INTERPOLATION BY CUBIC SPLINES

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1. Introduction

This paper is devoted to the study of interpolation by cubic spline functions, i.e., piecewise cubic polynomials. It will be shown that, given a suitable partition Δ of $I = [0,1]$ and the derivative values of a given function $f(x)$, at the knots of Δ, together with the function values at the end points there exists a unique cubic spline on I, of class $C^2(I)$, which is the interpolent of f. The same can be said if the derivative and function values are given consecutively rather than the derivative values at the end points of Δ. Moreover, given the function values at the even knots and the derivative values at the odd ones, or given the function and derivative values at the odd knots together with the function values at the end points of Δ, there exists a unique spline $s(x) \in C^1(I)$ which is a piecewise polynomial on each double subinterval. To this end the following notations, definitions and results will be needed (see [1], [2], and [3]) throughout this work.

$$\text{Let } I = [0,1] \text{ and } \Delta\colon 0 = x_0 < x_1 < \ldots < x_{N+1} = 1$$

be a partition of I, f_i either a real number given at the point x_i or the value of a given function $f(x)$ at this point, i.e., $f_i = f(x_i)$ and $Df(x_i) = f_i^1$. For each non-negative integer m and for each p, $1 \leq p \leq \infty$, let $PC^{m,p}(a, b)$ denote the collection of all real valued functions $\varphi(x)$ such that: $\varphi(x) \in C^{m-1}[a, b]$, and such that $D^m\varphi \in C(x_i, x_{i+1})$; (x_i, x_{i+1}) is an open subinterval and $D^m\varphi \in L^p[a, b]$ where

$$\|D^m\varphi\|_p = \left(\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |D^m\varphi(x)|^p\right)^{1/p} < \infty\,,$$

$$\|D^m\varphi\|_\infty = \max_{o \leq i \leq N} \sup_{x \in (x_i, x_{i+1})} |D^m\varphi| < \infty\,.$$

Definition 1.1

Given Δ, let the space of cubic splines with respect to Δ, $S(\Delta)$, be the vector space of all twice continuously differentiable, piecewise cubic polynomials on I with respect to Δ, i.e.,

$S(\Delta) \equiv \{p(x) \in C^2(I) \mid p(x)$ is a cubic polynomial on each subinterval $[x_i, x_{i+1}]$, $o \leqq i \leqq N$, defined by $\Delta\}$.

Definition 1.2

Given $f = \{f_0, \ldots, f_{N+1}, f_0^1, f_{N+1}^1\}$, let $\vartheta_s f$, the $S(\Delta)$ — interpolate of f, be the unique spline, $s(x)$, in $S(\Delta)$ such that $s(x_i) = f_i$, $o \leqq i \leqq N + 1$, and $Ds(x_i) = f_i^1$, $i = o, N + 1$.

It is a known result [2], that this procedure is well defined according to the following

Theorem 1.1

Given numbers f_i, $o \leqq i \leqq N + 1$, and f_i^1, $i = o$, $N + 1$, there exists a unique spline $s(x)$ such that

$$s(x_i) = f_i, \quad o \leqq i \leqq N + 1,$$

and

$$Ds(x_i) = f_i^1, \quad i = o, \quad N + 1.$$

2. Approximation theorems

2.1 Single step interpolation

Here, we are going to solve the following problem: given $f = \{f_0^1, f_1^1, \ldots, f_{N+1}^1, f_0, f_{N+1}\}$, let ${}^*\vartheta_s f$, the $S(\Delta)$ — interpolate of f, be the unique spline, $s(x)$, in $S(\Delta)$ such that

$$Ds(x_i) = f_i^1, \ o \leqq i \leqq N + 1,$$

and

$$s(x_i) = f_i, \ i = o, \ N + 1.$$

We have to prove that this procedure is well defined.

Theorem 2.1

Assume that $h_{i+1} > h_i$ for all i, or let N be even and the partition be uniform. Given numbers f_i^1, $0 \leqq i \leqq N + 1$, and f_i, $i = o$, $N + 1$, there exists a unique spline $s(x)$ such that $Ds(x_i) = f_i^1$, $o \leqq i \leqq N + 1$, and $s(x_i) = f_i$, $i = o$, $N + 1$, *Proof*

In the subinterval $[x_i, x_{i+1}]$, choose $s(x)$ to agree with the cubic polynomial $p(x)$ such that

$$p(x_i) = s_i,\ p(x_{i+1}) = s_{i+1},$$
$$Dp(x_i) = f_i^1,\ Dp(x_{i+1}) = f_{i+1}^1,$$

such a polynomial exists by the theory of Hermite interpolation (see [2]), therefore

$$D^2 s(x_i) = -\frac{6}{h_i^2} s_i + \frac{6}{h_i^2} s_{i+1} - \frac{4}{h_i} f_i^1 - \frac{2}{h_i} f_{i+1}^1.$$

A similar expression for $D^2 s(x_i)$ in $[x_{i-1}, x_i]$ is given by

$$D^2 s(x_i) = \frac{6}{h_{i-1}^2} s_{i-1} - \frac{6}{h_{i-1}^2} s_i - \frac{4}{h_{i-1}} f_i^1 + \frac{2}{h_{i-1}} f_{i-1}^1.$$

Hence, for $D^2 s(x)$ to be continuous at x_i, we obtain (with the notation $h_i = x_{i+1} - x_i$)

$$\begin{aligned} -3h_{i-1}^{-1} h_i s_{i-1} + 3(h_{i-1}^{-1} h_i - h_i^{-1} h_{i-1}) s_i + 3h_i^{-1} h_{i-1} s_{i+1} = \\ = h_i f_{i-1}^1 + 2(h_i + h_{i-1}) f_i^1 + h_{i-1} f_{i+1}^1 \end{aligned} \tag{2.1}$$

For $1 \leq i \leq N$, (2.1) is a system of N linear equations in the unknowns s_i, $i = 1, 2, \ldots, N$. It can be written in the matrix form $\mathbf{A}s = d$, where $\mathbf{A} = [a_{ij}]$ is

$$\mathbf{A} = \begin{bmatrix} 1 & \lambda_0 & 0 & & & & \\ \mu_1 & 1 & \lambda_1 & & \bigcirc & & \\ 0 & \mu_2 & 1 & \lambda_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mu_{N-2} & 1 & \lambda_{N-2} \\ & \bigcirc & & 0 & \mu_{N-1} & 1 \end{bmatrix} \quad (2.2$$

and

$$\lambda_i = \frac{h_i^2}{h_{i+1}^2 - h_i^2}, \qquad i = 0, \ldots, N-2,$$

$$\mu_i = \frac{k_{i+1}^2}{h_{i+1}^2 - h_i^2}, \qquad i = 1, \ldots, N-1,$$

$$\lambda_i + \mu_i = -1, \qquad i = 1, \ldots, N-2.$$

For the existence and uniqueness of the solution, the matrix **A** has to be non-singular, i.e., det $(A) \neq 0$, but this fact can be proved by Gauss elimination if $h_{i+1} > h_i$.

In the case of a *uniform partition*, i.e., $h_i = h_{i-1} = h$, $1 \leq i \leq N$, (2.1) takes the form

$$-3s_{i-1} + 3s_{i+1} = hf^1_{i-1} + 4hf^1_i + hf^1_{i+1},\ 1 \leq i \leq N, \tag{2.3}$$

and hence the matrix **A** will be:

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 0 & & \bigcirc \\ -3 & 0 & 3 & & \\ & \ddots & \ddots & \ddots & \\ & & & 0 & 3 \\ \bigcirc & & & -3 & 0 \end{bmatrix}$$

In the case where N *is even*, obviously the matrix will be *non-singular* and it will be *singular if* N *is odd.*

Remark

Equation (2.3) is exactly the Simpson's Rule applied to s' (x), which is a polynomial of degree two, in the interval $[x_{i-1}, x_{i+1}]$, i.e.,

$$\int_{x_{i-1}}^{x_{i+1}} s'(x)\, dx = \frac{h}{3}\left[s^1_{i-1} + 4s^1_i + s^1_{i+1}\right],$$

where $s^1_i = f^1_i$ where N is required to be an even integer.

Now, combining the two cases prescribed by the previous theorems we obtain:

Theorem 2.2

Let N be even and the partition be uniform. There exists unique spline $s \in S(\Delta)$ such that

$$(i)\ s(x_{2j}) = f_{2j},\ 0 \leq j \leq \frac{1}{2}N;\ s(x_{N+1}) = f_{N+1},$$

$$(ii)\ Ds(x_{2j+1}) = f^1_{2j+1},\ 0 \leq j \leq \frac{1}{2}(N-2),$$

$$(iii)\ Ds(x_i) = f^1_i,\ i = 0, N+1.$$

Proof

In the subinterval $[x, x_{i+1}]$, i even, we choose $s(x)$ to agree with the cubic polynomial $p(x)$ such that

$$p(x_i) = f_i, \qquad p(x_{i+1}) = s_{i+1},$$
$$Dp(x_i) = s_i^1, \quad Dp(x_{i+1}) = f_{i+1}^1,$$

where s_i^1 and s_{i+1} remain to be determined.

$$s(x) = f_i\left[\frac{(x-x_{i+1})^2}{h_i^2} + 2\,\frac{(x-x_i)(x-x_{i+1})^2}{h_i^3}\right] +$$
$$s_{i+1}\left[\frac{(x-x_{i+1})^2}{h_i^2} - 2\,\frac{(x-x_{i+1})(x-x_i)^2}{h_i^3}\right] + s_i^1\,\frac{(x-x_i)(x-x_{i+1})^2}{h_i^2} +$$
$$+ f_{i+1}^1\,\frac{(x-x_i)^2\,(x-x_{i+1})}{h_i^2}\,.$$

Hence

$$D^2\, s(x_i) = -\frac{6}{h_i^2} f_i + \frac{6}{h_i^2}\, s_{i+1} - \frac{4}{h_i}\, s_i^1 - \frac{2}{h_i} f_{i+1}^1\,.$$

A similar expression for $D^2 s(x_i)$ in the subinterval $[x_{i-1}, x_i]$ is given by

$$D^2\, s(x_i) = -\frac{6}{h_{i-1}^2}\, s_{i-1} - \frac{6}{h_i^2} f_i + \frac{4}{h_{i-1}}\, s_i^1 + \frac{2}{h_{i-1}} f_{i-1}^1.$$

So, for $D^2 s(x)$ to be continuous at the even knots, we obtain

$$\begin{aligned} &3h_{i-1}^{-2}\, s_{i-1} + 2\,(h_i^{-1} + h_{i-1}^{-1})\, s_i^1 - 3h_i^{-2}\, s_{i+1} - = \\ &= -3(h_i^{-2} - h_{i-1}^{-2})\, j_i - h_{i-1}^{-1} f_{i-1}^1 - h_i^{-1} f_{i+1}^1\,. \end{aligned} \tag{2.4}$$

In the case where i is odd, for $D^2 s(x)$ to be continuous at odd knots we have:

$$\begin{aligned} &h_{i-1}^{-1}\, s_{i-1}^1 + 3\big(h_i^{-2} - h_{i-1}^{-2}\big)\, s_i + h_i^{-1}\, s_{i+1}^1 = \\ &= -\,3\, h_{i-1}^{-2} f_{i-1} - 2\,(h_i^{-1} + h_{i-1}^{-1})\, f_i^1 + 3h_i^{-2} f_{i+1}\,. \end{aligned} \tag{2.5}$$

Equations (2.4), (2.5) form a system of N linear equations in the unknowns $s_1, s_3, \ldots, s_{N-1}$ and $s_2^1, s_4^1, \ldots, s_N^1$ for N even. The system (2.4), (2.5) can be written in the matrix form

$$\mathbf{B}s = k. \tag{2.6}$$

In the special case of a uniform partition $h_i = h_{i-1} = h$, $1 \leq i \leq N$, the system (2.4), (2.5) becomes

$$3s_{i-1} + 4hs_i^1 - 3s_{i+1} = -hf_{i-1}^1 - hf_{i+1}^1, \quad i \text{ even}, \tag{2.7}$$

$$hs_{i-1}^1 + hs_{i+1}^1 = -3f_{i-1} - 4hf_i^1 + 3f_{i+1}, \quad i \text{ odd}, \tag{2.8}$$

and the matrix **B** will be

$$\mathbf{B} = \begin{bmatrix} 0 & h & 0 & & \\ 3 & 4h & -3 & & \bigcirc \\ & \ddots & \ddots & \ddots & \\ \bigcirc & & h & 0 & h \\ & & 0 & 3 & 4h \end{bmatrix}$$

which is non-singular in the case N is even and otherwise the matrix is singular. Indeed, using the technique of symmetrization, denoting $D_N = \det(\mathbf{D}\tilde{\mathbf{B}}\mathbf{D}^{-1})$, we have the recurrence formula

$$D_k = a_k D_{k-1} - b_{k-1}^2 D_{k-2}$$

with $D_0 = 1$, $D_1 = 0$, where $a_{2k} = 4h$, $a_{2k+1} = 0$ and $b_{k-1} \neq 0$. It follows that $D_{2k} \neq 0$ and $D_{2k+1} = 0$, for all k. The vector k is

$$k = \begin{bmatrix} -3f_0 + 3f_2 - 4hf_1^1 - hf_0^1, \; i = 1 \\ -hf_{i-1}^1 - hf_{i+1}^1, \; i = 2j, 1 \leq j \leq \frac{1}{2}(N-2), N > 2 \\ -3f_{i-1} - 4hf_i^1 + 3f_{i+1}, \; i = 2j+1, \; 1 \leq j \leq \frac{1}{2}(N-2), N > 2 \\ -hf_{N-1}^1 - hf_{N+1}^1 + 3f_{N+1}, \; i = N \end{bmatrix}$$

and *we denote by $\tilde{\vartheta}_s f$ the unique spline defined by* (*i*), (*ii*), (*iii*).

Unfortunately, we have not yet found a general sufficient condition on the step sizes h_i to assure the non-singularity of matrix **B**. However, it is easy to see *that for each partition there exists a position of the knot x_N such that if the matrix is singular, then its shifting to the right or left will mean non-singularity of the matrix.*

2.2 Double step interpolation

Let us examine how to develop an interpolation procedure which uses only the values $f = \{f_0, f_1^1, f_2, \ldots, f_N, f_{N+1}\}$, N odd, when given Δ and such that the interpolent function $s(x)$ is a piecewise cubic polynomial on each subinterval $[x_{2i}, x_{2i+2}]$, $0 \leq i \leq (N-1)/2$ and is continuously differentiable, i.e., $s(x) \in C^1(I)$.

Let $s(x)$ be the unique spline such that

$$\begin{cases} s(x_{2i}) = f_{2i},\ 0 \leq i \leq (N+1)/2\,, \\ s(x_N) = f_N, \\ Ds(x_{2i+1}) = f^1_{2i+1},\ 0 \leq i \leq (N-1)/2. \end{cases} \tag{2.8}$$

We shall see that this procedure is well defined.

Theorem 2.3

Given numbers f_{2i}, f^1_{2i+1}, $0 \leq i \leq (N-1)/2$ *and* f_i, $i = N, N+i$, *N odd, there exists a unique spline* $s(x) \in C(I)$ *which satisfies (2.8), provided* $h_{2i-1} \neq 2h_{2i-2}$.

Proof

In the interval $[x_{2i}, x_{2i+2}]$, choose $s(x)$ to agree with the cubic polynomial $p(x)$ such that

$$p(x_{2i}) = f_{2i},\ p(x_{2i+1}) = s_{2i+1}\,,$$

$$p(x_{2i+2}) = f_{2i+2},\ Dp(x_{2i+1}) = f^1_{2i+1}\,,$$

where s_{2i+1} remain to be determined, thus $s(x)$ can be written as

$$\begin{aligned} s(x) = &-f_{2i}\,\frac{(x-x_{2i+2})(x-x_{2i+1})^2}{h^2_{2i}(h_{2i}+h_{2i+1})} + f_{2i+2}\,\frac{(x-x_{2i})(x-x_{2i+1})^2}{h^2_{2i+1}(h_{2i}+h_{2i+1})} - \\ &- f^1_{2i+1}\,\frac{(x-x_{2i})(x-x_{2i+1})(x-x_{2i+2})}{h_{2i}h_{2i+1}} - \\ &- s_{2i+1}(x-x_{2i})(x-x_{2i+2})\left\{(x-x_{2i+1})\,\frac{(h_{2i}-h_{2i+1})}{h^2_{2i}h^2_{2i+1}} + \frac{1}{h_{2i}h_{2i+1}}\right\}. \end{aligned}$$

Hence

$$\begin{aligned} Ds(x_{2i}) = &-\frac{(3h_{2i}+2h_{2i+1})}{h_{2i}(h_{2i}+h_{2i+1})}\,f_{2i} + \frac{h^2_{2i}}{h^2_{2i+1}(h_{2i}+h_{2i+1})}\,f_{2i+2} - \\ &- \frac{(h_{2i}+h_{2i+1})}{h_{2i+1}}\,f^1_{2i+1} - \frac{(2h_{2i+1}-h_{2i})(h_{2i+1}+h_{2i})}{h_{2i}h^2_{2i+1}}\,s_{2i+1}\,, \end{aligned}$$

where

$$h_{2i} = x_{2i+1} - x_{2i},\ h_{2i+1} = x_{2i+2} - x_{2i+1}\,.$$

A similar expression for $Ds(x_{2i})$ in the interval $[x_{2i-2}, x_{2i}]$ is given by

$$Ds(x_{2i}) = -\frac{h_{2i-1}^2}{h_{2i-2}^2(h_{2i-2}+h_{2i-1})} f_{2i-2} + \frac{(3h_{2i-1}+2h_{2i-2})}{h_{2i-2}(h_{2i-2}+h_{2i-1})} f_{2i} - $$
$$ - \frac{(h_{2i-2}+h_{2i-1})}{h_{2i-2}} f_{2i-1}^1 - \frac{(2h_{2i-2}-h_{2i-1})(h_{2i-2}+h_{2i-1})}{h_{2i-1}h_{2i-2}^2} s_{2i-1}.$$

Then $Ds(x)$ is continuous at x_{2i} iff

$$\frac{(2h_{2i-2}-h_{2i-1})(h_{2i-2}+h_{2i-1})}{h_{2i-1}\,h_{2i-2}^2} s_{2i-1} - \frac{(2h_{2i+1}-h_{2i})(h_{2i+1}+h_{2i})}{h_{2i}\,h_{2i+1}^2} s_{2i+1} =$$
$$= -\frac{h_{2i-1}^2}{h_{2i-2}^2(h_{2i-2}+h_{2i-1})} f_{2i-2} - \frac{(2h_{2i-2}+h_{2i-1})}{h_{2i-2}} f_{2i-1}^1 +$$
$$+ \left[\frac{(3h_{2i-1}+2h_{2i-2})}{h_{2i-1}(h_{2i-2}+h_{2i-1})} + \frac{(3h_{2i}+2h_{2i+1})}{h_{2i}(h_{2i}+h_{2i+1})}\right] f_{2i} +$$
$$+ \frac{(h_{2i}+h_{2i+1})}{h_{2i+1}} f_{2i+1}^1 - \frac{h_{2i}^2}{h_{2i+1}^2(h_{2i}+h_{2i+1})} f_{2i+2},$$
$$i = 1, \ldots, (N-1)/2\,. \tag{2.9}$$

The system (2.9) can be written in the matrix form

$$\mathbf{M}s = e,$$

where $\mathbf{M} = [m_{ij}]$ is

$$\mathbf{M} = \begin{bmatrix} \mu_1 & \lambda_1 & & & \bigcirc \\ & \mu_3 & \lambda_3 & & \\ & & \ddots & \ddots & \\ \bigcirc & & & \mu_{N-4} & \lambda_{N-4} \\ & & & & \mu_{N-2} \end{bmatrix}$$

and

$$\mu_{2i-1} = \frac{(2h_{2i-2}-h_{2i-1})(h_{2i-2}+h_{2i-1})}{h_{2i-2}^2\,h_{2i-1}} \qquad i = 1, \ldots, (N-1)/2,$$
$$\lambda_{2i-1} = \frac{(h_{2i}-2h_{2i+1})\,(h_{2i}+h_{2i+1})}{h_{2i+1}^2\,h_{2i}} \qquad i = 1, \ldots, (N-3)/2.$$

Hence the condition for the regularity of $\mathbf{M}$ is that

$$h_{2i-1} \neq 2h_{2i-2}\,. \;\square$$

In the special case of a uniform partition, i.e., $h_{2i-2} = h_{2i-1} = h_{2i+1} = h$, (2.9) will take the form

$$s_{2i-1} - s_{2i+1} = -\frac{1}{4} f_{2i-2} - hf^1_{2i-1} + \frac{5}{2} f_{2i} + hf^1_{2i+1} - \frac{1}{4} f_{2i+2}, \quad (2.10)$$

$$1 \leq i \leq (N-1)/2.$$

Similarly it is easy to show that the following theorem is also true.

Theorem 2.4

Given numbers f_i, $0 \leq i \leq N+1$ and f^1_N, N odd, there exists a unique cubic spline $s(x) \in C^1(I)$ such that

$s(x_i) = f_i$, $0 \leq i \leq N+1$,

$Ds(x_N) = f^1_N$.

It can be shown that no restriction on the partition is required.
Finally, given f $\{f_0, f_1, f^1_1, \ldots, f_N, f^1_N, f_{N+1}\}$, N odd, such that

$$s(x_{2i+1}) = f_{2i+1}, \; Ds(x_{2i+1}) = f^1_{2i+1}, \; 0 \leq i \leq (N-1)/2, \quad (2.11)$$
$$s(x_i) = f_i, \; i = 0, \; N+1.$$

We shall show that there exists a unique cubic spline $s(x) \in C^1(I)$ which is piecewise continuous on each subinterval $[x_{2i}, x_{2i+2}]$, $0 \leq i \leq (N-1)/2$.

Theorem 2.5

Given numbers f_{2i+1}, f^1_{2i+1}, $0 \leq i \leq (N-1)/2$ and f_i, $i = 0, N+1$, N odd, there exists a unique cubic spline $s(x) \in C^1(I)$ which satisfies (2.11), under some conditions described in the proof.

Proof: The proof can be handled as in Theorem 2.3.

The matrix form of it can be written as

$$\Lambda s = k, \quad (2.12)$$

where $\mathbf{\Lambda} = [\lambda_{ij}]$ is

$$\lambda_{ij} = \begin{cases} \dfrac{3h_{2i-2}^2}{h_{2i-2}^2}\left(\delta_i + \dfrac{2}{3}\alpha_i\right) + \dfrac{3h_{2i+1}^2}{h_{2i}^2}\left(\gamma_i + \dfrac{2}{3}\beta_i\right), & i = j \\ -\delta_i, \quad j = i-1, \\ -\gamma_i, \quad j = i+1, \\ 0, \qquad \text{otherwise}, \end{cases}$$

i.e., $\mathbf{\Lambda}$ is a diagonally dominant tridiagonal matrix if

$$\frac{3h_{2i-2}^2}{h_{2i-1}^2} \geq 1, \quad \frac{3h_{2i+1}^2}{h_{2i}^2} \geq 1,$$

consequently,

$$\frac{1}{\sqrt{3}} h_{2i} \leq h_{2i+1} \leq \sqrt{3}\, h_{2i}, \quad i = 1, \ldots, (N-3)/2,$$

and

$$h_1 \leq \sqrt{3}\, h_0, \quad h_{N-1} \leq \sqrt{3}\, h_N$$

where

$$\alpha_i = \frac{h_{2i-1}}{h_{2i-2}\,(h_{2i-2} + h_{2i-1})}, \quad \beta_i = \frac{h_{2i}}{h_{2i+1}(h_{2i} + h_{2i+1})},$$

and

$$\gamma_i = \frac{h_{2i}}{h_{2i+1}}\beta_i, \; \delta_i = \frac{h_{2i-1}}{h_{2i-2}}\alpha_i \,. \; \square$$

In the case of a uniform partition, (2.12) will be (2.13)

$$s_{2i-2} - 10s_{2i} + s_{2i+2} = -4hf_{2i-1}^1 + 4hf_{2i+1}^1 - 4f_{2i-1} + 4f_{2i+1}, \quad 0 \leq i \leq (N-1)/2,$$

with nonsingular coefficient matrix.

3. Error analysis

Here, we shall give *a priori* error bounds for the interpolation procedure introduced in item 2.1 in the L^2-norm. The spline interpolating function as defined in item 1 is known [2] to be characterized as the solution of a variational problem.

The question now arises, whether the same theorem is valid or not in the case of spline function defined by (i), (ii), (iii).

Theorem 3.1

Given Δ and $f = \{f_0, f_1^1, \ldots, f_{N+1}, f_0^1, f_{N+1}^1\}$, N even. Let

$$V = \Big\{ w \in PC^{2,2}(I) \mid w(x_{2j}) = f_{2j},\ 0 \leq j \leq \frac{1}{2} N;\ w(x_{N+1}) = f_{N+1},\ \text{and}$$

$$Dw(x_{2j+1}) = f_{2j+1}^1,\ 0 \leq j \leq \frac{1}{2}(N-2) \ \text{and}\ Dw(x_i) = f_i^1,\ i = 0, N+1 \Big\}.$$

Then the variational problem of finding the functions $p \in V$ which minimize $\| D^2 w \|^2$, for all $w \in V$, has the unique solution $\tilde{\vartheta}_s f$, whenever it exists.

Proof

As in the proof of Theorem 3.1 [2], $p \in V$ is a solution of the variational problem iff

$$(D^2 p, D^2 \delta)_2 = 0, \tag{3.1}$$

for all $\delta \in V_0 = \Big\{ w \in PC^{2,2}(I) \mid w(x_{2j}) = 0,\ 0 \leq j \leq \frac{1}{2} N;\ w(x_{N+1}) = 0$ and

$$Dw(x_{2j+1}) = 0.\ 0 \leq j \leq \frac{1}{2}(N-2) \ \text{and}\ Dw(x_i) = 0,\ i = 0, N+1 \Big\}.$$

Moreover, the variational problem has a unique solution. Now it remains to show that $\tilde{\vartheta}_s f$ is a solution of (3.1) i.e.,

$$(D\, \tilde{\vartheta}_s f, D^2 \delta)_2 = 0, \quad \delta \in V_0. \tag{3.2}$$

Since

$$(D^2 \tilde{\vartheta}_s f, D^2 \delta)_2 = \int_0^1 D^2 \tilde{\vartheta}_s f(x)\, D\, \delta(x)\, dx =$$

$$= \sum_{j=0}^{\frac{1}{2}N} \int_{x_{2j}}^{x_{2j+1}} D^2 \tilde{\vartheta}_s f(x)\, D^2 \delta(x)\, dx + \sum_{j=0}^{\frac{1}{2}(N-2)} \int_{x_{2j+1}}^{x_{2j+2}} D\, \tilde{\vartheta}_s f(x)\, D^2 \delta(x)\, dx\,.$$

Integrating by parts it is easy to see that (3.2) is satisfied, because $\tilde{\vartheta}_s f$ is a cubic polynomial on each subinterval.

Corollary 3.1

If $f \in PC^{2,2}(I)$, *then*

$$\| D^2 \tilde{\vartheta}_s f \|^2 + \| D^2 \tilde{\vartheta}_s f - D^2 f \|^2 = \| D^2 f \|^2 .$$

Lemma 3.1 (see [2])

If $f \in PC_0^{1,2}(a, b)$, *then*

$$\int_a^b f^2(x)\, dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b (Df(x))^2\, dx, \text{ where}$$

$$PC_0^{1,2}(a, b) = \{\varphi \in PC^{1,2} \mid \varphi(a) = \varphi(b) = 0\}.$$

Theorem 3.2

If $f \in PC^{2,2}(I)$, *then*

$$\| D^2 (f_n - \tilde{\vartheta}_s f) \|_2 \leq \| D^2 f \|_2, \tag{3.3}$$

$$\| D(f - \tilde{\vartheta}_s f) \|_2 \leq \frac{4h}{\pi} \| D^2 f \|_2 , \tag{3.4}$$

and

$$\| f - \tilde{\vartheta}_s f \|_2 \leq \frac{8h^2}{\pi^2} \| D^2 f \|_2 , \tag{3.5}$$

where $h = \max (x_{i+1} - x_i)$.

$$0 \leq i \leq N.$$

Proof

Inequality (3.3) follows immediately from Corollary 3.1. To prove (3.4), let $e(x) = f(x) - \tilde{\vartheta}_s f(x)$. Since $e(x_{2j}) = 0$, $0 \leq j \leq \frac{1}{2} N$, and $e(x_{N+1}) = 0$, N even, then by Rolle's Theorem

$$De(\xi_j) = 0,\ 0 \leq j \leq \frac{1}{2}(N-2),\ x_{2j} < \xi_j < x_{2j+2} ,$$

and

$$De(\xi_{N/2}) = 0,\ x_N < \xi_{N/2} < x_{N+1} .$$

Then using Lemma 3.1, we have

$$\int_{\xi_j}^{\xi_{j+1}} [De(x)]^2\,dx \le \frac{(4h)^2}{\pi^2} \int_{\xi_j}^{\xi_{j+1}} [D^2e(x)]^2\,dx,\ 0 \le j \le \frac{1}{2}(N-2), \tag{3.6}$$

$$\int_0^{\xi_0} [De(x)]^2\,dx \le \frac{(2h)^2}{\pi^2} \int_0^{\xi_0} [D^2 e(x)]^2\,dx\,, \tag{3.7}$$

and

$$\int_{\xi_{N/2}}^{1} [De(x)]^2\,dx \le \frac{h^2}{\pi^2} \int_{\xi_{N/2}}^{1} [D^2 e(x)]^2\,dx\,. \tag{3.8}$$

Hence

$$\left[\int_0^{\xi_0} + \sum_{j=0}^{\frac{1}{2}(N-2)} \int_{\xi_j}^{\xi_{j+1}} + \int_{\xi_{N/2}}^{1}\right](De(x))^2\,dx \le \frac{(4h)^2}{\pi^2}\left[\int_0^{\xi_0} + \sum_{j=0}^{\frac{1}{2}(N-2)} \int_{\xi_j}^{\xi_{j+1}} + \int_{\xi_{N/2}}^{1}\right](D^2e(x))^2\,dx,$$

i.e.,

$$\| De(x) \|_2 \le \frac{4h}{\pi} \| D^2e(x) \|_2, \tag{3.9}$$

by using (3.3), we have

$$\| De(x) \|_2 \le \frac{4h}{\pi} \|D^2f(x) \|_2 .$$

In a similar manner, it is easy to prove (3.5) using (3.9). □

*We now turn to the *a priori* error bounds for the interpolation error, $f - \overset{*}{\vartheta}_s f$, and its derivatives.

Theorem 3.3

Let Δ and $f = \{f_0^1, f_1^1, \dots, f_{N+1}^1, f_0, f_{N+1}\}$ be given, $V = \{w \in PC^{2,2}(I) \mid Dw(x_i) = f_i^1,\ 0 \le i \le N+1$ and $w(x_i) = f_i,\ i = 0, N+1\}$. Then the variational problem of finding the functions $p \in V$ which minimize $\| D^2w \|_2^2$, for all $w \in V$, has the unique solution $\overset{*}{\vartheta}_S f$, whenever it exists.

Proof

The proof is similar to that of Theorem 3.2 but for completeness it will be outlined here.

As in the proof of Theorem 3.1 [2], $p \in V$ is a solution of the variational problem iff

$$(D^2p, D^2\delta)_2 = 0, \tag{3.10}$$

for all $\delta \in V_0 = \{wPC^{2,2}(I) \mid Dw(x_i) = 0, 0 \leq i \leq N+1$ and $w(x_i) = 0, i = 0, N+1\}$, i.e., the variational problem has a unique solution. Now, we shall show that $\overset{*}{\vartheta}_s f$ is a solution of (3.10), i.e.,

$$(D^2 \overset{*}{\vartheta}_s f, D^2\delta)_2 = 0, \text{ for all } \delta \in V_0 . \tag{3.11}$$

But

$$(D^2 \overset{*}{\vartheta}_s f, D^2 \delta)_2 = \int_0^1 D^2 \overset{*}{\vartheta}_s f(x) D^2 \delta(x)\, dx =$$

$$= \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} D^2 \overset{*}{\vartheta}_s f(x) D^2 \delta(x)\, dx = \sum_{i=0}^{N} \Big[D\delta(x) \;\; D^2 \;\overset{*}{\vartheta}_s f(x) \Big]_{x_i}^{x_{i+1}} -$$

$$- \sum_{i=0}^{N} \Big[\delta(x) D^3 \overset{*}{\vartheta}_s f(x) \Big]_{x_i}^{x_{i+1}} - \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \delta(x) D^4 \overset{*}{\vartheta}_s f(x)\, dx \equiv 0 .$$

Theorem 3.4

If $f \in PC^{2,2}(I)$, then

$$\| D^2(f - \overset{*}{\vartheta}_s f) \|_2 \leq \| D^2 f \|_2, \tag{3.12}$$

$$\| D(f - \overset{*}{\vartheta}_s f) \|_2 \leq h/n \, \| D^2 f \|_2, \tag{3.13}$$

and

$$\| f - \overset{*}{\vartheta}_s f \|_2 \leq \frac{h}{\pi^2} \| D^2 f \|_2 . \tag{3.14}$$

Proof

(3.12) is a consequence of Corollary 3.1. To prove (3.13) we note that $Df(x_i) - D\overset{*}{\vartheta}_s f(x_i) = 0$, for all $0 \leq i \leq N+1$, and by Lemma 3.1 it follows that

$$\int_{x_i}^{x_{i+1}} [Df(x) - D \overset{*}{\vartheta}_s f(x)]^2\, dx \leq \frac{1}{\pi^2} (x_{i+1} - x_i)^2 \int_{x_i}^{x_{i+1}} [D^2 (f(x) - D^2 \overset{*}{\vartheta}_s f(x)]^2\, dx \tag{3.15}$$

for all $0 \leq i \leq N$.

Summing both sides of (3.15), (3.13) follows by taking the square root of both sides of the resulting inequality.

(3.14) is proved by using the fact that $f(x_i) - \overset{*}{\vartheta}_s f(x_i) = 0$, $i = 0, N+1$, and using Lemma 3.1 and (3.13), i.e.,

$$\int_0^1 [f(x) - \overset{*}{\vartheta}_s f(x)]^2 \, dx \leq \frac{1}{\pi^2} \int_0^1 [Df(x) - D \overset{*}{\vartheta}_s f(x)]^2 \, dx \leq$$

$$\leq \frac{1}{\pi^2} \| D(f - \overset{*}{\vartheta}_s f) \|_2^2 \leq \frac{h^2}{\pi^4} \| D(f - \overset{*}{\vartheta}_s f) \|_2^2 \cdot \square$$

Summary

It has been shown that, given a suitable partition of $I = [0, 1]$ and the derivative values of a given function $f(x)$, at the knots of Δ, together with the function values at the end points, there exists a unique cubic spline on I which is the interpolant of f. The same can be said if the derivative and function values are pairwise alternating. Similar questions have been investigated for double-step interpolation. *A priori* error bounds are also presented for single-step interpolation.

References

1. Ahlberg, J. H., E. N. Nilson and J. L. Walsh, Theory of Splines and Their Applications, Academic Press, New York 1967.
2. Schultz, M. H., Spline Analysis, Prentice-Hall, London 1973.
3. Rivilin, T. J. An Introduction to the Approximation of Functions, Blaisdell, New York 1969.
4. Sallam, S. M. M., Spline functions and stability questions for global approximations to ordinary differential equations, Dissertation, Budapest 1978.

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