CALCULATION OF THE DISPERSION FUNCTION OF WAVEGUIDES WITH INHOMOGENEOUS DIELECTRIC BY SERIES EXPANSION

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Introduction

The permittivity and permeability of the dielectric are constant in each region of the cross section of the lossless waveguides with inhomogeneous dielectric. A method will be presented suiting to determine the frequency characteristic of the propagation coefficient in form of power series. STEVENSON [1], [2] calculated the electromagnetic field of scattering problems by means of frequency power series choosing zero frequency as centre of the series expansion. The problem treated now is more complicated because not only the series of the field but also that one of the propagation coefficient have to be determined and in our case the centre of the series expansion can be an arbitrary frequency.

Differential equation and boundary conditions of the eigenvalue problem

Let the z axis of the co-ordinate system be parallel to the direction of the wave propagation and let the unit vector in this direction be denoted by $\mathbf{k}$. In the m-th region of the cross section, denoted by A_m, where the permittivity and permeability are the values ε_m and μ_m, resp., the complex expression of the electric and magnetic field strength is written as:

$$\mathbf{E}_m = (\mathbf{e}_{Tm} + \mathbf{e}_{zm}) \exp(-pz) \tag{1}$$

$$\mathbf{H}_m = (\mathbf{h}_{Tm} + \mathbf{h}_{zm}) \exp(-pz). \tag{2}$$

Here p denotes the propagation coefficient, vectors $\mathbf{e}_{Tm}$ and $\mathbf{h}_{Tm}$ are perpendicular to the z axis, vectors $\mathbf{e}_{zm}$ and $\mathbf{h}_{zm}$ are parallel to it, and all the four vectors depend only on the two co-ordinates in the cross section. If it causes no misunderstanding, the subscript m related to region A_m will be omitted in the following.

The three other vectors can be expressed in terms of vector $\mathbf{e}_T$·

$$\mathbf{e}_z = \frac{\mathbf{k}}{p} \operatorname{div} \mathbf{e}_T \tag{3}$$

$$\mathbf{h}_z = -\frac{1}{j\omega\mu} \operatorname{curl} \mathbf{e}_T \tag{4}$$

$$\begin{aligned} \mathbf{h}_T &= \frac{\mathbf{k}}{p} \times \left(j\omega\varepsilon \mathbf{e}_T + \frac{1}{j\omega\mu} \operatorname{curl} \operatorname{curl} \mathbf{e}_T \right) = \\ &= \frac{\mathbf{k}}{j\omega\mu} \times \left(p\mathbf{e}_T + \frac{1}{p} \operatorname{grad} \operatorname{div} \mathbf{e}_T \right). \end{aligned} \tag{5}$$

Inside every region $\mathbf{e}_T$ satisfies the equation

$$\Delta \mathbf{e}_T + (p^2 + \varepsilon\mu\omega^2)\mathbf{e}_T = 0 \tag{6}$$

and on the boundaries of the regions it satisfies the following boundary conditions. Along the outlines of perfect conductors:

$$\mathbf{n} \times \mathbf{e}_T = 0 \tag{7}$$

$$\operatorname{div} \mathbf{e}_T = 0 \tag{8}$$

and along the outlines separating regions A_m and A_k of the dielectric:

$$\mathbf{n} \times \mathbf{e}_{Tm} = \mathbf{n} \times \mathbf{e}_{Tk} \tag{9}$$

$$\varepsilon_m \mathbf{n} \mathbf{e}_{Tm} = \varepsilon_k \mathbf{n} \mathbf{e}_{Tk} \tag{10}$$

$$\operatorname{div} \mathbf{e}_{Tm} = \operatorname{div} \mathbf{e}_{Tk} \tag{11}$$

$$\frac{1}{\mu_m} \operatorname{curl} \mathbf{e}_{Tm} = \frac{1}{\mu_k} \operatorname{curl} \mathbf{e}_{Tk}\,, \tag{12}$$

where $\mathbf{n}$ denotes the unit vector normal to the outline [3].

Eq. (6) and boundary conditions (7) to (12) define a boundary value problem concerning the vectorial function $\mathbf{e}_T$, in which ω^2 acts as parameter. For all values of ω^2 an infinity of eigenvalues p^2 is obtained as solution of this boundary value problem. So the function $p^2(\omega^2)$ has an infinity of branches, which correspond to several modes. A function $\mathbf{e}_T(\mathbf{r}, \omega^2)$ belongs to every branch of function $p^2(\omega^2)$, i.e., to every mode, and it gives the transversal electric field in dependence on the frequency.

Solution of the eigenvalue problem as frequency power series

The examined branch of functions $\mathbf{e}_T(\mathbf{r}, \omega^2)$ and $\mathbf{p}^2(\omega^2)$ is determined in the neighbourhood of a point ω_0^2 as power series in ω^2. ω_0^2 may be an arbitrary positive value. The dimensionless quantities

$$w = \frac{\omega L}{c} \quad \text{and} \quad w_0 = \frac{\omega_0 L}{c} \tag{13}$$

are suitably introduced for the series expansion, where c is light velocity in vacuum and L is an arbitrary constant of length dimension. The power séries are written as:

$$p^2 = L^{-2} \sum_{i=0}^{\infty} a_i (w^2 - w_0^2)^i \tag{14}$$

$$\mathbf{e}_T = \sum_{i=0}^{\infty} \mathbf{e}_i (w^2 - w_0^2)^i \tag{15}$$

Here coefficients a_i are dimensionless.

Substituting relationships (20) to (22) into Eq. (6), from the coefficients of powers of $(w^2 - w_0^2)$ the following equations arise for vectors $\mathbf{e}_i$ inside the regions:

$$L^2 \Delta \mathbf{e}_0 + (w_0^2 \varepsilon_r \mu_r + a_0) \mathbf{e}_0 = 0 \tag{16}$$

$$L^2 \Delta \mathbf{e}_i + (w_0^2 \varepsilon_r \mu_r + a_0) \mathbf{e}_i = -\varepsilon_r \mu_r \mathbf{e}_{i-1} - \sum_{j=1}^{i} a_j \mathbf{e}_{i-j}, \tag{17}$$

where ε_r and μ_r denote the relative permittivity and permeability, resp., valid in the region. As vector $\mathbf{e}_T$ satisfies the boundary conditions at all frequencies, also vectors $\mathbf{e}_i$ have to satisfy them separately for all values of i.

Eq. (16) and the boundary conditions define a boundary value problem yielding a_0 as an eigenvalue. To determine a_0 is the same as to solve the eigenvalue problem related to the waveguide at frequency ω_0. If the eigenvalue a_0 related to the examined mode is simple, coefficient $\mathbf{e}_0$ of series (15) is uniquely determined by the boundary value problem up to a constant multiplier. This multiplier may be chosen arbitrarily. First the case of such simple eigenvalues is discussed.

Determination of the coefficients of the power series in the case of simple eigenvalues

With knowledge of a_0 and $\mathbf{e}_0$, coefficients a_i and $\mathbf{e}_i$ can be determined successively by recursion procedure. Vector $\mathbf{e}_i$ has to be determined from the boundary value problem defined by Eq. (17) and the boundary conditions. As the homogeneous equivalent of this boundary value problem has a non-trivial solution, the inhomogeneous problem has a solution only if the right-hand side of Eq. (17) satisfies some condition [5]. Coefficient a_i can be determined from this condition.

The condition can be formulated in terms of vector $\mathbf{h}$ which gives the vector $\mathbf{h}_T$ in (2) at frequency ω_0. Vector $\mathbf{h}$ can be determined from the equation:

$$L^2\Delta\mathbf{h}+(w_0^2\varepsilon_r\mu_r+a_0)\mathbf{h}=0 \tag{18}$$

considering the following boundary conditions [3]. Along the outline of the perfect conductor:

$$\mathbf{nh}=0 \tag{19}$$

$$\text{curl}\,\mathbf{h}=0\,. \tag{20}$$

Along the outline separating regions A_m and A_k:

$$\mathbf{n}\times\mathbf{h}_m=\mathbf{n}\times\mathbf{h}_k \tag{21}$$

$$\mu_m\mathbf{nh}_m=\mu_k\mathbf{nh}_k \tag{22}$$

$$\text{div}\,\mathbf{h}_m=\text{div}\,\mathbf{h}_k \tag{23}$$

$$\frac{1}{\varepsilon_m}\,\text{curl}\,\mathbf{h}_m=\frac{1}{\varepsilon_k}\,\text{curl}\,\mathbf{h}_k\,. \tag{24}$$

As the eigenvalue a_0 is simple, $\mathbf{h}$ can be uniquely determined up to a constant multiplier from the boundary value problem above. If the examined mode is not a quasi-TE or quasi-TEM mode, the cut-off frequency of which is just ω_0, i.e. at least one of the quantities a_0 and $\text{div}\,\mathbf{e}_0$ is non-zero, it is simpler to calculate $\mathbf{h}$ from the following relationship derived from (5):

$$\begin{aligned}\mathbf{h}&=\mathbf{k}\times\left(w_0^2\varepsilon_r-\frac{L^2}{\mu_r}\,\text{curl curl}\,\mathbf{e}_0\right)=\\&=\frac{1}{\mu_r}\,(a_0\mathbf{e}_0+L^2\,\text{grad div}\,\mathbf{e}_0)\times\mathbf{k}\,.\end{aligned} \tag{25}$$

It can be proved,—but the proof is omitted here—that a function $\mathbf{e}_i$ satisfying Eq. (17) and the boundary conditions exists only if the condition

$$\iint_A \left[\left(\varepsilon_r\mu_r\mathbf{e}_{i-1}+\sum_{j=1}^{i} a_j\mathbf{e}_{i-j}\right)\times\mathbf{h}\right]\mathrm{d}\mathbf{A}=0 \tag{26}$$

is fulfilled, where the surface A denotes the cross section of the dielectric. Integration over the surface A means, of course, that the integration has to be performed over each region A_m and these integrals have to be summarized. Let us introduce the following notations:

$$u_j=\int_A(\mathbf{e}_j\times\mathbf{h})\mathrm{d}\mathbf{A} \tag{27}$$

$$v_j=\int_A\varepsilon_r\mu_r(\mathbf{e}_j\times\mathbf{h})\mathrm{d}\mathbf{A}\,. \tag{28}$$

If $\mathbf{h}$ can be calculated by means of relationship (25), u_j and v_j can also be given in the following form:

$$u_j=\int_A\left(w_0^2\varepsilon_r\mathbf{e}_0\mathbf{e}_j-\frac{L^2}{\mu_r}\operatorname{curl}\mathbf{e}_0\operatorname{curl}\mathbf{e}_j\right)\mathrm{d}A \tag{29}$$

$$v_j=\int_A\varepsilon_r(L^2\operatorname{div}\mathbf{e}_0\operatorname{div}\mathbf{e}_j-a_0\mathbf{e}_0\mathbf{e}_j)\mathrm{d}A\,. \tag{30}$$

With these notations the value

$$a_i=-\left(v_{i-1}+\sum_{j=1}^{i-1}a_ju_{i-j}\right)\Big/u_0 \tag{31}$$

arises from condition (26) for the coefficient a_i supposing that u_0 is non-zero. The case $u_0=0$ indicates that the function $p^2(\omega^2)$ cannot be expanded in Taylor's series about the point ω_0^2. This occurs if ω_0^2 is a branch point of function $p^2(\omega^2)$ [3].

On the basis of this a recursion procedure can be constructed. At the beginning of the i-th step of the procedure, coefficients $a_0\ldots a_{i-1}$ and $\mathbf{e}_0\ldots\mathbf{e}_{i-1}$ are known. From these a_i can be computed by means of (31) and also $\mathbf{e}_i$ by means of relationships (17) and (7) to (12). It is to be noted here that a similar method can be constructed on the basis of the transversal component of the magnetic field strength. It is more convenient to use if the examined mode is a quasi-TE mode the cut-off frequency of which is just ω_0.

Determination of the coefficients of the power series in the case of multiple eigenvalues

If the eigenvalue a_0 is a multiple one, i.e., if for the same a_0 the boundary value problem concerning function $\mathbf{e}_0$ has several linearly independent solutions, several modes exist, the propagation coefficient of which is the same value $p=\sqrt{a_0}/L$ at frequency ω_0.

Let a_0 be an eigenvalue of multiplicity k. Let us determine some way k linearly independent solutions $\mathbf{e}_{0l}$ of the eigenvalue problem defined by relationships (16) and (7) to (12) and thereafter k linearly independent solutions $\mathbf{h}_l$ of the boundary value problem defined by relationships (18) to (24). The coefficient $\mathbf{e}_0$ belonging to the examined mode can be given as a linear combination of functions $\mathbf{e}_{0l}$:

$$\mathbf{e}_0=\sum_{l=1}^{k} b_{0l}\mathbf{e}_{0l}. \tag{32}$$

Similarly, if $\mathbf{e}_i^*$ is a particular solution satisfying Eq. (17) and the boundary conditions, the coefficient $\mathbf{e}_i$ sought for can be given as:

$$\mathbf{e}_i=\mathbf{e}_i^*+\sum_{l=1}^{k} b_{il}\mathbf{e}_{0l}. \tag{33}$$

Now condition (26) has to be satisfied for all the k functions $\mathbf{h}_l$. These k conditions will be assembled into one matrix equation. To this aim coefficients b_{jl} in (32) and (33) are assembled into a vector $\bar{b}_j$ of k elements. Vectors $\bar{u}_j=[u_{jl}]$; $\bar{u}_j^*=[u_{jl}^*]$ and $\bar{v}_j=[v_{jl}]$ also of k elements and matrices $\bar{\bar{s}}=[s_{lq}]$ and $\bar{\bar{t}}=[t_{lq}]$ of $k\times k$ elements are defined, with elements

$$u_{jl}=\int_A(\mathbf{e}_j\times\mathbf{h}_l)\mathrm{d}\mathbf{A} \tag{34}$$

$$u_{jl}^*=\int_A(\mathbf{e}_j^*\times\mathbf{h}_l)\mathrm{d}\mathbf{A} \tag{35}$$

$$v_{jl}=\int_A\varepsilon_r\mu_r(\mathbf{e}_j\times\mathbf{h}_l)\,\mathrm{d}\mathbf{A} \tag{36}$$

$$s_{lq}=\int_A(\mathbf{e}_{0q}\times\mathbf{h}_l)\mathrm{d}\mathbf{A} \tag{37}$$

$$t_{lq}=\int_A\varepsilon_r\mu_r(\mathbf{e}_{0q}\times\mathbf{h}_l)\mathrm{d}\mathbf{A}. \tag{38}$$

With these notations conditions (26) can be rewritten as:

$$(\bar{\bar{t}}+a_1\bar{\bar{s}})\bar{b}_0=0 \tag{39}$$

$$(\bar{\bar{t}}+a_1\bar{\bar{s}})\bar{b}_i=-\bar{v}_i-a_1\bar{u}_i^*-\sum_{j=2}^{i+1}a_j\bar{u}_{i-j+1}\,. \tag{40}$$

Eq. (39) has non-trivial solution only if the matrix $\bar{\bar{t}}+a_1\bar{\bar{s}}$ is singular. a_1 can be determined from the resulting algebraic equation. Only the case when a_1 is a simple root of the algebraic equation concerning it, is treated here. In this case the respective vector $\bar{b}_0$ and with it the coefficient e_0 can be determined uniquely from Eq. (39) up to a constant multiplier. Also the vector $\bar{c}$ can be determined uniquely from the following equation up to a constant multiplier:

$$\bar{c}^{\mathrm{T}}(\bar{\bar{t}}+a_1\bar{\bar{s}})=0 \tag{41}$$

Eq. (40) has a solution only if its right-hand side is orthogonal to the vector $\bar{c}$. From this the relationship

$$a_{i+1}=-\bar{c}^{\mathrm{T}}(\bar{v}_i+a_1\bar{u}_i^*+\sum_{j=2}^{i}a_j\bar{u}_{i-j+1})/\bar{c}^{\mathrm{T}}\bar{u}_0 \qquad i=1,2\ldots \tag{42}$$

arises for the coefficient a_{i+1}.

On the basis of this the recursion procedure can be constructed. At the beginning of the i-th step of the procedure coefficients $a_0\ldots a_i$ and $e_0\ldots e_{i-1}$ are known. In knowledge of these a particular solution of Eq. (17) can be found, then coefficient a_{i+1} can be determined from (42), vector $\bar{b}_i$ from (41) and with it coefficient e_i.

If a_1 is a multiple root of the relevant equation, generally two or more linearly independent vectors $\bar{b}_0$, which satisfy Eq. (39), can be found. This indicates that there are two or more modes for which both coefficients a_0 and a_1 are equal, i.e. the dispersion curves of which not only intersect but also join at frequency ω_0. These modes cannot be distinguished if not on the basis of coefficient a_2. So besides a_0, a_1 also coefficient a_2 must be known for determining vector $\bar{b}_0$, similarly also coefficient a_{i+2} for determining vector $\bar{b}_i$. As this case does not occur but for special combinations of parameter values, the description of the calculation method to be followed is omitted. It occurs for some symmetrical arrangements that entire dispersion curves of two or more modes are the same, i.e. all coefficients a_i are equal. These modes cannot be distinguished from each other but on the basis of additional view-points. Taking these view-points into consideration the calculation procedure is generally the same as in the case of a simple eigenvalue.

Finally, it is worth mentioning that a method more convenient for the numerical technique can be given for determining the power series if ω_0 is the cut-off frequency of the examined mode. It has been treated in two previous papers [3], [4], illustrated on numerical examples.

Summary

A method has been given for determining the frequency Taylor's series of the square of the propagation coefficient of waveguides if the permittivity is constant in each region of the cross section. First the eigenvalue problem concerning the transversal component of the electric field strength has to be solved at the frequency about which the series expansion is performed. Starting from this the coefficients of the Taylor's series can be computed by means of a recursion procedure involving no more eigenvalue problem.

References

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