THE COVARIANCE FUNCTION OF A STOCHASTIC PROCESS

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To make a mathematical description of the transmission of radio-locator signs through a wide-band micro-wave channel, it is necessary to study the following type of stochastic processes:

$$\xi_t = \left|\cos\left(a - t + \sum_{c=1}^{n} c_i\, \varrho_i \cos\,(w_i t + \varphi_i)\right)\right|$$

where

- t is the time parameter,
- $a,\ c_i,\ w_i\ (i = 1, \ldots, n)$ are deterministic constants,
- $\varrho_i,\ \varphi_i\ (i = 1, \ldots, n)$ are idependent random variables,
- φ_i is of uniform distribution between 0 and 2π $(i = 1, \ldots, n)$,
- ϱ_i is a non-negative random variable that likes the low values, that is, its density function is high at the neighbourhood of zero.

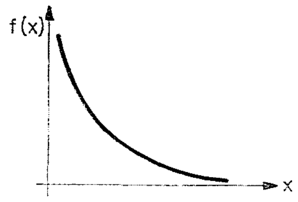
It is primarily important to determine the covariance function of this process. In this article the necessary calculations are written down. As it will be seen, the calculations are based on the assumption that ϱ_i has a second order χ-distribution $(i = 1, \ldots, n)$. In this case normally distributed random variables arise, permitting in fact to calculate the expected value of some random variables. The second-order χ-distribution fits the description of the phenomenon since its density function is

$$f(x) = \begin{cases} \dfrac{1}{x} \cdot e^{-\frac{x^2}{2}} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leqslant 0 \,. \end{cases}$$

The calculation of the covariance function

The covariance function of the stochastic process ξ_t is defined in the following way:

$$b(t, s) = M(\xi_t\, \xi_s) - M(\xi_t) \cdot M(\xi_s).$$



PROPOSITION:

$$b(t, s) = \frac{e^{-\sum_{i=1}^{n} c_i^2}}{2}\left[\left(e^{-\sum_{i=1}^{n} c_i^2 \cos w_i (t-s)} - 1\right)\cos a(t+s) + \right.$$

$$\left. + \left(e^{\sum_{i=1}^{n} c_i^2 \cos w_i (t-s)} - 1\right)\cos a(t-s)\right].$$

Proof:

Let us assume that ξ and η are independent random variables and their common distribution is standard normal distribution. Let ϱ and φ, — also random variables — be the polar co-ordinates of the point (ξ, η). ϱ and φ are known to be independent, ϱ to be of second-order χ-distribution, and φ of uniform distribution.

It is obvious that also the contrary is true: if ϱ has a second-order χ-distribution, φ a uniform distribution, and ϱ and φ are independent, then $\xi = \varrho \cdot \cos \varphi$ and $\eta = \varrho \cdot \sin \varphi$ are independent and their common distribution is a normal distribution.

It follows that if t is fixed, then $\varrho_i \cos (w_i t + \varphi_i)$ has a standard normal distribution. So if t is fixed, then

$$\alpha_t = a \cdot t + \sum_{i=1}^{n} c_i \varrho_i \cos (w_i t + \varphi_i)$$

is normally distributed, and its expected value is $a \cdot t$, and its variance is

$$\sqrt{\sum_{i=1}^{n} c_i^2}.$$

Lemma 1:

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \cos (c \cdot t)\, dt = \sqrt{2\pi}\, e^{-\frac{c^2}{2}}.$$

***Proof*:**

From the identity $\cos(ct) = \frac{1}{2}[e^{ict} + e^{-ict}]$ we get

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \cos(ct)\, dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{ict}\, dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{ic(-t)}\, dt.$$

The substitution $u = -t$ transforms the second integral on the right side into the first one, so:

$$= \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{ict}\, dt = e^{-\frac{c^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(t-ic)^2}{2}} dt = \sqrt{2\pi}\, e^{-\frac{c^2}{2}}.$$

At the last step the well-known — identity was used:

$$\int_{-\infty}^{\infty} e^{-\frac{(t-ic)^2}{2}} dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi}.$$

Lemma 2:

If the random variable ε has a normal distribution, and its expected value and variance are m and σ respectively, then

$$M(\cos\varepsilon) = e^{-\frac{\sigma^2}{2}} \cos m.$$

Proof:

$$M(\cos\varepsilon) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}} \cos x\, dx =$$

Substituting $t = \frac{x-m}{\sigma}$, then using the identity $\cos(m + \sigma t) =$

$= \cos m \cos \sigma t - \sin m \sin \sigma t$ we get

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \cos(m + \sigma t)\, dt =$$

$$= \frac{\cos m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \cos \sigma t\, dt - \frac{\sin m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \sin \sigma t\, dt.$$

Here the second integral is equal to zero, since the integrand is an odd function. Applying lemma 1 to the first integral we get the proposition of lemma 2.

Now the proof of the PROPOSITION will be continued. From lemma 2 it follows that

$$M(\xi_t) = M(\cos \alpha_t) = e^{-\frac{\sum_{i=1}^{n} c^2}{2}} \cos (at) .$$

Hence:

$$M(\xi_t)\cdot M(\xi_s) = e^{-\frac{\sum_{i=1}^{n} c_i^2}{2}} \cos (at)\, e^{-\frac{\sum_{i=1}^{n} c_i^2}{2}} \cos as =$$

$$= \frac{e^{-\sum_{i=1}^{n} c_i^2}}{2} (\cos a(t+s) + \cos a(t-s)).$$

$M(\xi_t\xi_s)$ will be determined from the identity

$$\xi_t\, \xi_s = \cos \alpha_t \cos \alpha_s = \frac{1}{2} \cos (\alpha_t + \alpha_s) + \frac{1}{2} \cos (\alpha_t - \alpha_s) .$$

If t and s are fixed, then the distribution of the random variable $\alpha_t \pm \alpha_s$ is normal distibution since

$$\alpha_t \pm \alpha_s = a \cdot (t \pm s) + \sum_{i=1}^{n} c_i\, \varrho_i \left(\cos (w_i t + \varphi_i) \pm \cos (w_i s + \varphi_i)\right) =$$

$$= a \cdot (t \pm s) + \sum_{i=1}^{n} c_i\, \varrho_i \left((\cos w_i t \pm \cos w_i s) \cos\varphi_i - (\sin w_i t \pm \sin w_i s)\cdot \sin \varphi_i\right).$$

Here $\varrho_i \cos \varphi_i$, $\varrho_i \sin \varphi_i$ $(i = 1, \ldots, n)$ are independent random variables with normal distribution, so $\alpha_t \pm \alpha_s$ is in fact normally distributed, and its expected value and variance are: $a(t \pm s)$ and

$$\sqrt{\sum_{i=1}^{n} c_i^2 \left[(\cos w_i t \pm \cos w_i s)^2 + (\sin w_i t \pm \sin w_i s)^2\right]} =$$

$$= \sqrt{2 \sum_{i=1}^{n} c_i^2 \left(1 \pm \cos w_i (t-s)\right)} .$$

So, lemma 2 yields:

$$M(\xi_t \cdot \xi_s) = \frac{1}{2} M\big(\cos (\alpha_t + \alpha_s)\big) + \frac{1}{2} M\big(\cos (\alpha_t - \alpha_s)\big) =$$

$$= \frac{1}{2} e^{-\sum_{i=1}^{n} c_i^2 (1 + \cos \omega_i(t-s))} \cdot \cos a(t + s) +$$

$$+ \frac{1}{2} e^{-\sum_{i=1}^{n} c_i^2 (1 + \cos \omega_i(t-s))} \cdot \cos a(t - s) =$$

$$= \frac{e^{-\sum_{i=1}^{n} c_i^2}}{2} \left[e^{-\sum_{i=1}^{n} c_i^2 \cos \omega_i (t-s)} \cos a(t + s) + \right.$$

$$\left. + e^{-\sum_{i=1}^{n} c_i^2 \cos \omega_i (t-s)} \cos a(t - s) \right].$$

Using this result it is easy to derive that $b(t, s) = M(\xi_t \xi_s) - M(\xi_t) M(\xi_s)$ equals the formula given in the PROPOSITION.

Summary

Theoretically it is no problem to determine the covariance function of a stochastic process but in case of actual stochastic processes it needs sometimes long calculations and some skill. In this work the covariance function of an actual stochastic process is determined. At each step it is indicated what theoretical assumptions the calculations are based on.

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