\ -0.3374 \\ -2.8416 \\ 4.4716 \\ -1.5672 \end{bmatrix}$$

Equipotential curves of magnetic scalar potential $\Phi(x, y) = \Phi_\delta(x, y) + w_D(x, y)\, \Phi_\alpha(x, y)$ are shown in Fig. 5.

Summary

In this paper the solution of planar problems of electrostatic and stationary magnetic fields is dealt with. The region limited by electrodes and lines of force is filled by a homogeneous substance. Electrode potentials are given, boundary curves can be described analytically. The solution of the Laplace equation is produced by using variational calculus. At satisfying boundary conditions, the R-functions are used for the description of planar regions. The application of the method for the calculation of the electromagnetic field is illustrated by an example.

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